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# On a time-dependent subdifferential evolution inclusion with a nonconvex upper-semicontinuous perturbation

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**Abstract.** We investigate the existence of local approximate and strong solutions for a time-dependent subdifferential evolution inclusion with a nonconvex upper-semicontinuous perturbation.

**Keywords.** Convex function, Evolution equation, Nonconvex perturbation, Subdifferential, Upper-semicontinuous operator.

**AMS (MOS) Subject Classifications.** 35G25, 47J35, 47H14

## 1 Introduction

For a given family of convex lower-semicontinuous functions $(f^t)_{t\in[0,T]}$, defined on a separable real Hilbert space $X$ with range in $\mathbb{R}\cup\{\infty\}$, and a family of multivalued operators $(B(t,.))_{t\in[0,T]}$ on $X$, we shall prove an existence theorem for evolution equations of type:

$$u'(t) + \partial f^t(u(t)) + B(t, u(t)) \ni 0\ ,\quad t \in [0, T]. \qquad (1)$$

For each $t$, $\partial f^t$ denotes the ordinary subdifferential of convex analysis. The operator $B(t, .) : X \rightrightarrows X$ is a multivalued perturbation of $\partial f^t$, dependent on the time $t$.

When the perturbation $B(t, .)$ is single valued and monotone, many existence, uniqueness and regularity results have been established, *see* Brezis [3] (if $f^t$ is independent of

$t$), Attouch-Damlamian [2] and Yamada [18]. The study of case $B(t,.)$ nonmonotone and upper-semicontinuous with convex closed values has been developed under some assumptions of compactness on $\operatorname{dom} f^t = \{x \in X \mid f^t(x) < \infty\}$ the *effective domain of* $f^t$. For example, Attouch-Damlamian [1] have studied the case $f$ independent of time. Otani [15] has extended this result with more general assumptions (the convex function $f^t$ depends on time). He has also studied the case where $-B(t,.)$ is the subdifferential of a lower semicontinuous convex function, *see* [14].
In this article, the operator $B(t,.)$ will be assumed upper-semicontinuous with compact values which are not necessary convex, and it is not assumed be a contraction map. Nevertheless, $-B(t,.)$ will be assumed cyclically monotone. Cellina and Staicu [7] have studied this type of inclusion when $f^t$ and $B(t,.)$ are not dependent on $t$.

This paper is organized as follows. In Section 2 we recall some definitions and results on time-dependent subdifferential evolution inclusions and upper-semicontinuity of operators which will be used in the sequel. We also introduce the assumptions of our main result. In Section 3 we obtain existence of approximate solutions for the problem (1) and give properties of these solutions. In Section 4 we establish existence theorem for the problem (1). We particularly study two cases where the family $(f^t)_t$ satisfies more restricted assumptions. Examples illustrate our results in Section 5.

## 2 Perturbed problem

Assume that $X$ is a real separable Hilbert space. We denote by $\|.\|$ the norm associated with the inner product $\langle .,. \rangle$ and the topological dual space is identified with the Hilbert space. Let $T > 0$ and $(f^t)_{t\in[0,T]}$ be a family of convex lower-semicontinuous (lsc, in short) proper functions on $X$. We will denote by $\partial f^t$ the ordinary subdifferential of convex analysis.

**Definition 2.1** *A function* $u : [0,T] \to X$ *is said* strong solution *of*

$$u' + \partial f^t(u) + B(t,u) \ni 0$$

*if* [1]*:* (i) *there exists* $\beta \in L^2(0,T;X)$ *such that* $\beta(t) \in B(t,u(t))$ *for a.e.* $t \in [0,T]$,

(ii) *$u$ is a solution of* $\begin{cases} u'(t) + \partial f^t(u(t)) + \beta(t) \ni 0 & \textit{for a.e. } t \in [0,T] \\ u(t) \in \operatorname{dom} f^t & \textit{for any } t \in [0,T]. \end{cases}$

The aim result of this article is, for each $u_0 \in \operatorname{dom} f^0$, the existence of a local strong solution $u$ of $u' + \partial f^t(u) + B(t,u) \ni 0$ with $u(0) = u_0$, when the values of the upper-semicontinuous multiapplication $B(t,.)$ are not convex.

We shall consider the following assumption on $(f^t)_{t\in[0,T]}$, *see* Kenmochi [10, 11]:

[1] As usual, $L^r(0,T;X)$ ($T \in ]0,\infty]$) denotes the space of $X$-valued measurable functions on $[0,T)$ which are $r^{th}$ power integrable (if $r = \infty$, then essentially bounded). For $r = 2$, $L^2(0,T;X)$ is a Hilbert space, in which $\|.\|_{L^2(0,T;X)}$ and $\langle .,. \rangle_{L^2(0,T;X)}$ are the norm and the scalar product.

$\mathbf{(H_0)}$: *for each* $r \geqslant 0$, *there are absolutely continuous real-valued functions* $h_r$ *and* $k_r$ *on* $[0,T]$ *such that:*

> *(i)* $h_r' \in L^2(0,T)$ *and* $k_r' \in L^1(0,T)$,
> *(ii) for each* $s,t \in [0,T]$ *with* $s \leqslant t$ *and each* $x_s \in \operatorname{dom} f^s$ *with* $\|x_s\| \leqslant r$ *there exists* $x_t \in \operatorname{dom} f^t$ *satisfying*
>
> $$\begin{cases} \|x_t - x_s\| \leqslant |h_r(t) - h_r(s)|(1 + |f^s(x_s)|^{1/2}) \\ f^t(x_t) \leqslant f^s(x_s) + |k_r(t) - k_r(s)|(1 + |f^s(x_s)|). \end{cases}$$

or the slightly stronger assumption, *see* Yamada [18], denoted by **(H)**, when (ii) holds for any $s,t$ in $[0,T]$.

The following existence theorem have been proved in [19]:

**Theorem 2.1** *Let* $T > 0$ *and* $\beta \in L^2(0,T;X)$. *Let* $u_0 \in \operatorname{dom} f^0$. *If* $(H_0)$ *holds, then the problem*

$$\begin{cases} u'(t) + \partial f^t(u(t)) + \beta(t) \ni 0\ , \quad \text{a.e. } t \in [0,T] \\ u(t) \in \operatorname{dom} f^t\ , \quad t \in [0,T] \\ u(0) = u_0 \end{cases}$$

*has a unique solution* $u : [0,T] \to X$ *which is absolutely continuous.*

Furthermore, we have the following type of energy inequality, *see* [11, Chapter 1]: if $\|u(t)\| < r$ for $t \in [0,T]$, then

$$f^t(u(t)) - f^s(u(s)) + \frac{1}{2}\int_s^t \|u'(\tau)\|^2\, d\tau \leqslant \frac{1}{2}\int_s^t \|\beta(\tau)\|^2\, d\tau + \int_s^t c_r(\tau)\,[\,1 + |f^\tau(u(\tau))|\,]\, d\tau \tag{2}$$

for any $s \leqslant t$ in $[0,T]$, where $c_r : \tau \mapsto 4|h_r'(\tau)|^2 + |k_r'(\tau)|$ is an element of $L^1(0,T)$.

Let us add a compactness assumption on each $f^t$ by using the following definition:

**Definition 2.2** *A function* $f : X \to \mathbb{R} \cup \{+\infty\}$ *is said* of compact type *if the set* $\{x \in X \mid |f(x)| + \|x\|^2 \leq c\}$ *is compact at each level* $c$.

Denote by $L^2_w(0,T;X)$ the space $L^2(0,T;X)$ endowed with the weak topology. Under this compactness assumption on each $f^t$, the map

$$p : \begin{pmatrix} L^2_w(0,T;X) & \to & \mathcal{C}([0,T];X) \\ \beta & \mapsto & u \end{pmatrix}$$

is continuous and maps bounded set into relatively compact sets following [9, proposition 3.3], $\beta$ and $u$ being defined in Theorem 2.1.

Recall the definition of upper-semicontinuity of operators.

**Definition 2.3** *Let $E_1$ and $E_2$ be two Hausdorff topological sets. A multivalued operator $B : E_1 \rightrightarrows E_2$ is said* upper-semicontinuous *(***usc** *in short) at $x \in \operatorname{Dom} B$ if for all neighborhood $\mathcal{V}_2$ of the subset $Bx$ of $E_2$, there exists a neighborhood $\mathcal{V}_1$ of $x$ in $E_1$ such that $B(\mathcal{V}_1) \subset \mathcal{V}_2$.*

Furthermore, if $E_1$ and $E_2$ are two Hausdorff topological spaces with $E_2$ compact and $B : E_1 \rightrightarrows E_2$ is a multivalued map with $Bx$ closed for any $x \in E_1$, then $B$ is usc if and only if the graph of $B$ is closed in $E_1 \times E_2$. We introduce following conditions on the multifunction $B : [0,T] \times X \rightrightarrows X$:

**($\mathbf{B}_o$)** *:* (i) $\operatorname{Dom}(\partial f^t) \subset \operatorname{Dom} B(t,.)$ *for any* $t \in [0,T]$,
(ii) *there exist nonnegative constants* $\rho, M$ *such that* $\|x - u_0\| \leqslant \rho$ *implies* $B(t,x) \subset M\mathbb{B}_X$ *for any* $t \in [0,T]$ *and* $x \in \operatorname{Dom} \partial f^t$.

