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Existence of Solutions for Nonconvex Third Order Differential Inclusions.

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Abstract

This paper proves the existence of solutions for a third order initial value nonconvex differential inclusion. We start with an upper semicontinuous compact valued multifunction F which is contained in a lower semicontinuous convex function ∂V and show that,

$$x^{(3)}(t) \in F(x(t), x'(t), x''(t)),\ x(0) = x_0,\ x'(0) = y_0,\ x''(0) = z_0.$$

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1 Introduction

The origins of boundary and initial value problems for differential inclusions are in the theory of differential equations and serve as models for a variety of applications including control theory. Existence results for the second order differential inclusion,

$$x'' \in F(x, x'),\ x(0) = x_0,\ x'(0) = y_0,$$

have been obtained by many authors (see [4], [5] and the references therein). In [5], Lupulescu showed existence for the problem

$$x'' \in F(x, x') + f(t, x, x'),\ x(0) = x_0,\ x'(0) = y_0$$

for the case in which F is an upper semicontinuous compact valued multifunction such the $F(x, y) \subset \partial V(y)$ and f is a Carathéodory function.

In this paper, we prove an existence result for the third order differential inclusion,

$$x^{(3)}(t) \in F(x(t), x'(t), x''(t)),\ x(0) = x_0,\ x'(0) = y_0,\ x''(0) = z_0,$$

where F is an upper semicontinuous compact valued multifunction and $F(x, y, z) \subset \partial V(z)$ for some proper lower semicontinuous convex function V. Expounding upon the methods used to establish existence by Lupulescu in [4] and [5], we define a sequence of approximate solutions on a given interval and show that the sequence converges to an actual solution.

2 Preliminaries

Let $\mathbb{R}^m$ be an m dimensional Euclidean space with an inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$.

Let $x \in \mathbb{R}^m$ and $r > 0$. The open ball centered at x with radius r is defined by

$$B_r(x) = \{y \in \mathbb{R}^m : \|x - y\| < r\},$$

where $\overline{B}_r(x)$ denotes its closure.

For the proper lower semicontinuous convex function $V : \mathbb{R}^m \to \mathbb{R}$, the multifunction $\partial V : \mathbb{R}^m \to 2^{\mathbb{R}^m}$ defined by

$$\partial V(x) = \{\gamma \in R^m : V(y) - V(x) \geq \langle \gamma, y - x \rangle, \forall y \in \mathbb{R}^m\},$$

is the subdifferential of V.

Let $L^2[a, b]$ be a Hilbert space with the inner product defined by

$$\langle x, y \rangle = \int_a^b x(t)\overline{y(t)}dt,$$

where $\overline{y(t)}$ denotes the complex conjugate of $y(t)$, and the norm is defined as

$$\|x\| = \sqrt{\int_a^b |x(t)|^2 dt}.$$

Let $\overline{co}F(x, y, z)$ denote the closed convex hull of F and $x_n \rightrightarrows x$ denote that x_n converges uniformly to x.

We need the following theorems from Aubin and Cellina [1].

Theorem 0.3.4 Consider a sequence of absolutely continuous functions $x_k(\cdot)$ from an interval I to a Banach Space X satisfying

(i) for every $t \in I$, ${x_k(t)}_k$ is a relatively compact subset of X;

(ii) there exists a positive function $c(\cdot) \in L^2(I)$ such that, for almost all $t \in I$, $\|x'_k(t)\| \leq c(t)$.

Then there exists a subsequence, again denoted by $x_k(\cdot)$, converging to an absolutely continuous function $x(\cdot)$ from I to X in the sense that

(i) $x_k(\cdot)$ converges uniformly to $x(\cdot)$ over compact subsets of I;

(ii) $x'_k(\cdot)$ converges weakly to $x'_k(\cdot)$ in $L^2(I, X)$.

