Electronic Journal of Qualitative Theory of Differential Equations
2007, No. **2**, 1-15; **http://www.math.u-szeged.hu/ejqtde/**

ON THE EXISTENCE OF MILD SOLUTIONS FOR NEUTRAL FUNCTIONAL DIFFERENTIAL INCLUSIONS IN BANACH SPACE

L.GUEDDA

Abstract. A theorem on existence of mild solutions for partial neutral functional differential inclusions with unbounded linear part generating a noncompact semigroup in Banach space is established.

1. Introduction

Semilinear neutral functional differential inclusion has been the object of many studies by many researchers in the recent years. The method which consists in defining an integral multioperator for wich fixed points set coincides whith the solutions set of differential inclusion has been often applied to existence problems. In the case of inclusions on infinite dimensional spaces its direct application is complicated by the fact that the integral multioperator is noncompact except if one impose a severe compactness assumption.

In this paper using the method of condensing integral multioperators and fractional power of closed operators theory, we study the existence of mild solutions for initial value problems for first order semilinear neutral functional differential inclusions in a separable Banach space E for the form:

$$\frac{d}{dt}\left[x(t) - h(t, x_t)\right] \in Ax(t) + F(t, x_t), a.e.\ t \in [0, T] \quad (1.1)$$

$$x(t) = \varphi(t),\ t \in [-r, 0], \quad (1.2)$$

where $A : D(A) \subset E \to E$ is the infinitesimal generator of an uniformly bounded analytic semigroup of linear operators, $\{e^{At}\}_{t\geq 0}$ on a separable Banach space E; the multimap $F : [0, T] \times C([-r, 0], E) \to P(E)$ and $h : [0, T] \times C([-r, 0], E) \to E$, are given functions, $0 < r < \infty, \varphi \in C([-r, 0], E)$, where $P(E)$ denotes the class of all nonempty subsets of E, and $C([-r, 0], E)$ denotes the space of continuous functions from $[-r, 0]$ to E.

For any continuous function x defined on $[-r, T]$ and any $t \in [0, T]$, we denote by x_t the element of $C([-r, 0], E)$ defined by

$$x_t(\theta) = x(t + \theta), \theta \in [-r, 0].$$

For any $u \in C([-r, 0], E)$ the norm of u is defined by

$$\|u\| = \sup\{\|u(\theta)\| : \theta \in [-r, 0]\}.$$

Date:
2000 *Mathematics Subject Classification.* 34A60, 34K05.
Key words and phrases. Functional inclusions, condensing operators, semigroup of linear operators.

The function $x_t(.)$ represents the history of the state from time $t-r$, up the present time t.

Our work was motivated by the paper of E. Hernandez [3]. Using the theory of condensing operators, one can clarify certain conditions given in [3] in the form of estimates. Let us mention that existence results for semilinear differential inclusion with χ-regularity condition for the multivalued nonlinearity, where χ is the Hausdorf measure of noncompactness, were obtained in the works of N.S. Papageorgiou [9, 10], and existence results for impulsive neutral functional differential inclusion by S.K. Ntouyas [7]. See also [2, 5, 6]. A general existence theorem was given by V.V. Obukhovskii [8] for a semilinear functional differential inclusions with an analytic semigroup and upper Carathéodory type nonlinearity. In case where the linear part generates a strongly continuous semigroup, and the multivalued nonlinearity satisfies simple and general conditions of boundedness and χ-regularity, existence results were obtained in the paper of J.F. Couchouron and M.I. Kamenskii [1]. In this paper we use the results given in [1] and in the book of M. Kamenskii *et al.* [4] to study the multivalued part of our integral multioperators.

2. Preliminaries

Along this work, E will be a separable Banach space provided with norm $\|.\|$, $A : D(A) \subset E \to E$ is the infinitesimal generator of an uniformly bounded analytic semigroup of linear operators, $\{e^{At}\}_{t\geq 0}$, on a separable Banach space E. We will assume that $0 \in \rho(A)$ and that $\left\|e^{At}\right\| \leq M$ for all $t \in [0,T]$. Under these conditions it is possible to define the fractional power $(-A)^\alpha, 0 < \alpha \leq 1$, as closed linear operator on its domain $D(-A)^\alpha$. Furthermore, $D(-A)^\alpha$ is dense in E and the function $\|x\|_\alpha = \|(-A)^\alpha x\|$ defines a norm in $D(-A)^\alpha$. If X_α is the space $D(-A)^\alpha$ endowed with the norm $\|.\|_\alpha$, then X_α is a Banach space and there exists $c_\alpha > 0$ such that $\left\|(-A)^\alpha e^{At}\right\| \leq \frac{c_\alpha}{t^\alpha}$, for $t > 0$. Also the inclusion $X_\alpha \hookrightarrow X_\beta$ for $0 < \beta \leq \alpha \leq 1$ is continuous.

For additional details respect of semigroup theory, we refer the reader to Pazy [11].