**(B)** *:*
(i) $\operatorname{Dom}(\partial f^t) \subset \operatorname{Dom} B(t,.)$ *and the set* $B(t,x)$ *is compact for any* $t \in [0,T]$ *and* $x \in \operatorname{Dom}(\partial f^t)$,
(ii) *there exist a nonnegative real* $\rho$ *and a convex lsc function* $\varphi : X \to \mathbb{R}$ *such that* $\|x - u_0\| \leqslant \rho$ *implies* $B(t,x) \subset -\partial\varphi(x)$ *for any* $t \in [0,T]$ *and* $x \in \operatorname{Dom}(\partial f^t)$,
(iii) *for a.e.* $t \in [0,T]$, *the restriction of* $B(t,.)$ *to* $\operatorname{Dom}(\partial f^t)$ *is usc,*
(iv) *for each* $r \geqslant 0$, *there is a nonnegative real-valued function* $g_r$ *on* $[0,T]^2$ *such that*
*(a)* $\lim_{t\to s_-} g_r(t,s) = 0$,
*(b) for each* $s, t \in [0,T]$ *with* $t \leqslant s$ *and each* $x_s \in \operatorname{Dom} \partial f^s$ *and* $\beta_s \in B(s, x_s)$ *with* $\|x_s\| \vee \|\beta_s\| \leqslant r$ *there exists* $x_t \in \operatorname{Dom} B(t,.)$ *and* $\beta_t \in B(t, x_t)$ *satisfying*

$$\|x_t - x_s\| \vee \|\beta_t - \beta_s\| \leqslant g_r(t,s).$$

By convexity, the function $\varphi$ of (B)(ii) is $M$-Lipschitz continuous on some closed ball $u_0 + \rho\mathbb{B}_X$ and the inclusion $\partial\varphi(x) \subset M\mathbb{B}_X$ holds for any $x \in u_0 + \rho\mathbb{B}_X$. In fact, we could take $\varphi$ with extended real values and $u_0$ in the interior of the effective domain of $\varphi$. Thus, (B)(ii) implies ($\mathrm{B}_o$)(ii).
The condition (B)(ii) means that $-B(t,.)$ is cyclically monotone uniformly in $t$. An example is the multiapplication $B(t,.) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defined by

$$\beta = (\beta_1, \dots, \beta_n) \in B(t,x) \iff \beta_1 \in \begin{cases} \{1\} & \text{if } x_1 < 0 \\ \{-1, 1\} & \text{if } x_1 = 0 \\ \{-1\} & \text{if } x_1 > 0 \end{cases} \quad \text{and} \quad \beta_2 = \cdots = \beta_n = 0$$

for any $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. When $B(t,.) = -\partial\psi^t$ with $\psi^t : X \to \mathbb{R} \cup \{+\infty\}$ a lsc proper function, then $\psi^t$ is convex if this operator is monotone and (B)(ii) is equivalent to the existence of a real constant $\alpha_t$ with $\psi^t = \varphi + \alpha_t$. In this case we deals with the problem $u' + \partial f^t(u) - \partial\varphi(u) \ni 0$, *see* Otani [14] when $f^t$ is not dependent on $t$. This condition (ii) could be extended to a function $\varphi^t$ which depends on the time $t$, and also with a nonconvex function: for example, a convex composite function, *see* [8].

The condition (B)(iii) is always satisfied if $B(t,.)$ or $-B(t,.)$ is a maximal monotone operator of $X$, and more generally if they are $\phi$-monotone of order 2.

The condition (B)(iv) is always satisfied if $B(t,.) = B : X \rightrightarrows X$ is not depending on the time $t$. It can also be written for any $t \leqslant s$ in $[0,T]$:

$$\lim_{t \to s_-} e(\mathrm{gph}B(t,.) \cap r\mathbb{B}_{X^2}, \mathrm{gph}B(s,.)) = 0,$$

$e$ standing for the excess between two sets. When $B(t,.)$ or $-B(t,.)$ is the subdiffferential of a convex lsc function $\psi^t$ which satisfies ($\mathrm{H}_0$), the condition (iv) is satisfied.

# 3 Existence of approximate solutions

For any real $\lambda > 0$ and $t \in [0,T]$, the function $f^t_\lambda$ shall denote the *Moreau-Yosida proximal function* of index $\lambda$ of $f^t$, and we set

$$J^t_\lambda = (I + \lambda \partial f^t)^{-1}\ ,\ \ Df^t_\lambda = \lambda^{-1}(I - J^t_\lambda).$$

We first prove the approximate result of existence :

**Theorem 3.1** *Let $(f^t)_{t\in[0,T]}$ be a family of proper convex lsc functions on $X$ with each $f^t$ of compact type. Assume that* (H) *and* ($\mathrm{B}_o$) *are satisfied. For each $u_0 \in \mathrm{dom}\, f^0$, there exists $T_0 \in ]0,T]$ such that $u' + \partial f^t(u) + B(t,u) \ni 0$ has at least an approximate solution $x : [0,T_0] \to X$ with $x(0) = u_0$ in the following sense: there exist sequences $(x_n)_n$ of absolutely continuous functions from $[0,T_0]$ to $X$, $(u_n)_n$ and $(\beta_n)_n$ of piecewise constant functions from $[0,T_0]$ to $X$ which satisfy:*

1. *for a.e. $t \in [0,T_0]$*

$$\begin{cases} x'_n(t) + \partial f^t(x_n(t)) + \beta_n(t) \ni 0 \\ x_n(0) = u_0 \end{cases} \qquad \text{and} \qquad \beta_n(t) \in B(\theta_n(t), u_n(t))$$

   *where* $0 \leqslant t - \theta_n(t) \leqslant 2^{-n}T$,

2. *there exists $N \in \mathbb{N}$ such that for any $n \geqslant N$:*

$$\forall t \in [0,T] \qquad \|x_n(t) - u_0\| \leqslant \rho \quad \text{and} \quad \|\beta_n(t)\| \leqslant M,$$

3. *$(x_n)_n$ and $(u_n)_n$ converge uniformly to $x$ on $[0,T_0]$, $(\beta_n)_n$ converges weakly to $\beta$ in $L^2(0,T_0;X)$, $(x'_n)_n$ converges weakly to $x'$ in $L^2(0,T_0;X)$ and $x$ is the solution of $x'(t) + \partial f^t(x(t)) + \beta(t) \ni 0$, $x(0) = u_0$ on $[0,T_0]$.*

## 3.1 Proof of Theorem 3.1

**Lemma 3.1** *We can find a set* $\{z_t : t \in [0,T]\}$ *and* $\rho_0 > 0$ *such that* $z_t \in \rho_0\mathbb{B}$, $f^t(z_t) \leq \rho_0$ *for every* $t \in [0,T]$.

**Proof.** Let $z_0 \in \operatorname{dom} f^0$ and $r > 0$ such that $r \geqslant \|z_0\| \vee |f^0(z_0)|$. For all $t \in [0,T]$, there exists $z_t \in \operatorname{dom} f^t$ satisfying

$$\begin{cases} \|z_t - z_0\| \leqslant |h_r(t) - h_r(0)|(1 + |f^0(z_0)|^{1/2}) \\ f^t(z_t) \leqslant f^0(z_0) + |k_r(t) - k_r(0)|(1 + |f^0(z_0)|). \end{cases}$$

The lemma holds with $\rho_0 = (\ r + \|h_r'\|_{L^1}(1 + r^{1/2})\ ) \vee (\ r + \|k_r'\|_{L^1}(1 + r)\ )$. □

**Lemma 3.2** *[18, Proposition 3.1]. Let* $x \in X$ *and* $\lambda > 0$. *The map* $t \mapsto J_\lambda^t x$ *is continuous on* $[0,T]$.

**Proof.** ¿From Kenmochi [11, Chapter 1, Section 1.5, Theorem 1.5.1], there is a nonnegative constant $\alpha$ such that $f^t(x) \geqslant -\alpha(\|x\| + 1)$ for all $x \in X$ and $t \in [0,T]$. Thus,

$$f_\lambda^t(x) - \frac{1}{2\lambda}\|x - J_\lambda^t x\|^2 = f^t(J_\lambda^t x) \geqslant -\alpha(1 + \|J_\lambda^t x\|),$$

which implies

$$\|x - J_\lambda^t x\|^2 \leqslant 2\lambda\alpha(1 + \|J_\lambda^t x - x\| + \|x\|) + 2\lambda f_\lambda^t(x). \tag{3}$$

Since $2\lambda f_\lambda^t(x) \leqslant 2\lambda f^t(z_t) + \|z_t - x\|^2 \leqslant 2\lambda\rho_0 + (\rho_0 + \|x\|)^2$ by Lemma 3.1, we can conclude:

$$\begin{aligned} \sup\{\|J_\lambda^t x\| \mid t \in [0,T]\ ,\ \lambda \in ]0,1]\ ,\ x \in r\mathbb{B}\} < \infty \\ \sup\{|f^t(J_\lambda^t x)| \mid t \in [0,T]\ ,\ x \in r\mathbb{B}\} < \infty \end{aligned}$$

for any $r > 0$.