Theorem 1.1.4 (the Convergence Theorem) Let F be a proper hemicontinuous map from a Hausdorff locally convex space X to the closed convex subsets of a Banach Space Y. Let I be an interval of $\mathbb{R}$ and $x_k(\cdot)$ and $y_k(\cdot)$ be measurable functions from I to Y respectively satisfying for almost all t in I and for every neighborhood $\aleph$ of 0 in $X \times Y$, there exists a $k_0 = k_0(t, \aleph)$ such that for every $k_0 \leq k$, $(x_k(t), y_k(t)) \in graph(F) + \aleph$. If,

(i) $x_k(\cdot)$ converges almost everywhere to a function $x(\cdot)$ from I to X;

(ii) $y_k(\cdot)$ belongs to $L^2(I, Y)$ and converges weakly to $y(\cdot)$ in $L^2(I, Y)$,

then, for almost all $t \in I$,

$$(x(t), y(t)) \in graph(F) \ \ i.e. y(t) \in F(x(t)).$$

We also need the following lemma from Brezis [2].

Lemma 3.3 Let $u \in D(V)$ almost everywhere on $[0, T]$ and suppose $g \in L^2([0, T], \mathbb{R})$ such that $g(t) \in \partial V(u(t))$ almost everywhere on $[0, T]$. Then, the function $t \longmapsto V(u(t))$ is absolutely continuous on $[0, T]$.

Also, let $t \in [0, T]$ such that $u(t) \in D(V)$ and let u and *V(u)* be differentiable. Then for all $t \in [0, T]$

$$\frac{d}{dt} V(u(t)) = \left\langle h, \frac{du}{dt}(t) \right\rangle \forall h \in \partial V(u(t)).$$

3 The Main Result

THEOREM: If $F : \Omega \to 2^{\mathbb{R}^m}$ and $V : \mathbb{R}^m \to \mathbb{R}$ satisfy the assumptions

$(A1)$ $\Omega \subset \mathbb{R}^{2m}$ where Ω is open and $F : \Omega \to 2^{\mathbb{R}^m}$ is a compact valued upper semicontinuous multifunction;

$(A2)$ there exists a lower semicontinuous proper convex function $V : \mathbb{R}^m \to \mathbb{R}$ such that $F(x, y, z) \subset \partial V(z)$ for every $(x, y, z) \in \Omega$.

Then, for every $(x_0, y_0, z_0) \in \Omega$, there exists a $T > 0$ and a solution $x : [0, T] \to \mathbb{R}^m$ of

$$x^{(3)}(t) \in F(x(t), x'(t), x''(t)),\ x(0) = x_0,\ x'(0) = y_0,\ x''(0) = z_0. \tag{1}$$

By a solution we are referring to an absolutely continuous function $x : [0,T] \to \mathbb{R}^m$ with absolutely continuous first and second derivatives with the initial values $x(0) = x_0$, $x'(0) = y_0$, $x''(0) = z_0$, and $x^{(3)}(t) \in F(x(t), x'(t), x''(t))$, *a.e.* on $[0,T]$.

PROOF: Suppose $(x_0, y_0, z_0) \in \Omega$. Then, $K = \bar{B}_r(x_0, y_0, z_0) \subset \Omega$ for some $r > 0$ since Ω is open. By assumption $(A1)$

$$F(K) = \bigcup_{(x,y,z)\in K} F(x,y,z)$$

is compact. Then there exists an $M > 0$ such that

$$\sup\{\|v\| : \ v \in F(x,y,z), \ (x,y,z) \in K)\} \leq M. \tag{2}$$

Set

$$T < \min\left\{ \frac{r}{M}, \left(\frac{r}{M}\right)^{\frac{1}{2}}, \left(\frac{r}{M}\right)^{\frac{1}{3}}, \frac{r}{2\|z_0\|}, \frac{r}{2\|y_0\|}, \left(\frac{2r}{3\|z_0\|}\right)^{\frac{1}{2}} \right\}. \tag{3}$$

Let n, j be integers where $1 \leq j \leq n$. Set $t_n^j = \frac{jT}{n}$. For $t \in \left[t_n^{j-1}, t_n^j\right]$ define,

$$x_n(t) = x_n^j + \left(t - t_n^j\right) y_n^j + \frac{1}{2}\left(t - t_n^j\right)^2 z_n^j + \frac{1}{6}\left(t - t_n^j\right)^3 v_n^j, \tag{4}$$

where $x_n^0 = x_0$, $y_n^0 = y_0$, and $z_n^0 = z_0$.