Let Y^+ be the positive cone of an ordered Banach space $(Y^+, \leq)$. A function Ψ defined on the set of all bounded subsets of the Banach space X with values in Y^+ is called a measure of noncompactness on X if $\Psi(\Omega) = \Psi(\bar{co}\Omega)$ for all bounded subsets $\Omega \subset X$, where $\bar{co}\Omega$ denote the closed convex hull of Ω. The measure Ψ is called nonsingular if for every $a \in X$, $\Omega \in P(X)$, $\Psi(\{a\} \cup \Omega) = \Psi(\Omega)$, monotone, if $\Omega_0, \Omega_1 \in P(X)$ and $\Omega_0 \subseteq \Omega_1$implly $\Psi(\Omega_0) \leq \Psi(\Omega_1)$. One of most important example of measure of noncompactness, is the Hausdorf measure of noncompactness defined on each bounded set Ω of X by:

$$\chi(\Omega) = \inf\{\varepsilon > 0; \Omega \text{ has a finite } \varepsilon\text{-net in } X\}$$

Let $K(X)$ denotes the class of compact subsets of X, $Kv(X)$ denotes the class of compact convex subsets of X, and (Q, d) a metric space.

A multimap $G : Z \to K(X)$ is called $\Psi-$condensing if for every bounded set $\Omega \subset E$, that is not relatively compact we have $\Psi(G(\Omega)) \ngeq \Psi(\Omega)$, where $Z \subset X$.

A multivalued $G : X \to K(Q)$ is u.s.c.at a point $x \in X$, if for every $\varepsilon > 0$ there exists neighborhood $V(x)$ such that $G(x') \subset W_\varepsilon(G(x))$, for every $x' \in V(x)$. Here by $W_\varepsilon(A)$ we denote the ε-neighborhood of a set A, i.e. $W_\varepsilon(A) = \{y \in Y : d(y, A) < \varepsilon\}$, where $d(y, A) = \inf_{x \in A} d(x, y)$.

A multimap $G : X \to Q$ is quasicompact if its restriction to every compact subset $A \subset X$ is compact.

The sequence $\{f_n\}_{n=1}^\infty \subset L^1([0, T], X)$ is semicompact if it is integrably bounded and the set $\{f_n(t)\}_{n=1}^\infty$is relatively compact for almost every $t \in [0, T]$.

Any semicompact sequence in $L^1([0, T], X)$ is weakly compact in $L^1([0, T], X)$.

A function $f : [0, T] \to X$ is said to be strongly measurable if there exists a sequence $\{f_n\}$ of step functions such that $\|f(t) - f_n(t)\| \to 0$ as $n \to \infty$ for *a.e.* $t \in [0, T]$.

For all this definitions see for example [4].

In the following $C([-r, T], E)$ is the space of continuous functions from $[-r, T]$ to E endowed with the supremum norm. For any $x \in C([-r, T], E)$,

$$\|x\|_1 = \sup \{\|x(t)\| : t \in [-r, T]\}.$$

In section 3 we establish some existence results to the problem (1.1)-(1.2) using the following well known results. (See [4]).

Lemma 2.1. *Let E be a separable Banach space and $G : [0, T] \to P(E)$ an integrable, integrably bounded multifunction such that*

$$\chi(G(t)) \leq q(t)$$

for a.e. $t \in [0, T]$ where $q \in L^1_+([0, T])$. Then for all $t \in [0, T]$

$$\int_0^t \chi(G(s))\, ds \leq \int_0^t q(s) ds.$$

Lemma 2.2. *Let E be a separable Banach space and $\tilde{S}$ an operator*

$$\tilde{S} : L^1([0, T], E) \to C([0, T], E)$$

which satisfies the following conditions:

$S1)$ There exists $D > 0$ such that

$$\left\|\tilde{S}\, f(t) - \tilde{S}\, g(t)\right\| \leq D \int_0^t \|f(s) - g(s)\|\, ds, \;\; 0 \leq t \leq T$$

for every $f, g \in L^1([0, T], E)$.

$S2)$ For any compact $K \subset E$ and sequence $\{f_n\}_{n=1}^\infty \subset L^1([0, T], E)$ such that $\{f_n(t)\}_{n=1}^\infty \subset K$ for a.e.$t \in [0, T]$ the weak convergence $f_0 \underset{w}{\to} f_n$ implies $\tilde{S}f_n \to \tilde{S}f_0$.