Let $t \in [0,T]$ and $r \geqslant \|J_\lambda^t x\|$. By assumption $(\mathrm{H}_0)$, for each $s \in [0,T]$ with $s \geqslant t$ there exists $x_s \in \operatorname{dom} f^s$ satisfying

$$\begin{cases} \|J_\lambda^t x - x_s\| \leqslant |h_r(t) - h_r(s)|(1 + |f^t(J_\lambda^t x)|^{1/2}) \\ f^s(x_s) \leqslant f^t(J_\lambda^t x) + |k_r(t) - k_r(s)|(1 + |f^t(J_\lambda^t x)|). \end{cases}$$

Since $\lambda^{-1}(x - J_\lambda^s x) \in \partial f^s(J_\lambda^s x)$, we have

$$f^s(J_\lambda^s x) + \frac{1}{\lambda}\langle x - J_\lambda^s x, x_s - J_\lambda^s x\rangle \leqslant f^s(x_s) \leqslant f^t(J_\lambda^t x) + |k_r(t) - k_r(s)|(1 + |f^t(J_\lambda^t x)|).$$

Hence, for any $s \geqslant t$, we have

$$\begin{aligned} & \frac{1}{\lambda}\langle x - J_\lambda^s x, J_\lambda^t x - J_\lambda^s x\rangle \\ \leqslant\ & \frac{1}{\lambda}\langle x - J_\lambda^s x, J_\lambda^t x - x_s\rangle + f^t(J_\lambda^t x) - f^s(J_\lambda^s x) + |k_r(t) - k_r(s)|(1 + |f^t(J_\lambda^t x)|) \\ \leqslant\ & \|Df_\lambda^s(x)\||h_r(t) - h_r(s)|(1 + |f^t(J_\lambda^t x)|^{1/2}) + f^t(J_\lambda^t x) - f^s(J_\lambda^s x) \\ & +|k_r(t) - k_r(s)|(1 + |f^t(J_\lambda^t x)|). \end{aligned}$$

By symmetry it is true for any $s \in [0,T]$. In the same way for $t, s$ in $[0,T]$, we have

$$\frac{1}{\lambda}\langle x - J^t_\lambda x, J^s_\lambda x - J^t_\lambda x\rangle \leqslant \|Df^t_\lambda(x)\| |h_r(t) - h_r(s)|(1 + |f^s(J^s_\lambda x)|^{1/2}) + f^s(J^s_\lambda x) - f^t(J^t_\lambda x) \\ + |k_r(t) - k_r(s)|(1 + |f^s(J^s_\lambda x)|).$$

Adding these two inequalities we obtain

$$\frac{1}{\lambda}\|J^s_\lambda x - J^t_\lambda x\|^2 \leqslant [\, \|Df^t_\lambda(x)\| \vee \|Df^s_\lambda(x)\|\,]\, |h_r(t) - h_r(s)|(1 + |f^s(J^s_\lambda x)|^{1/2} \vee |f^t(J^t_\lambda x)|^{1/2}) \\ + |k_r(t) - k_r(s)|(1 + |f^s(J^s_\lambda x)| \vee |f^t(J^t_\lambda x)|).$$

Since both $\|Df^t_\lambda(x)\|$ and $|f^t(J^t_\lambda x)|$ are bounded, $t \mapsto J^t_\lambda x$ is continuous on $[0,T]$. □

By [11, Lemma 1.5.3], for $r \geqslant \|u_0\| + 1$, $M_1 \geqslant |f^0(u_0)| + \alpha r + \alpha + 1$ and $T_1 \in ]0,T[$ such that

$$\left[1 + M_1 \exp \int_0^T |k'_r|\right] \int_0^{T_1} |h'_r| \leqslant 1,$$

there exists an absolutely continuous function $v$ on $[0,T_1]$ satisfying:

* $v(0) = u_0$ and $\limsup_{t\to 0_+} f^t(v(t)) \leqslant f^0(u_0)$
* $\|v(t)\| \leqslant r$ for any $t \in [0,T_1]$
* for any $t \in [0,T_1]$, $|f^t(v(t))| \leqslant M_1 + M_1 \exp \int_0^T |k'_r| \int_0^t |h'_r|$
* for almost any $t \in [0,T_1]$, $\|v'(t)\| \leqslant \left[1 + M_1 \exp \int_0^T |k'_r|\right] |h'_r(t)|$.

For $r \geqslant \|u_0\| + \rho$, let us choose $T_2 > 0$ such that

$$\left(|f^0(u_0)| + \frac{M^2}{2}T_2 + \int_0^{T_2} c_r(\tau)\, d\tau\right)\left(1 + T_2 \exp \int_0^{T_2} c_r(\tau)\, d\tau\right) \leqslant |f^0(u_0)| + \rho.$$

*Let $r \geqslant (\|u_0\| \vee |f^0(u_0)|) + \rho + 1$ be fixed. Let us choose $T_0 > 0$ small enough in order to have*

$$(1 + r^{1/2})^2 T_0 \int_0^{T_0} |h'_r| \leqslant \frac{\rho^2}{32}\,, \quad T_0 \leqslant T_1 \wedge T_2 \quad \textit{and}$$

$$M_T\sqrt{T_0} + [M + \alpha]T_0 + \left[1 + M_1 \exp(\int_0^T |k'_r|)\right] \int_0^{T_0} |h'_r(s)|\, ds \leqslant \frac{\rho}{4}$$

*where* $M_T = 2\left[M_1 + M_1\left(\exp \int_0^T |k'_r|\right) \int_0^T |h'_r(s)|\, ds + \alpha r + \alpha\right]^{1/2}$.

**Lemma 3.3** *Let $\beta : [0,T] \to X$ be a measurable function with $\|\beta(t)\| \leqslant M$ for a.e. $t \in [0,T]$. Then,*

$$\forall t \in [0,T_0] \qquad \|p(\beta)(t) - u_0\| \leqslant \frac{\rho}{2},$$

*the map $p$ being defined in Section 2.*

**Proof.** The curve $u = p(\beta)$ exists on $[0,T]$ following Theorem 2.1. We have for a.e. $t \in [0,T_0]$:

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\|u(t)-v(t)\|^2 &\leqslant f^t(v(t)) - f^t(u(t)) + [M + \|v'(t)\|]\,\|u(t)-v(t)\| \\ &\leqslant \frac{1}{2}M_T^2 + [M + \|v'(t)\| + \alpha]\,\|u(t)-v(t)\|.\end{aligned}$$

We thus obtain for any $t \in [0,T_0]$

$$\frac{1}{2}\|u(t)-v(t)\|^2 \leqslant \frac{1}{2}M_T^2 T_0 + \int_0^t [M + \|v'(s)\| + \alpha]\,\|u(s)-v(s)\|\,ds.$$

Gronwall's lemma yields for any $t \in [0,T_0]$

$$\begin{aligned}\|u(t)-v(t)\| &\leqslant M_T\sqrt{T_0} + \int_0^t [M + \|v'(s)\| + \alpha]\,ds \\ &\leqslant M_T\sqrt{T_0} + [M+\alpha]T_0 + \left[1 + M_1 \exp\int_0^T |k_r'|\right]\int_0^{T_0} |h_r'(s)|\,ds \leqslant \frac{\rho}{4}.\end{aligned}$$

Furthermore,

$$\|v(t)-u_0\| \leqslant \int_0^t \|v'(s)\|\,ds \leqslant \left[1 + M_1 \exp\int_0^T |k_r'|\right]\int_0^{T_0} |h_r'(s)|\,ds \leqslant \frac{\rho}{4}.$$

By choice of $T_0 > 0$, we obtain $\|u(t)-u_0\| \leqslant \rho/2$ for any $t \in [0,T_0]$. □

*For simplicity of notation, we now write $T$ instead of $T_0$. We also assume that $f^t(x) \geqslant 0$ for any $x \in X$ with $\|x - u_0\| \leqslant \rho$, since we have $f^t(x) \geqslant -\alpha(\|u_0\| + \rho + 1)$.*

Let $n \in \mathbb{N}^\star$ such that :

$$\alpha^2 2^{-6n} + 2^{-3n+1}\left[r + (1+r)(\int_0^T |k_r'| + \alpha)\right] \leqslant \frac{\rho^2}{32}.$$

Let us set $f_n^t = f_{2^{-3n}}^t$ and $J_n^t = J_{2^{-3n}}^t$. Let $\mathcal{P}$ be a partition of $[0,T]$:

$$\mathcal{P} = \{0 = t_0^n < t_1^n < \cdots < t_{2^n}^n = T\}$$

where $t_k^n = k2^{-n}T$ for $k = 0,\ldots,2^n$.