For $0 \leq j \leq n-1$ and $v_n^j \in F\left(x_n^j, y_n^j, z_n^j\right)$, define

$$\begin{cases} x_n^{j+1} &= x_n^j + \left(\dfrac{T}{n}\right) y_n^j + \dfrac{1}{2}\left(\dfrac{T}{n}\right)^2 z_n^j + \dfrac{1}{6}\left(\dfrac{T}{n}\right)^3 v_n^j \\ y_n^{j+1} &= y_n^j + \left(\dfrac{T}{n}\right) z_n^j + \dfrac{1}{2}\left(\dfrac{T}{n}\right)^2 v_n^j \\ z_n^{j+1} &= z_n^j + \left(\dfrac{T}{n}\right) v_n^j. \end{cases} \tag{5}$$

We claim that $\left(x_n^1, y_n^1, z_n^1\right) \in K$. Using (5), we have

$$\left\|z_n^1 - z_0\right\| = \left\|z_n^0 + \left(\frac{T}{n}\right) v_n^0 - z_0\right\| \leq \left(\frac{T}{n}\right) M < r.$$

As well as,

$$\begin{aligned}\left\|y_n^1 - y_0\right\| &= \left\|y_n^0 + \left(\frac{T}{n}\right) z_n^0 + \frac{1}{2}\left(\frac{T}{n}\right)^2 v_n^0 - y_0\right\| \\ &\le \left(\frac{T}{n}\right)\|z_0\| + \frac{1}{2}\left(\frac{T}{n}\right)^2 M \\ &< \frac{1}{2}r + \frac{1}{2}r = r.\end{aligned}$$

Also,

$$\begin{aligned}\left\|x_n^1 - x_0\right\| &= \left\|x_n^0 + \left(\frac{T}{n}\right) y_n^0 + \frac{1}{2}\left(\frac{T}{n}\right)^2 z_n^0 + \frac{1}{6}\left(\frac{T}{n}\right)^3 v_n^0 - x_0\right\| \\ &\le \left(\frac{T}{n}\right)\|y_0\| + \frac{1}{2}\left(\frac{T}{n}\right)^2 \|z_0\| + \frac{1}{6}\left(\frac{T}{n}\right)^3 M \\ &< T\|y_0\| + \frac{1}{2}T^2\|z_0\| + \frac{1}{6}T^3 M \\ &< \frac{1}{2}r + \frac{1}{3}r + \frac{1}{6}r = r.\end{aligned}$$

Hence the claim holds. Now suppose $j \ge 1$. We make the assumption that,

$$\begin{cases} x_n^j = x_n^0 + j\left(\frac{T}{n}\right) y_n^0 + \frac{1}{2}\left(\frac{jT}{n}\right)^2 z_n^0 + \frac{1}{6}\left(\frac{T}{n}\right)^3 [\left(3j^2 - 3j + 1\right) v_n^0 + \left(3j^2 - 9j + 7\right) v_n^1 \\ \qquad + \left(3j^2 - 15j + 19\right) v_n^2 + \cdots + 7v_n^{j-2} + v_n^{j-1}], \\ y_n^j = y_n^0 + j\left(\frac{T}{n}\right) z_n^0 + \frac{1}{2}\left(\frac{T}{n}\right)^2 \left[(2j-1)v_n^0 + (2j-3)v_n^1 + \cdots + 3v_n^{j-2} + v_n^{j-1}\right], \\ z_n^j = z_n^0 + \left(\frac{T}{n}\right)\left[v_n^0 + v_n^1 + \cdots + v_n^{j-1}\right]. \end{cases} \tag{6}$$

To see this, let $j = 1$. Then,

$$z_n^1 = z_n^0 + \left(\frac{T}{n}\right) v_n^0 = z_n^{0+1},$$

$$y_n^1 = y_n^0 + \frac{T}{n} z_n^0 + \frac{1}{2}\left(\frac{T}{n}\right)^2 v_n^0 = y_n^{0+1},$$

$$x_n^1 = x_n^0 + \frac{T}{n} y_n^0 + \frac{1}{2}\left(\frac{T}{n}\right)^2 z_n^0 + \frac{1}{6}\left(\frac{T}{n}\right)^3 v_n^0 = x_n^{0+1}.$$

Thus, (6) holds when $j = 1$. Let's suppose assumption (6) holds for $j > 1$. Using (5) we see that,

$$\begin{aligned}
z_n^{j+1} &= z_n^j + \left(\frac{T}{n}\right) v_n^j \\
&= z_n^0 + \left(\frac{T}{n}\right) \left[v_n^0 + v_n^1 + \cdots + v_n^{j-1}\right] + \left(\frac{T}{n}\right) v_j^n \\
&= z_n^0 + \left(\frac{T}{n}\right) \left[v_n^0 + v_n^1 + \cdots + v_n^j\right].
\end{aligned}$$

Thus the assumption holds for z_n^j. Using this and (5) we have,

$$\begin{aligned}
y_n^{j+1} &= y_n^j + \left(\frac{T}{n}\right) z_n^j + \frac{1}{2}\left(\frac{T}{n}\right)^2 v_n^j \\
&= y_n^0 + j\left(\frac{T}{n}\right) z_n^0 + \frac{1}{2}\left(\frac{T}{n}\right)^2 \left[(2j-1)v_n^0 + (2j-3)v_n^1 + \cdots + 3v_n^{j-2} + v_n^{j-1}\right] \\
&\quad + \left(\frac{T}{n}\right)\left(z_n^0 + \frac{T}{n}\left[v_n^0 + v_n^1 + \cdots + v_n^{j-1}\right]\right) + \frac{1}{2}\left(\frac{T}{n}\right)^2 v_n^j \\
&= y_n^0 + (j+1)\left(\frac{T}{n}\right) z_n^0 + \left(\frac{T}{n}\right)^2 \left[\left(j - \frac{1}{2}\right) v_n^0 + \left(j - \frac{3}{2}\right) v_n^1 + \cdots + \frac{1}{2} v_n^{j-1} + v_n^1 + \ldots v_n^{j-1}\right] \\
&\quad + \frac{1}{2}\left(\frac{T}{n}\right)^2 v_n^j \\
&= y_n^0 + (j+1)\left(\frac{T}{n}\right) z_n^0 + \left(\frac{T}{n}\right)^2 \left[\left(j + \frac{1}{2}\right) v_n^0 + \left(j - \frac{1}{2}\right) v_n^1 + \cdots + \frac{3}{2} v_n^{j-1} + \frac{1}{2} v_n^j\right] \\
&= y_n^0 + (j+1)\left(\frac{T}{n}\right) z_n^0 + \frac{1}{2}\left(\frac{T}{n}\right)^2 \left[(2j+1)v_n^0 + (2j-1)v_n^1 + \cdots + v_n^j\right] \\
&= y_n^0 + (j+1)\left(\frac{T}{n}\right) z_n^0 + \frac{1}{2}\left(\frac{T}{n}\right)^2 \left[(2(j+1)-1)v_n^0 + (2(j-1)-3)v_n^1 + \cdots + v_n^j\right].
\end{aligned}$$

Thus the assumption holds for y_n^j. Finally, with this and (5) we have,

$$\begin{aligned}
x_n^{j+1} &= x_n^j + \left(\frac{T}{n}\right) y_n^j + \frac{1}{2}\left(\frac{T}{n}\right)^2 z_n^j + \frac{1}{6}\left(\frac{T}{n}\right)^3 v_n^j \\
&= x_n^0 + j\left(\frac{T}{n}\right) y_n^0 + \frac{1}{2}\left(\frac{jT}{n}\right)^2 z_n^0 + \frac{1}{6}\left(\frac{T}{n}\right)^3 \left[(3j^2 - 3j + 1)v_n^0 + (3j^2 - 9j + 7)v_n^1 + \cdots + v_n^{j-1}\right] \\
&\quad + \left(\frac{T}{n}\right)\left(y_n^0 + j\left(\frac{T}{n}\right) z_n^0 + \frac{1}{2}\left(\frac{T}{n}\right)^2 \left[(2j-1)v_n^0 + (2j-3)v_n^1 + \cdots + 3v_n^{j-2} + v_n^{j-1}\right]\right) \\
&\quad + \frac{1}{2}\left(\frac{T}{n}\right)^2 \left(z_n^0 + \frac{T}{n}\left[v_n^0 + \cdots + v_n^{j-1}\right]\right) + \frac{1}{6}\left(\frac{T}{n}\right)^3 v_n^j
\end{aligned}$$