Let us set $u_0^n = J_n^{t_0^n} u_0$. By assumption $(\mathrm{B}_o)$(i), $B(t_0^n, u_0^n)$ is non empty and contains an element $\beta_0^n$. Let $t \in [0,T]$. Under the assumption $(\mathrm{H}_0)$, there exists $u_{n,t} \in \operatorname{dom} f^t$ satisfying

$$\begin{cases}\|u_{n,t} - u_0\| \leqslant |h_r(t) - h_r(0)|(1 + r^{1/2}) \\ f^t(u_{n,t}) \leqslant r + |k_r(t) - k_r(0)|(1+r).\end{cases}$$

Using the definition of the Moreau-Yosida approximate, we obtain

$$\begin{aligned}\frac{2^{3n}}{2}\|J_n^t u_0 - u_0\|^2 &= f_n^t(u_0) - f^t(J_n^t u_0)\\ &\leqslant f^t(u_{n,t}) + \frac{2^{3n}}{2}\|u_{n,t} - u_0\|^2 + \alpha\|J_n^t u_0 - u_0\| + \alpha(1+\|u_0\|)\\ &\leqslant r + (1+r)\int_0^t |k_r'| + \frac{2^{3n}}{2}(1+r^{1/2})^2 t\int_0^t |h_r'|^2 + \alpha\|J_n^t u_0 - u_0\| + \alpha(1+r).\end{aligned}$$

Thus, by choice of $r$, $T$ and $n$ we obtain

$$\begin{aligned}&\|J_n^t u_0 - u_0\|\\ \leqslant\ & \alpha 2^{-3n} + \sqrt{\alpha^2 2^{-6n} + 2r2^{-3n} + 2(1+r)2^{-3n}(\int_0^T |k_r'| + \alpha) + (1+\sqrt{r})^2 T\int_0^T |h_r'|^2}\\ \leqslant\ & \frac{\rho}{2}.\end{aligned}$$

In particular, $\|u_0^n - u_0\| \leqslant \rho/2$. Under (B$_o$)(ii) it follows $\|\beta_0^n\| \leqslant M$. Let us set $x_0^n = p(\beta_0^n)$. By Lemma 3.3,

$$\forall t \in [0,T] \qquad \|x_0^n(t) - u_0\| \leqslant \frac{\rho}{2}.$$

Let us set $u_1^n = J_n^{t_1^n} x_0^n(t_1^n)$ and take $\beta_1^n \in B(t_1^n, u_1^n)$. Since $J_n^{t_1^n}$ is 1-Lipschitz continuous, we have

$$\|u_1^n - u_0\| \leqslant \|x_0^n(t_1^n) - u_0\| + \|J_n^{t_1^n} u_0 - u_0\| \leqslant \rho.$$

Next, (B$_o$)(ii) implies $\|\beta_1^n\| \leqslant M$. We then set

$$\beta_1^n(t) = \begin{cases} \beta_0^n & \text{if } t \in [t_0^n, t_1^n[ \\ \beta_1^n & \text{if } t \in [t_1^n, T] \end{cases}$$

Let us set $x_1^n = p(\beta_1^n)$. By unicity and continuity of the curve it follows $x_1^n(t) = x_0^n(t)$ if $t \in [t_0^n, t_1^n]$. Furthermore, $\|\beta_1^n(t)\| \leqslant M$ for any $t \in [0,T]$. By Lemma 3.3,

$$\forall t \in [0,T] \qquad \|x_1^n(t) - u_0\| \leqslant \frac{\rho}{2}.$$

Let $k \in \mathbb{N}^\star$. Assume that there exists a map $\beta_{k-1}^n : [0,T] \to X$ which is constant on each $[t_{k-1}^n, t_k^n[$ with $\|\beta_{k-1}^n(t)\| \leqslant M$ for any $t \in [0,T]$. Set $x_{k-1}^n = p(\beta_{k-1}^n)$. Then,

$$\forall t \in [0,T] \qquad \|x_{k-1}^n(t) - u_0\| \leqslant \frac{\rho}{2}.$$

Let us set $u_k^n = J_n^{t_k^n} x_{k-1}^n(t_k^n)$ and take $\beta_k^n \in B(t_k^n, u_k^n)$. Since

$$\|u_k^n - u_0\| \leqslant \|x_{k-1}^n(t_k^n) - u_0\| + \|J_n^{t_k^n} u_0 - u_0\| \leqslant \rho$$

we have $\|\beta_k^n\| \leqslant M$. We then set

$$\beta_k^n(t) = \begin{cases} \beta_{k-1}^n(t) & \text{if } t \in [t_0^n, t_k^n[ \\ \beta_k^n & \text{if } t \in [t_k^n, T] \end{cases}$$

Let us set $x_k^n = p(\beta_k^n)$. By unicity it follows $x_k^n(t) = x_{k-1}^n(t)$ if $t \in [0, t_k^n]$. Furthermore, $\|\beta_k^n(t)\| \leqslant M$ for any $t \in [0,T]$. By Lemma 3.3,

$$\forall t \in [0,T] \qquad \|x_k^n(t) - u_0\| \leqslant \frac{\rho}{2}.$$

We then set

$$x_n := x_{2^n-1}^n = \sum_{k=0}^{2^n} x_k^n \, \chi_{[t_k^n, t_{k+1}^n[} \qquad \text{and} \qquad \beta_n := \beta_{2^n-1}^n = \sum_{k=0}^{2^n} \beta_k^n \, \chi_{[t_k^n, t_{k+1}^n[},$$

where $\chi_{[t_k^n, t_{k+1}^n[}(t) = 1$ if $t \in [t_k^n, t_{k+1}^n[$ , and $= 0$ otherwise. For all $t \in [0, T[$, there exists $0 \leqslant k \leqslant 2^n$ with $t \in [t_k^n, t_{k+1}^n[$ and we set

$$\theta_n(t) = t_k^n \quad \text{and} \quad \theta_n(T) = T.$$

So, $x_n : [0,T] \to X$ is an absolutely continuous function and $\beta_n : [0,T] \to X$ is a measurable map which satisfy for a.e. $t \in [0,T]$

$$\begin{cases} x_n'(t) + \partial f^t(x_n(t)) + \beta_n(t) \ni 0 \\ x_n(0) = u_0 \end{cases} \qquad \text{and} \qquad \beta_n(t) \in B(\theta_n(t), u_n(t))$$

where we set $u_n(t) = J_n^{\theta_n(t)} x_n(\theta_n(t))$. By construct, there exists $N \in \mathbb{N}$ such that for any $n \geqslant N$:

$$\forall t \in [0,T] \qquad \|x_n(t) - u_0\| \leqslant \rho \quad \text{and} \quad \|\beta_n(t)\| \leqslant M.$$

A subsequence of $(\beta_n)_n$, again denoted by $(\beta_n)_n$, converges weakly to $\beta$ in $L^2(0,T;X)$. By continuity of the map $p$, the sequence $x_n = p(\beta_n)$ converges uniformly to a curve $x = p(\beta)$ on $[0,T]$ and a subsequence of $(x_n')_n$ converges weakly to $x'$ in $L^2(0,T;X)$.
In other words, the curve $x$ is the solution of $x'(t) + \partial f^t(x(t)) + \beta(t) \ni 0$, $x(0) = u_0$ on $[0,T]$.

Let $n \in \mathbb{N}^\star$ and $t \in [0,T]$. We have

$$\|u_n(t) - x(t)\| \leqslant \|x_n(\theta_n(t)) - x(\theta_n(t))\| + \|x(\theta_n(t)) - x(t)\| + \|J_n^{\theta_n(t)} x(t) - x(t)\|. \quad (4)$$

Under the assumption ($\mathrm{H}_0$), there exists $u_{n,t} \in \operatorname{dom} f^{\theta_n(t)}$ satisfying

$$\begin{cases} \|u_{n,t} - x(t)\| \leqslant |h_r(\theta_n(t)) - h_r(t)|(1 + r^{1/2}) \\ f^{\theta_n(t)}(u_{n,t}) \leqslant r + |k_r(\theta_n(t)) - k_r(t)|(1 + r). \end{cases}$$

Using the definition of the Moreau-Yosida approximate, we obtain

$$\begin{aligned}
\frac{2^{3n}}{2}\|J_n^{\theta_n(t)}x(t)-x(t)\|^2 &= f_n^{\theta_n(t)}(x(t)) - f^t(J_n^{\theta_n(t)}x(t)) \\
&\leqslant f^{\theta_n(t)}(u_{n,t}) + \frac{2^{3n}}{2}\|u_{n,t}-x(t)\|^2 + \alpha\|J_n^{\theta_n(t)}x(t)-x(t)\| + \alpha(1+r) \\
&\leqslant r + (1+r)\int_{\theta_n(t)}^t |k_r'| + \frac{2^{3n}}{2}(1+r^{1/2})^2(t-\theta_n(t))\int_{\theta_n(t)}^t |h_r'|^2 \\
&\qquad +\alpha\|J_n^{\theta_n(t)}x(t)-x(t)\| + \alpha(1+r).
\end{aligned}$$

Thus,

$$\|J_n^{\theta_n(t)}x(t)-x(t)\| \leqslant \alpha 2^{-3n} + \sqrt{\alpha^2 2^{-6n} + 2r2^{-3n} + 2(1+r)2^{-3n}(\int_0^T |k_r'| + \alpha) + (1+r^{1/2})^2 2^{-n}\int_0^T |h_r'|^2}$$

and $(J_n^{\theta_n(.)}x)_n$ converges uniformly to $x$ on $[0,T]$. Since $(x_n)_n$ converges uniformly to $x$ on $[0,T]$ and $x$ is continuous on $[0,T]$, (4) assures the uniform convergence of $(u_n)_n$ to $x$ on $[0,T]$.
□

## 3.2 Properties of approximate solutions

**Lemma 3.4** *We have $\|x(t)\| \vee |f^t(x(t))| \leqslant r$ for all $t \in [0,T]$. Under the assumption (B)(ii), the element $\beta(t)$ belongs to $-\partial\varphi(x(t))$ for a.e. $t \in [0,T]$.*

**Proof.** By inequality (2), we have for any $s \leqslant t$ in $[0,T]$

$$f^t(x(t)) - f^s(x(s)) + \frac{1}{2}\int_s^t \|x'(\tau)\|^2\,d\tau \leqslant \frac{1}{2}M^2T + \int_s^t c_r(\tau)\,[\,1+|f^\tau(x(\tau))|\,]\,d\tau.$$

Since $\|x(t)-u_0\| \leqslant \rho$, we have assumed for simplicity that $f^t(x(t)) \geqslant 0$ for any $t \in [0,T]$. Gronwall's lemma yields for any $t \in [0,T]$

$$f^t(x(t)) \leqslant \left(f^0(u_0) + \frac{M^2}{2}T + \int_0^T c_r(\tau)\,d\tau\right)\left(1 + T\exp\int_0^T c_r(\tau)\,d\tau\right) \leqslant |f^0(u_0)|+\rho. \quad (5)$$

by assumption on $T$.