$$
\begin{aligned}
&= x_n^0 + (j+1)\left(\frac{T}{n}\right) y_n^0 + \frac{1}{2}(j+1)^2 \left(\frac{T}{n}\right)^2 z_n^0 \\
&\quad + \frac{1}{6}\left(\frac{T}{n}\right)^3 \left[\left(3j^2 - 3j + 1\right) v_n^0 + \left(3j^2 - 9j + 7\right) v_n^1 + \cdots + v_n^{j-1}\right] \\
&\quad + \left(\frac{T}{n}\right)^3 \left[j v_n^0 + (j-1) v_n^1 + \cdots + v_n^{j-1}\right] + \frac{1}{6}\left(\frac{T}{n}\right)^3 v_n^j \\
&= x_n^0 + (j+1)\left(\frac{T}{n}\right) y_n^0 + \frac{1}{2}(j+1)^2 \left(\frac{T}{n}\right)^2 z_n^0 \\
&\quad + \frac{1}{6}\left(\frac{T}{n}\right)^3 \left[\left(3j^2 - 3j + 1\right) v_n^0 + \left(3j^2 - 9j + 7\right) v_n^1 + \cdots + v_n^{j-1}\right] \\
&\quad + \frac{1}{6}\left(\frac{T}{n}\right)^3 \left[6j v_n^0 + (6j-6) v_n^1 + \cdots + 6 v_n^{j-1}\right] + \frac{1}{6}\left(\frac{T}{n}\right)^3 v_n^j \\
&= x_n^0 + (j+1)\left(\frac{T}{n}\right) y_n^0 + \frac{1}{2}(j+1)^2 \left(\frac{T}{n}\right)^2 z_n^0 \\
&\quad + \frac{1}{6}\left(\frac{T}{n}\right)^3 \left[\left(3j^2 + 3j + 1\right) v_n^0 + \left(3j^2 - 3j + 1\right) v_n^1 + \cdots + 7 v_n^{j-1} + v_n^j\right].
\end{aligned}
$$

Thus the assumption holds for x_n^j. Using (2), (3) and the relations in (6), we show that $\left(x_n^j, y_n^j, z_n^j\right) \in K$.

$$
\begin{aligned}
\left\|z_n^j - z_0\right\| &= \left\| z_n^0 + \left(\frac{T}{n}\right)\left[v_n^0 + v_n^1 + \cdots + v_n^{j-1}\right] - z_0 \right\| \\
&\le j\left(\frac{T}{n}\right) M \\
&\le TM \\
&< r.
\end{aligned}
$$

And,

$$
\begin{aligned}
\left\|y_n^j - y_0\right\| &= \left\| y_n^0 + j\left(\frac{T}{n}\right) z_n^0 + \frac{1}{2}\left(\frac{T}{n}\right)^2 \left[(2j-1) v_n^0 + (2j-3) v_n^1 + \cdots + 3 v_n^{j-2} + v_n^{j-1}\right] - y_0 \right\| \\
&\le j\left(\frac{T}{n}\right)\|z_0\| + \frac{1}{2}\left(\frac{jT}{n}\right)^2 M \\
&\le T\|z_0\| + \frac{1}{2} T^2 M \\
&< \frac{1}{2} r + \frac{1}{2} r \\
&= r.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\left\|x_n^j - x_0\right\| &= \Bigg\| x_n^0 + j\left(\frac{T}{n}\right) y_n^0 + \frac{1}{2}\left(\frac{jT}{n}\right)^2 z_n^0 \\
&\quad + \frac{1}{6}\left(\frac{T}{n}\right)^3 \left[\left(3j^2 - 3j + 1\right) v_n^0 + \left(3j^2 - 9j + 7\right) v_n^1 + \cdots + v_n^{j-1}\right] - x_0 \Bigg\| \\
&\le \frac{jT}{n}\|y_0\| + \frac{1}{2}\left(\frac{jT}{n}\right)^2 \|z_0\| + \frac{1}{6}\left(\frac{jT}{n}\right)^3 M \\
&\le T\|y_0\| + \frac{1}{2}T^2\|z_0\| + \frac{1}{6}T^3 M \\
&< \frac{1}{2}r + \frac{1}{2}\left(\frac{2}{3}\right) r + \frac{1}{6}r = r.
\end{aligned}
$$