Next, $\beta_n(t)$ belongs to $-\partial\varphi(u_n(t))$ with the uniform convergence of $(u_n)_n$ to $x$ on $[0,T]$.
Let us define $\tilde\varphi : L^2(0,T;X) \to \mathbb{R}\cup\{+\infty\}$ by $\tilde\varphi(u) = \int_0^T \varphi(u(t))\,dt$. It is known that $\tilde\varphi$ is proper lsc convex and

$$\alpha \in \partial\tilde\varphi(u) \iff \alpha(t) \in \partial\varphi(u(t)) \text{ for a.e. } t \in [0,T].$$

Thus, $-\beta_n \in \partial\tilde\varphi(u_n)$. Passing to the limit we obtain $-\beta \in \partial\tilde\varphi(x)$. Hence, $\beta(t)$ belongs to $-\partial\varphi(x(t))$ for a.e. $t \in [0,T]$. □

**Lemma 3.5** *For almost any $t \in [0,T]$ we have $f^t(x(t)) = \liminf_{n\to+\infty} f^t(x_n(t))$. Furthermore,*

$$\lim_{n\to+\infty} \int_0^T f^t(x_n(t))\,dt = \int_0^T f^t(x(t))\,dt \quad \textit{and} \quad \lim_{n\to+\infty} \int_0^T (f^t)^\star(y_n(t))\,dt = \int_0^T (f^t)^\star(y(t))\,dt,$$

*where we set $y_n(t) = -x_n'(t) - \beta_n(t)$ and $y(t) = -x'(t) - \beta(t)$ for a.e. $t$ in $[0,T]$.*

**Proof.** By lower semicontinuity of $f^t$, the inequality

$$f^t(x(t)) \leqslant \liminf_{n\to+\infty} f^t(x_n(t))$$

holds for any $t \in [0,T]$. The maps $v \mapsto \int_0^T f^t(v(t))\,dt$ and $w \mapsto \int_0^T (f^t)^\star(w(t))\,dt$ are proper lsc convex on $L^2(0,T;X)$. So,

$$\liminf_{n\to+\infty} \int_0^T f^t(x_n(t))\,dt \geqslant \int_0^T f^t(x(t))\,dt \quad \text{and} \quad \liminf_{n\to+\infty} \int_0^T (f^t)^\star(y_n(t))\,dt \geqslant \int_0^T (f^t)^\star(y(t))\,dt.$$

But, $f^t(x_n(t)) + (f^t)^\star(y_n(t)) = \langle y_n(t), x_n(t)\rangle$ for any $t \in [0,T]$, with

$$\lim_{n\to+\infty} \int_0^T \langle y_n(t), x_n(t)\rangle\,dt = \int_0^T \langle y(t), x(t)\rangle\,dt.$$

□

**Lemma 3.6** *We have the inequality*

$$\int_0^T \langle \beta(s), x'(s)\rangle\,ds \leqslant \liminf_{n\to+\infty} \int_0^T \langle \beta_n(s), x_n'(s)\rangle\,ds. \tag{6}$$

**Proof.** Let $n \in \mathbb{N}^\star$. The maps $x_n$, $\beta_n$ and $u_n$ are constant on $[t_k^n, t_{k+1}^n[$, $k = 0,\ldots,2^n-1$. Hence,

$$\int_0^T \langle \beta_n(s), x_n'(s)\rangle\,ds = \sum_{k=0}^{2^n-1} \int_{t_k^n}^{t_{k+1}^n} \langle \beta_k^n, (x_k^n)'(s)\rangle\,ds = \sum_{k=0}^{2^n-1} \langle \beta_k^n, x_k^n(t_{k+1}^n) - x_k^n(t_k^n)\rangle.$$

Since $\beta_k^n \in -\partial\varphi(u_k^n)$ for any $k = 0,\ldots,2^n-1$, we obtain:

$$\begin{aligned}
\int_0^T \langle \beta_n(s), x_n'(s)\rangle\,ds &\geqslant \sum_{k=0}^{2^n-1} \varphi(u_k^n) - \varphi(x_k^n(t_{k+1}^n)) - M\|u_k^n - x_k^n(t_k^n)\| \\
&= \varphi(u_0^n) - \varphi(x_{2^n-1}^n(t_{2^n}^n)) + \sum_{k=1}^{2^n-1} \varphi(u_k^n) - \varphi(x_k^n(t_k^n)) - M\|u_k^n - x_k^n(t_k^n)\| \\
&\geqslant \varphi(J_n^0 u_0) - \varphi(x_n(T)) - 2M \sum_{k=1}^{2^n-1} \|u_k^n - x_k^n(t_k^n)\|.
\end{aligned}$$

Since $\|u_k^n - u_0\| \leqslant \|u_n(t_k^n) - x_n(t_k^n)\| + \rho/2 \leqslant \rho$ for $n$ large enough, we have $f^{t_k^n}(u_k^n) \geqslant 0$. Furthermore, inequality (5) assures that $f^{t_k^n}(x_k^n(t_k^n)) \leqslant f^0(u_0)+\rho \leqslant r$. Using the definition of the Moreau-Yosida approximate, we obtain

$$\frac{2^{3n}}{2}\|u_k^n - x_k^n(t_k^n)\|^2 = f_n^{t_k^n}(x_k^n(t_k^n)) - f^{t_k^n}(u_k^n) \leqslant r.$$

Thus, $\|u_k^n - x_k^n(t_k^n)\| \leqslant \sqrt{r2^{-3n+1}}$ and

$$\sum_{k=1}^{2^n-1} \|u_k^n - x_k^n(t_k^n)\| \leqslant \sqrt{r2^{-n+1}}.$$

Consequently,

$$\lim_{n\to+\infty} \sum_{k=1}^{2^n-1} \|u_k^n - x_k^n(t_k^n)\| = 0.$$

By continuity of $\varphi$ and convergence of $(x_n)_n$ to $x$, we obtain

$$\liminf_{n\to+\infty} \int_0^T \langle \beta_n(s), x_n'(s)\rangle \, ds \geqslant \varphi(u_0) - \varphi(x(T)).$$

Since $\beta(s) \in -\partial\varphi(x(s))$ almost everywhere, $\langle \beta(s), x'(s)\rangle = -(\varphi \circ x)'(s)$ holds for a.e. $s$ and we obtain the inequality (6) . □

# 4 Existence of strong solutions

We now prove the existence of strong solutions.

## 4.1 General case

Let us set

$$\Phi(x,y) = \int_0^T \langle y(t), x'(t)\rangle \, dt - f^T(x(T)) + f^0(u_0)$$

for any absolutely continuous function $x : [0,T] \to X$ with $x' \in L^2(0,T;X)$ and any fonction $y \in L^2(0,T;X)$.

**Theorem 4.1** *Let $(f^t)_{t\in[0,T]}$ be a family of proper convex lsc functions on $X$ with each $f^t$ of compact type which satifies the assumption* (H)*. Assume that for any sequence $(x_n)_n$ in $H^1(0,T;X)$ which converges uniformly to the absolutely continuous function $x$ with the weak convergence of $(x_n')_n$ to $x'$ in $L^2$, and for any $(y_n)_n$ which converges weakly to $y$ in $L^2$ with $y_n(t) \in \partial f^t(x_n(t))$ for almost all $t$, there exists $n_k \to +\infty$ such that*

$$\liminf_{k\to+\infty} \Phi(x_{n_k}, y_{n_k}) \geqslant \Phi(x,y).$$

*Then, for each $u_0 \in \operatorname{dom} f^0$, there exists $T_0 \in ]0,T]$ such that $u' + \partial f^t(u) + B(t,u) \ni 0$ has at least a strong solution $u : [0,T_0] \to X$ with $u(0) = u_0$.*

**Proof.** Consider $x$ an approximate solution. We prove $x'(t)+\partial f^t(x(t))+B(t,x(t))\ni 0$ for a.e. $t$ in $[0,T]$. So, we begin by prove that $(x'_n)_n$ converges strongly to $x'$ in $L^2(0,T;X)$.
Step 1. - Let us set $y_n(t)=-x'_n(t)-\beta_n(t)$ and $y(t)=-x'(t)-\beta(t)$ for a.e. $t$ in $[0,T]$. It is easy to see that for any $n\in\mathbb{N}$ and almost any $t\in[0,T]$:

$$\|x'_n(t)\|^2+\langle y_n(t),x'_n(t)\rangle+\langle\beta_n(t),x'_n(t)\rangle=0 \quad \text{and} \quad \|x'(t)\|^2+\langle y(t),x'(t)\rangle+\langle\beta(t),x'(t)\rangle=0.$$

The sequence $(x'_n)_n$ converges weakly to $x'$ in $L^2(0,T;X)$. The strong convergence of $(x'_n)_n$ to $x'$ in $L^2(0,T;X)$ is equivalent to

$$\limsup_{n\to+\infty}\int_0^T\|x'_n(t)\|^2\,dt\leqslant\int_0^T\|x'(t)\|^2\,dt.$$

¿From Lemma 3.6 it follows:

$$\begin{aligned}\limsup_{n\to+\infty}\int_0^T\|x'_n(t)\|^2\,dt &\leqslant -\liminf_{n\to+\infty}\int_0^T\langle y_n(t),x'_n(t)\rangle\,dt-\liminf_{n\to+\infty}\int_0^T\langle\beta_n(t),x'_n(t)\rangle\,dt\\ &\leqslant -\liminf_{n\to+\infty}\int_0^T\langle y_n(t),x'_n(t)\rangle\,dt-\int_0^T\langle\beta(t),x'(t)\rangle\,dt.\end{aligned}$$

Since $\int_0^T\langle\beta(t),x'(t)\rangle\,dt=-\int_0^T\|x'(t)\|^2\,dt+\int_0^T\langle y(t),x'(t)\rangle\,dt$, it suffices to show that

$$\int_0^T\langle y(t),x'(t)\rangle\,dt\leqslant\liminf_{n\to+\infty}\int_0^T\langle y_n(t),x'_n(t)\rangle\,dt. \tag{7}$$

Step 2. - Under the assumption on $\Phi$, it follows by lower semicontinuity of $f^T$ :

$$\liminf_{k\to+\infty}\int_0^T\langle y_{n_k}(t),x'_{n_k}(t)\rangle\,dt\geqslant f^T(x(T))-f^0(u_0)+\liminf_{k\to+\infty}\Phi(x_{n_k},y_{n_k})\geqslant\int_0^T\langle y(t),x'(t)\rangle\,dt.$$

Step 3. - Let $\mathcal{N}$ be the negligeable subset of $[0,T]$ such that, for any $t\in[0,T]\backslash\mathcal{N}$, we have $x'_n(t)+\partial f^t(x_n(t))+\beta_n(t)\ni 0$, $\beta_n(t)\in B(\theta_n(t),u_n(t))$ and $(x'_n(t))_n$ converges to $x'(t)$. Since each $f^t$ are of compact type, the sets $X(t):=\mathrm{cl}\{x_n(t)\mid n\in\mathbb{N}^\star\}$ and $U(t):=\mathrm{cl}\{u_n(t)\mid n\in\mathbb{N}^\star\}$ are compact in $\mathrm{Dom}(\partial f^t)$. Let $r\geqslant M\vee(\rho+\|u_0\|)$. Under the assumption (B)(iv), for each $n\geqslant N$ and $t\in[0,T]$ with $t\neq\theta_n(t)$, there exists $z_n^t\in\mathrm{Dom}\,B(t,.)$ and $\alpha_n^t\in B(t,z_n^t)$ satisfying

$$\|z_n^t-u_n(t)\|\vee\|\alpha_n^t-\beta_n(t)\|\leqslant g_r(\theta_n(t),t).$$

When $t=\theta_n(t)$, we simply take $z_n^t=u_n(t)$ and $\alpha_n^t=\beta_n(t)$. Then, $(z_n^t)_n$ converges to $x(t)$ and $Z(t):=\mathrm{cl}\{z_n^t\mid n\in\mathbb{N}^\star\}$ is compact in $\mathrm{Dom}(\partial f^t)$.

The restriction of $B(t,.)$ to $\mathrm{Dom}(\partial f^t)$ being usc, the set $\{B(t,z)|z\in Z(t)\}$ is compact in $X$. Hence, $\Gamma(t):=\mathrm{cl}\{\alpha_n^t\mid n\in\mathbb{N}^\star\}$, and thus $\mathrm{cl}\{\beta_n(t)\mid n\in\mathbb{N}^\star\}$, are compact. So, $Y(t):=\mathrm{cl}\{y_n(t)\mid n\in\mathbb{N}^\star\}$ is compact in $X$.

Let us set $F^t(x) = \partial f^t(x) \cap Y(t)$ and $G^t(x) = B(t,x) \cap \Gamma(t)$ for any $x \in \mathrm{Dom}(\partial f^t)$ and $t \in [0,T]$. The multimaps $F^t$ and $G^t$ are upper semicontinuous on $\mathrm{Dom}(\partial f^t)$ with compact values in $X$. Let us denote by $e$ the excess between two sets. We have:

$$\begin{aligned} d(-x_n'(t), F^t(x(t)) + G^t(x(t))) &\leqslant d(y_n(t), F^t(x(t))) + d(\beta_n(t), G^t(x(t))) \\ &\leqslant e(F^t(x_n(t)), F^t(x(t))) + \|\beta_n(t) - \alpha_n^t\| + d(\alpha_n^t, G^t(x(t))) \\ &\leqslant e(F^t(x_n(t)), F^t(x(t))) + g_r(\theta_n(t), t) + e(G^t(z_n^t), G^t(x(t))). \end{aligned}$$

The upper-semicontinuity of $F^t$ and $G^t$ assures that

$$\lim_{n \to +\infty} d(-x_n'(t), F^t(x(t)) + G^t(x(t))) = 0.$$

Since $(x_n')_n$ converges to $x'$ a.e. on $[0,T]$, the equality $d(-x'(t), F^t(x(t)) + G^t(x(t))) = 0$ holds for a.e. $t \in [0,T]$ and we obtain by closedness of $F^t(x(t)) + G^t(x(t))$:

$$-x'(t) \in F^t(x(t)) + G^t(x(t)) \quad \text{for a.e.} \quad t \in [0,T].$$

Consequently, $x$ is a local solution to $x' + \partial f^t(x) + B(t,x) \ni 0$ with $x(0) = u_0$. □

## 4.2 Two particular cases

We consider two particular cases for which we can apply Theorem 4.1. These cases contains those of $f^t$ not depending on $t$.
First,

**Corollary 4.1** *Let $(f^t)_{t\in[0,T]}$ be a family of proper convex lsc functions on $X$ with each $f^t$ of compact type. Let $u_0 \in \mathrm{dom}\, f^0$. Assume that $f^t = g \circ F^t$ where $g$ is a proper convex lsc function on a Hilbert space $Y$ and $(F^t)_{t\in[0,T]}$ is a family of differentiable maps from $X$ to $Y$ such that $(DF^t)_t$ is equilipschitz continuous on a neighborhood of $u_0$ and such that:*

1. *for each $r \geqslant 0$, there is absolutely continuous real-valued function $b_r$ on $[0,T]$ such that:*
   *(a) $b_r' \in L^2(0,T)$,*
   *(b) for each $s,t \in [0,T]$, $\sup_{\|x\|\leqslant r} \|F^t(x) - F^s(x)\| \leqslant |b_r(t) - b_r(s)|$,*
2. *the qualification condition $\mathbb{R}_+[\mathrm{dom}\, g - F^0(u_0)] - DF^0(u_0)X = Y$ holds,*
3. *for each $r \geqslant 0$, there exists a negligible subset $N$ of $[0,T]$ such that the mapping $t \mapsto F^t(x)$ admits a derivative $\Delta^t(x)$ on $[0,T]\backslash N$ for any $x \in r\mathbb{B}_X$ and $\Delta^t$ is continuous on $r\mathbb{B}_X$ for any $t \in [0,T]\backslash N$,*
4. *the mapping $(t,x) \mapsto DF^t(x)$ is bounded on $[0,T] \times r\mathbb{B}_X$ for each $r > 0$ and it is continuous at $t$ for each $x$.*

*Assume that* (H) *and* (B) *are satisfied. Then, there exists* $T_0 \in ]0, T]$ *such that* $u' + \partial f^t(u) + B(t, u) \ni 0$ *has at least a strong solution* $u : [0, T_0] \to X$ *with* $u(0) = u_0$*.*

Remark under assumption *1.*, the mapping $t \mapsto F^t(x)$ is absolutely continuous and admits a derivative at a.e. $t \in [0, T]$ for each $x$. With the uniformly inequality *1.(b)*, we can hope that the almost derivability of $t \mapsto F^t(x)$ at $t$ is uniform in $x \in r\mathbb{B}_X$ thanks to the regularity of $F^t$ at $x$. Illustrate the importance of differentiability of $F^t$ by the following example : $F(t, x) = h(t - x)$ where $X = Y = \mathbb{R}$ and the real function $h$ is convex, Lipschitz continuous and non differentiable on $[0, T]$.

**Proof of Corollary 4.1.** Consider $x : [0, T] \to X$ an approximate solution. Let us set $y_n(t) = -x_n'(t) - \beta_n(t)$ and $y(t) = -x'(t) - \beta(t)$ for a.e. $t$ in $[0, T]$, $z_n(t) = F^t(x_n(t))$ and $z(t) = F^t(x(t))$ for a.e. $t \in [0, T]$. Then, $z_n$ and $z$ are absolutely continuous on $[0, T]$, thus are derivable at a.e. $t \in [0, T]$ and

$$z_n'(t) = \Delta^t(x_n(t)) + DF^t(x_n(t))x_n'(t) \quad , \quad z'(t) = \Delta^t(x(t)) + DF^t(x(t))x'(t).$$