Thus, $\left(x_n^j, y_n^j, z_n^j\right) \in K = B_r\left(x_0, y_0, z_0\right)$ for $1 \le j \le n$. Now, from the definition of x_n in (4) we have,

$$
\begin{cases}
x_n'(t) = y_n^j + \left(t - t_n^j\right) z_n^j + \frac{1}{2}\left(t - t_n^j\right)^2 v_n^j, \\
x_n''(t) = z_n^j + \left(t - t_n^j\right) v_n^j, \\
x_n^{(3)}(t) = v_n^j.
\end{cases}
\tag{7}
$$

By (2) we have that $\left\|x_n^{(3)}(t)\right\| = \left\|v_n^j\right\| \le M$. Similarly, (2) and (3) give the following,

$$
\begin{aligned}
\|x_n''(t)\| &= \left\|z_n^j + \left(t - t_n^j\right) v_n^j\right\| \\
&= \left\| z_n^0 + \left(\frac{T}{n}\right)\left[v_n^0 + v_n^1 + \cdots + v_n^{j-1}\right] + \left(t - t_n^j\right) v_n^j \right\| \\
&\le \|z_0\| + \left(\frac{jT}{n}\right) M + \left(\frac{T}{n}\right) M \\
&< \|z_0\| + 2r.
\end{aligned}
$$

As well as,

$$
\begin{aligned}
\|x_n'(t)\| &= \left\| y_n^j + \left(t - t_n^j\right) z_n^j + \frac{1}{2}\left(t - t_n^j\right)^2 v_n^j \right\| \\
&\le \|y_0\| + \left(\frac{jT}{n}\right)\|z_0\| + \frac{1}{2}\left(\frac{jT}{n}\right)^2 M + \left(\frac{T}{n}\right)\|z_0\| + \left(\frac{jT}{n}\right)^2 M + \frac{1}{2}\left(\frac{T}{n}\right)^2 M \\
&\le \|y_0\| + T\|z_0\| + 2T^2 M + T\|z_0\| \\
&< \|y_0\| + 3r.
\end{aligned}
$$

And finally,

$$
\begin{aligned}
\|x_n(t)\| &= \left\| x_n^j + \left(t - t_n^j\right) y_n^j + \frac{1}{2}\left(t - t_n^j\right)^2 z_n^j + \frac{1}{6}\left(t - t_n^j\right)^3 v_n^j \right\| \\
&\le \|x_0\| + T\|y_0\| + \frac{1}{2}T^2\|z_0\| + \frac{1}{6}T^3 M + \left(\frac{T}{n}\right)\left(\|y_0\| + T\|z_0\| + \frac{1}{2}T^2 M\right) \\
&\quad + \frac{1}{2}\left(\frac{T}{n}\right)^2 (\|z_0\| + TM) + \frac{1}{6}\left(\frac{T}{n}\right)^3 M \\
&\le \|x_0\| + T\|y_0\| + \frac{1}{2}T^2\|z_0\| + \frac{1}{6}T^3 M + T\|y_0\| + T^2\|z_0\| \\
&\quad + \frac{1}{2}T^3 M + \frac{1}{2}T^2 M\|z_0\| + \frac{1}{2}T^3 M + \frac{1}{6}T^3 M \\
&= \|x_0\| + 2T\|y_0\| + 2T^2\|z_0\| + T^3 M + \frac{1}{3}T^3 M \\
&< \|x_0\| + r + \frac{4}{3}r + r + \frac{1}{3}r \\
&< \|x_0\| + 4r.
\end{aligned}
$$

Since $\left\|x_n^{(3)}(t)\right\| \le M\ \forall t \in [0,T]$ the sequence $(x_n^{(3)}(t))$ is bounded in $L^2([0,T], R^m)$. Furthermore, suppose $\varepsilon > 0$ and $\forall t \in [0,T]$, and $\forall \tau \in [0,T]$, $|t - \tau| < \frac{\varepsilon}{M}$. Then,

$$
\begin{aligned}
\|x_n''(t) - x_n''(\tau)\| &\le \left| \int_\tau^t \|x_n^{(3)}(s)\| ds \right| \\
&\le \left| \int_\tau^t M ds \right| \\
&= M|t - \tau| \\
&\le M\left(\frac{\varepsilon}{M}\right) \\
&= \varepsilon.
\end{aligned}
$$

Thus, (x_n'') is equicontinuous. Similarly (x_n') and (x_n) are equicontinuous. Theorem 0.3.4 in [1] gives the following:

> There exists a subsequence, again denoted $(x_n)_n$ that converges to an absolutely continuous function $x : [0,T] \to R^m$ such that:
>
> (i) $(x_n) \rightrightarrows x$ on $[0,T]$,
> (ii) $(x_n') \rightrightarrows x'$ on $[0,T]$,
> (iii) $(x_n'') \rightrightarrows x''$ on $[0,T]$,
> (iv) $(x_n^{(3)})$ converges weakly to x^3 in $L^2([0,T], R^m)$.

By the Convergence Theorem, theorem 1.4.1 in [1], we have that

$$x^{(3)}(t) \in \overline{co}F(x(t), x'(t), x''(t)) \subset \partial V(x''(t)) \text{ a.e., } t \in [0, T].$$

Also, by the above and lemma 3.3 in [2],

$$\tfrac{d}{dt}V(x''(t)) = \langle x^{(3)}(t), x^{(3)}(t)\rangle = \left\|x^{(3)}(t)\right\|^2.$$

Since, $\int_0^T \frac{d}{dt}V\left(x''(t)\right)dt = \int_0^T \left\|x^{(3)}(t)\right\|^2 dt$ we have,

$$V\left(x''(T)\right) - V\left(x''(0)\right) = \int_0^T \left\|x^{(3)}(t)\right\|^2 dt. \tag{8}$$

However, by (7) we also have $x_n^{(3)}(t) = v_n^j \in F\left(x_n^j, y_n^j, z_n^j\right) \subset \partial V\left(x_n''(t_n^j)\right), \forall t \in \left[t_n^{j-1}, t_n^j\right]$. Which, from the definition of subdifferential, gives the following,

$$\begin{aligned}
V\left(x_n''\left(t_n^j\right)\right) - V\left(x_n''\left(t_n^{j-1}\right)\right) &\geq \left\langle x_n^{(3)}(t), x_n''\left(t_n^j\right) - x_n''\left(t_n^{j-1}\right)\right\rangle \\
&= \left\langle x_n^{(3)}(t), \int_{t_n^{j-1}}^{t_n^j} x_n^{(3)}(s)ds\right\rangle \\
&= \int_{t_n^{j-1}}^{t_n^j} \left\langle x_n^{(3)}(t), x_n^{(3)}(t)\right\rangle dt \\
&= \int_{t_n^{j-1}}^{t_n^j} \left\|x_n^{(3)}(t)\right\|^2 dt.
\end{aligned}$$

Combining the above inequalities with (8), we get the following inequality,

$$V\left(x_n''(T)\right) - V\left(z_0\right) \geq \int_0^T \left\|x_n^{(3)}(t)\right\|^2 dt.$$

If we let n approach infinity, we have

$$\begin{aligned}
V\left(x''(T)\right) - V\left(z_0\right) &\geq \limsup_{n\to\infty} \int_0^T \left\|x_n^{(3)}(t)\right\|^2 dt \\
\int_0^T \left\|x^{(3)}(t)\right\|^2 dt &\geq \limsup_{n\to\infty} \int_0^T \left\|x_n^{(3)}(t)\right\|^2 dt \\
\left\|x^{(3)}(t)\right\|^2 &\geq \limsup_{n\to\infty} \left\|x_n^{(3)}(t)\right\|^2.
\end{aligned}$$

However, the weak lower semicontinuity of the norm gives,

$$\left\|x^{(3)}(t)\right\|^2 \leq \liminf_{n\to\infty} \left\|x_n^{(3)}(t)\right\|^2.$$

Thus we have,

$$\left\|x^{(3)}(t)\right\|^2 = \lim_{n\to\infty}\left\|x_n^{(3)}(t)\right\|^2.$$

Hence the sequence $\left(x_n^{(3)}\right) \to x^{(3)}$ pointwise. By assumption (A1), F is closed, implying

$$x^{(3)}(t) \in F(x(t), x'(t), x''(t)),\ a.e.\ t \in [0, T].$$

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