Under the qualification condition, we have for any $x \in X$

$$\partial f^t(x) = DF^t(x)^\star \partial g(F^t(x)).$$

Let us write $y_n(t) = DF^t(x_n(t))^\star w_n(t)$ and $y(t) = DF^t(x(t))^\star w(t)$ where $w_n(t) \in \partial g(z_n(t))$ and $w(t) \in \partial g(z(t))$ for almost all $t \in [0, T]$. Hence, $g \circ z_n$ and $g \circ z$ are absolutely continuous with $\langle w_n(t), z_n'(t) \rangle = (g \circ z_n)'(t)$ for almost all $t \in [0, T]$. So,

$$\langle y_n(t), x_n'(t) \rangle = \frac{d}{dt}(f^t \circ x_n)(t) - \langle w_n(t), \Delta^t(x_n(t)).$$

and $\Phi(x_n, y_n) = -\int_0^T \langle w_n(t), \Delta^t(x_n(t)) \rangle \, dt$.
Let $r \geqslant \|x_n(t)\| \vee \|x(t)\|$ for any $n \in \mathbb{N}$ and $t \in [0, T]$. By continuity of $\Delta^t$ on $r\mathbb{B}_X$, $(\Delta^t(x_n(t)))_n$ converges to $\Delta^t(x(t))$ for a.e $t \in [0, T]$. Next, for a.e. $t$ and any $x \in r\mathbb{B}_X$, we have $\|\Delta^t(x)\| \leqslant |b_r'(t)|$. By Lebesgue's theorem $\Delta^\cdot(x_n(.))$ converges to $\Delta^\cdot x(.))$ in $L^2(0, T, Y)$. Since a subsequence of $(w_n)_n$ converges weakly to $w$, we can apply Theorem 4.1. □

For example, if $F^t$ is the affine mapping $x \mapsto A(t)x + b(t)$ where $A(t) : X \to Y$ is linear continuous and $b(t) \in Y$, the assumption of Corollary 4.1 becomes :

1. $b$ is absolutely continuous on $[0, T]$ and there is absolutely continuous real-valued function $a$ on $[0, T]$ such that:
   (a) $a' \in L^2(0, T)$,
   (b) for each $s, t \in [0, T]$, $\|A(t) - A(s)\| \leqslant |a(t) - a(s)|$.
2. the qualification condition $\mathbb{R}_+ \operatorname{dom} g - A(0)X = Y$ holds.

3. for each $r \geqslant 0$, there exists a negligible subset $N$ of $[0,T]$ such that $A'(t)$ is continuous on $r\mathbb{B}_X$ for any $t \in [0,T]\backslash N$.

Second, we use the conjugate of $f^t$.

**Lemma 4.1** *Let $(f^t)_{t\in[0,T]}$ be a family of proper convex lsc functions on $X$ satisfying* (H). *Assume that :*

> *for each $r \geqslant 0$, there exists a negligible subset $N$ of $[0,T]$ such that for any $t \in [0,T]\backslash N$, the mapping $s \mapsto (f^s)^\star(y)$ admits a derivative $\dot\gamma(t,y)$ at $t$ for any $y \in \operatorname{Dom}\partial(f^t)^\star$.*

*Let $x : [0,T] \to X$ be an absolutely continuous function and $y : [0,T] \to Y$ be such that $y(t) \in \partial f^t(x(t))$ for a.e. $t \in [0,T]$. For almost all $t \in [0,T]$, we have*

$$\dot\gamma(t, y(t)) + \frac{d}{dt} f^t(x(t)) = \langle y(t), x'(t)\rangle. \tag{8}$$

**Proof.** Let $s$ and $t$ be in $[0,T]\backslash N$ where $N$ is a suitable negligible subset of $[0,T]$. We have :

$$(f^s)^\star(y(s)) - (f^t)^\star(y(s)) \leqslant (f^s)^\star(y(s)) - (f^t)^\star(y(t)) - \langle y(s) - y(t), x(t)\rangle$$

since $x(t) \in \partial(f^t)^\star(y(t))$. From $f^t(x(t)) + (f^t)^\star(y(t)) = \langle y(t), x(t)\rangle$, we deduce

$$(f^s)^\star(y(s)) - (f^t)^\star(y(s)) \leqslant f^t(x(t)) - f^s(x(s)) + \langle y(s), x(s) - x(t)\rangle.$$

In the same way, for almost every $t, s$ in $[0,T]$ we have

$$(f^s)^\star(y(s)) - (f^t)^\star(y(s)) \leqslant f^t(x(t)) - f^s(x(s)) + \langle y(s), x(s) - x(t)\rangle.$$

Changing the role of $s$ and $t$, we also have:

$$\begin{aligned}(f^t)^\star(y(t)) - (f^s)^\star(y(t)) &\leqslant f^s(x(s)) - f^t(x(t)) + \langle y(t), x(t) - x(s)\rangle \\ &\leqslant (f^t)^\star(y(s)) - (f^s)^\star(y(s)) + \langle y(t) - y(s), x(t) - x(s)\rangle.\end{aligned}$$

The function $t \mapsto f^t(x(t))$ being absolutely continuous on $[0,T]$, *see* [11, Chapter 1], we obtain (8). □

The existence of $\dot\gamma$ implies some regularity on the domain of $(f^t)^\star$. For example, consider $(f^t)^\star(y) = h(t-y)$ where $X = Y = \mathbb{R}$ and the real function $h$ is convex, Lipschitz continuous and non differentiable on $[0,T]$. Then, we can not apply above lemma. The domain of $(f^t)^\star$ changes with $t$. But, we can apply Corollary 4.1 since $f^t(x) = tx + h^\star(-x)$. However, we have the absolutely continuity of $s \mapsto (f^s)^\star(y)$ in the following sense :

**Proposition 4.1** *Let $t \in [0,T]$, $y \in Y$, $\eta > 0$ and $r > 0$ such that if $|t-s| \leqslant \eta$, the set $\partial(f^s)^\star(y) \cap r\mathbb{B}_X$ is nonempty. Then, $s \mapsto (f^s)^\star(y)$ is absolutely continuous on $]t-\eta, t+\eta[$.*

**Proof.** 1) Lemma 3.1 with $\beta = \rho_o$ assures that $(f^t)^\star(y) \geqslant \langle y, z_t\rangle - f^t(z_t) \geqslant -\|y\|\beta - \beta$ for any $t \in [0,T]$ and $y \in X$. For $y \in \partial f^t(x)$, it follows

$$-\alpha(\|x\|+1) \leqslant f^t(x) = \langle y, x\rangle - (f^t)^\star(y) \leqslant \|y\|[\,\|x\| + \beta\,] + \beta.$$

So, there is a nonnegative constant $\beta$ such that $(f^t)^\star(y) \geqslant -\beta(\|y\|+1)$ for all $y \in X$ and $t \in [0,T]$.
Furthermore, for each $r > 0$, there is a nonnegative constant $c$ such that $|f^t(x)| \leqslant c(\|y\|+1)$ for all $x \in r\mathbb{B}_X$, $t \in [0,T]$ and $y \in \partial f^t(x)$.

2) Let $t$ be fixed in $[0,T]$ and $y \in \partial f^t(x)$. Let $r \geqslant \|x\|$ and $s \in [t,T]$. Under the assumption $(\mathrm{H}_0)$, there exists $x_s \in \operatorname{dom} f^s$ satisfying

$$\begin{cases} \|x - x_s\| \leqslant |h_r(t) - h_r(s)|(1 + |f^t(x)|^{1/2}) \\ f^s(x_s) \leqslant f^t(x) + |k_r(t) - k_r(s)|(1 + |f^t(x)|). \end{cases}$$

By definition of conjugate of a convex function, it follows

$$\begin{aligned} (f^t)^\star(y) - (f^s)^\star(y) &\leqslant \langle y, x - x_s\rangle + f^s(x_s) - f^t(x) \\ &\leqslant \|y\||h_r(t) - h_r(s)|(1 + |f^t(x)|^{1/2}) + |k_r(t) - k_r(s)|(1 + |f^t(x)|). \end{aligned}$$

We conclude thanks to 1):

$$\begin{aligned} (f^t)^\star(y) - (f^s)^\star(y) &\leqslant \|y\||h_r(t) - h_r(s)|(1 + |f^t(x)|^{1/2}) + |k_r(t) - k_r(s)|(1 + |f^t(x)|) \\ &\leqslant \|y\||h_r(t) - h_r(s)|(1 + \sqrt{c} + \sqrt{c\|y\|}) + |k_r(t) - k_r(s)|(1 + c + c\|y\|) \end{aligned}$$

3) Let $y \in Y$, $r > 0$ and $s, t \in [0,T]$. If the intersections of $\partial(f^t)^\star(y)$ and $\partial(f^s)^\star(y)$ with $r\mathbb{B}_X$ are non empty, let $x_s \in \partial(f^s)^\star(y)$ and $x_t \in \partial(f^t)^\star(y)$ with $r \geqslant \|x_s\| \vee \|x_t\|$. Step 2) implies

$$|(f^s)^\star(y) - (f^t)^\star(y)| \leqslant$$
$$\|y\||h_r(t) - h_r(s)|(1 + |f^s(x_s)|^{1/2} \vee |f^t(x_t)|^{1/2}) + |k_r(t) - k_r(s)|(1 + |f^s(x_s)| \vee |f^t(x_t)|).$$

By 1) we conlude:

$$|(f^s)^\star(y) - (f^t)^\star(y)| \leqslant \|y\||h_r(t) - h_r(s)|(1 + c^{1/2} + (c\|y\|)^{1/2}) + |k_r(t) - k_r(s)|(1 + c + c\|y\|).$$

□

**Corollary 4.2** *Let $(f^t)_{t\in[0,T]}$ be a family of proper convex lsc functions on $X$ with each $f^t$ of compact type. Assume that* (H) *and* (B) *are satisfied and that:*

1. *for each $r \geqslant 0$, there exists a negligible subset $N$ of $[0,T]$ such that for any $t$ in $[0,T]\backslash N$, the mapping $s \mapsto (f^s)^\star(y)$ admits a derivative $\dot\gamma(t,y)$ at $t$ for any $y \in \operatorname{Dom}\partial(f^t)^\star$.*

2. *for any* $(y_n)_n$ *which converges weakly to* $y$ *in* $L^2(0,T;X)$ *with* $y_n(t) \in \partial f^t(x_n(t))$ *where* $(x_n)_n$ *converges uniformly, there exists* $n_k \to +\infty$ *such that*

$$\liminf_{k\to+\infty} \int_0^T \dot\gamma(t, y_{n_k}(t))\,dt \geqslant \int_0^T \dot\gamma(t, y(t))\,dt.$$

*Then, for each* $u_0 \in \operatorname{dom} f^0$*, there exists* $T_0 \in ]0,T]$ *such that* $u' + \partial f^t(u) + B(t,u) \ni 0$ *has at least a strong solution* $u : [0,T_0] \to X$ *with* $u(0) = u_0$*.*

The assumption 2. is true when $\dot\gamma(t,.)$ is lsc and convex on $\operatorname{Dom}\partial(f^t)^\star$.

**Proof.** Lemma 4.1 implies $\Phi(x,y) = \int_0^T \dot\gamma(t,y(t))\,dt$. By assumption 2., we can apply Theorem 4.1. □

# 5 Examples of families $(f^t)_t$

## 5.1 Rafle

*See* Castaing, Valadier and Moreau [4, 17, 13]. Let $(C(t))_{t\in[0,T]}$ a family of nonempty closed convex subsets of $X$ whose intersection with bounded closed sets is compact. Consider the indicator function $f^t = \delta_{C(t)}$ of $C(t)$. Assume that for each $r \geqslant 0$, there is an absolutely continuous real-valued function $a_r$ on $[0,T]$ such that:

(i) $a_r' \in L^2(0,T)$;
(ii) for each $s,t$ in $[0,T]$, we have $e(C(s)\cap r\mathbb{B}_X, C(t)) \leqslant |a_r(s) - a_r(t)|$.

Under the assumption (B), we can apply Theorem 4.1: *Assume that for any sequence* $(x_n)_n$ *in* $H^1(0,T;X)$ *which converges uniformly to the absolutely continuous function* $x$ *with the weak convergence of* $(x_n')_n$ *to* $x'$ *in* $L^2$*, and for any* $(y_n)_n$ *which converges weakly to* $y$ *in* $L^2$ *with* $y_n(t) \in N_{C(t)}(x_n(t))$ *for almost all* $t$*, there exists* $n_k \to +\infty$ *such that*

$$\liminf_{k\to+\infty} \int_0^T \langle x_{n_k}'(t), y_{n_k}(t)\rangle\,dt \geqslant \int_0^T \langle x'(t), y(t)\rangle\,dt.$$

*Then, for each* $u_0 \in C(0)$*, there exists* $T_0 \in ]0,T]$ *such that* $u' + N_{C(t)}(u) + B(t,u) \ni 0$ *has at least a strong solution* $u : [0,T_0] \to X$ *with* $u(0) = u_0$*.*

Corollary 4.1 becomes:

**Corollary 5.1** *Let* $u_0 \in C(0)$ *and* $(F^t)_{t\in[0,T]}$ *be a family of differentiable maps from* $X$ *to an other Hilbert space* $Y$ *such that* $(DF^t)_t$ *is equilipschitz continuous on a neighborhood of* $u_0$*. Assume that* $C(t) = (F^t)^{-1}(C)$*,* $C$ *being a nonempty closed convex set. Under the assumptions (B) and:*

1. *for each* $r \geqslant 0$*, there is absolutely continuous real-valued function* $b_r$ *on* $[0,T]$ *such that:*

*(a)* $b'_r \in L^2(0,T)$,

*(b) for each* $s,t \in [0,T]$, $\sup_{\|x\|\leqslant r} \|F^t(x) - F^s(x)\| \leqslant |b_r(t) - b_r(s)|$,

2. *the qualification condition* $\mathbb{R}_+[C - F^0(u_0)] - DF^0(u_0)X = Y$ *holds,*

3. *for each* $r \geqslant 0$*, there exists a negligible subset* $N$ *of* $[0,T]$ *such that the mapping* $t \mapsto F^t(x)$ *admits a derivative* $\Delta^t(x)$ *on* $[0,T]\backslash N$ *for any* $x \in r\mathbb{B}_X$ *and* $\Delta^t$ *is continuous on* $r\mathbb{B}_X$ *for any* $t \in [0,T]\backslash N$*,*

4. *the mapping* $(t,x) \mapsto DF^t(x)$ *is bounded on* $[0,T] \times r\mathbb{B}_X$ *for each* $r > 0$ *and it is continuous at* $t$ *for each* $x$*,*

*there exists* $T_0 \in ]0,T]$ *such that* $u' + N_{C(t)}(u) + B(t,u) \ni 0$ *has at least a strong solution* $u : [0,T_0] \to X$ *with* $u(0) = u_0$*.*

On the other hand, $(f^t)^\star$ is the support function of $C(t)$, denoted by $\sigma_{C(t)}$. Moreau have proved in [13] that when $C$ is absolutely continuous, the map $t \mapsto \sigma_{C(t)}(y)$ is absolutely continuous on $[0,T]$ for any $y \in D$, where $D$ is the domain of $\sigma_{C(t)}$ which is not dependent of $t$. Corollary 4.2 becomes:

**Corollary 5.2** *Assume that* (B) *is satisfied and that:*

1. *for each* $r \geqslant 0$*, there exists a negligible subset* $N$ *of* $[0,T]$ *such that for any* $t$ *in* $[0,T]\backslash N$*, the mapping* $s \mapsto \sigma_{C(s)}(y)$ *admits a derivative* $\dot\gamma(t,y)$ *at* $t$ *for any* $y \in \operatorname{Dom}\partial\sigma_{C(t)}$*.*

2. *for any* $(y_n)_n$ *which converges weakly to* $y$ *in* $L^2(0,T;X)$ *with* $y_n(t) \in N_{C(t)}(x_n(t))$ *where* $(x_n)_n$ *converges uniformly, there exists* $n_k \to +\infty$ *such that*

$$\liminf_{k\to+\infty} \int_0^T \dot\gamma(t, y_{n_k}(t))\,dt \geqslant \int_0^T \dot\gamma(t,y(t))\,dt.$$

*Then, for each* $u_0 \in \operatorname{dom} f^0$*, there exists* $T_0 \in ]0,T]$ *such that* $u' + N_{C(t)}(u) + B(t,u) \ni 0$ *has at least a strong solution* $u : [0,T_0] \to X$ *with* $u(0) = u_0$*.*

By example, consider the affine map $F^t(x) = a(t)x + b(t)$ where $a(t) \in \mathbb{R}_+^\star$ is derivable nonincreasing at $t \in [0,T]$ and $b(t) \in Y$ is absolutely continuous at $t \in [0,T]$. We can apply both corollary 4.1 and 4.2 since

$$\dot\gamma(t,y) = \frac{-1}{a(t)^2}\left[a(t)\langle y, b'(t)\rangle + a'(t)(\sigma_C(y) - \langle y, b(t)\rangle)\right]$$

for a.e. $t \in [0,T]$ and any $y \in Y$. So, $\dot\gamma(t,.)$ is convex l.s.c. on $X$ and

$$\lim_{n\to+\infty} \int_0^T \dot\gamma(t, y_n(t))\,dt = \int_0^T \dot\gamma(t,y(t))\,dt.$$

## 5.2 Viscosity

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a convex lsc proper function of compact type. Consider $f^t(x) = f(x) + \frac{\varepsilon(t)}{2}\|x\|^2$ where $\varepsilon$ is an absolutely continuous real-valued function on $[0,T]$ with nonnegative values and $\varepsilon' \in L^1(0,T)$.
We can write $f^t = g \circ F^t$ with $g(x,r) = f(x) + r$ for any $(x,r) \in X \times \mathbb{R}$ and $F^t(x) = (x, \frac{\varepsilon(t)}{2}\|x\|^2)$ for any $x \in X$. By absolutely continuity of $\varepsilon$, $\Delta^t$ exists and is continuous on $X$ for a.e. $t \in [0,T]$ and

$$\Delta^t(x) = (0, \frac{\varepsilon'(t)}{2}\|x\|^2).$$

Furthermore,

$$DF^t(x)y = (y, \varepsilon(t)\langle x, y\rangle)$$

and $DF^t$ satifies assumption 5. We can apply Corollary 4.1.

On the other hand, $(f^t)^\star = (f^\star)_{\varepsilon(t)}$ is a $\mathcal{C}^1$-function on $X$ and, for any $y \in X$, the map $t \mapsto (f^t)^\star(y)$ is absolutely continuous on $[0,T]$ with for a.e. $t \in [0,T]$

$$\dot\gamma(t,y) = -\frac{\varepsilon'(t)}{2}\|D(f^t)^\star(y)\|^2.$$

By definition of $y_n$, $x_n(t) = D(f^t)^\star(y_n(t))$ holds for a.e. $t \in [0,T]$ and any $n \in \mathbb{N}$, hence

$$\dot\gamma(t,y_n(t)) = -\frac{\varepsilon'(t)}{2}\|x_n(t)\|^2.$$

In the same way,

$$\dot\gamma(t,y(t)) = -\frac{\varepsilon'(t)}{2}\|x(t)\|^2.$$

By uniform convergence of $(x_n)_n$ to $x$ on $[0,T]$ it follows

$$\lim_{n\to+\infty} \int_0^T \dot\gamma(t,y_n(t))\,dt = \int_0^T \dot\gamma(t,y(t))\,dt.$$

We can apply Corollary 4.2.

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