Then:

(i) *If the sequence of functions* $\{f_n\}_{n=1}^{\infty} \subset L^1([0,T],E)$ *is such that* $\|f_n(t)\| \leq \delta(t)$ *for all* $n = 1,2,\ldots$ *a.e.* $t \in [0,T]$ *and* $\chi(\{f_n\}_{n=1}^{\infty}) \leq \zeta(t)$ *a.e.* $t \in [0,T]$, *where* $\zeta \in L^1_+([0,T]$, *then*

$$\chi(\tilde{S}\ \{f_n(t)\}_{n=1}^{\infty}) \leq 2D \int_0^t \zeta(s)ds.$$

(ii) *For every semicomapct sequence* $\{f_n\}_{n=1}^{\infty} \subset L^1([0,T];E)$ *the sequence* $\{Sf_n\}_{n=1}^{\infty}$ *is relatively compact in* $C([0,T];E)$, *and; moreover, if* $f_n \underset{w}{\to} f_0$ *then* $\tilde{S}\ f_n \to \tilde{S}\ f_0$.

An example of this operator is the operator $\tilde{S} : L^1([0,T],E) \to C([0,T],E)$ defined for every $f \in L^1([0,T],E)$ by

$$\tilde{S}f(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}f(s)ds,$$

where $x_0 \in E$, and A is an unbounded linear operator generating a C_0-semigroup in E (see [1]).

Lemma 2.3. *If* G *is a convex closed subset of a Banach space* E, *and* $\Gamma : G \to Kv(G)$ *is closed* Θ *condensing, where* Θ *is nonsingular measure of noncompactness defined on subsets of* G, *then* $Fix\Gamma \neq \emptyset$.

Lemma 2.4. *Let* Z *be a closed subset of a Banach space* E *and* $F : Z \to K(E)$ *a closed multimap, which is* α*-condensing on every bounded subset of* Z, *where* α *is a monotone measure of noncompactness. If the fixed points set* $FixF$ *is bounded, then it is compact.*

3. Existence Results

Let us define what we mean by a mild solution of the problem (1.1)-(1.2).

Definition 3.1. *A function* $x \in C([-r,T],E])$ *is said to be a mild solution of the problem (1.1)-(1.2) if the function* $s \to Ae^{A(t-s)}h(s,x_s)$ *is integrable on* $[0,t)$ *for each* $0 \leq t < T$, *and there exists* $f \in L^1([0,T],E)$, $f(t) \in F(t,x_t)$ *a.e.* $t \in [0,T]$, *such that*

$$\begin{aligned} x(t) &= e^{At}(\varphi(0) - h(0,\varphi)) + h(t,x_t) + \int_0^t e^{A(t-s)}f(s)ds \\ &\quad + \int_0^t Ae^{A(t-s)}h(s,x_s)ds, \quad t \in [0,T], \end{aligned}$$

and

$$x(t) = \varphi(t), \quad t \in [-r,0].$$

To establish our results we consider the following conditions:

Suppose that the multimap $F : [0,T] \times C([-r,0],E) \to Kv(E)$ satisfies the following properties:

$F1)$ The multifunction $F(\cdot,u)$ has a strongly measurable selection for every $u\in C([-r,0],E)$.

$F2)$ The multimap $F:(t,\cdot)\to Kv(E)$ is upper semicontinuous for e.a. $t\in[0,T]$.

$F3)$ There exists a function $\beta\in L^1([0,T],\mathbb{R}^+)$ such that, for all $u\in C([-r,0],E)$,

$$\|F(t,u)\|\le\beta(t)(1+\|u(0)\|), a.e.\ t\in[0,T].$$

$F4)$ There exists a function $\kappa\in L^1([0,T],\mathbb{R}^+)$ such that for all $\Omega\subset C([-r,0],E)$, we have

$$\chi(F(t,\Omega))\le k(t)\chi(\Omega(0)), a.e.\ t\in[0,T],$$

where, for $t\in[0,T]$, $\Omega(0)=\{u(0);u\in\Omega\}$.

Assume also that

$H)$ There exist constants $0<\alpha<1$, $0\le d_1<1, d_2\ge 0$, $\omega>0$, and $\theta>0$, such that h is X_α-valued, and

(i) For all $u\in C([-r,0],E]$ and any $t\in[0,T]$

$$\|(-A)^\alpha h(t,u)\|\le d_1\|u(0)\|+d_2.$$

(ii) For all $u,v\in C([-r,0],E]$ and $t\in[0,T]$

$$\|(-A)^\alpha h(t,u)-(-A)^\alpha h(s,v)\|\le\theta\|u(0)-v(0)\|.$$

(iii) for all bounded set $\Omega\subset C([-r,0],E)$

$$\chi((-A)^\alpha h(t,\Omega)\le\omega\chi(\Omega(0)),\ a.e.\ t\in[0,T].$$

We note that from assumptions $(F1)-(F3)$ it follows that the superposition multioperator

$$sel_F:C([-r,T],E]\to P(L^1([0,T],E))$$

defined for $x\in C([-r,T],E]$ by:

$$sel_F(x)=\{f\in L^1([0,T],E), f(t)\in F(t,x_t), a.e.\ t\in[0,T]\}$$

is correctly defined (see [4]) and is weakly closed in the following sense: if the sequences $\{x^n\}_{n=1}^\infty\subset C([-r,T],E],\{f_n\}_{n=1}^\infty\subset L^1([0,T],E), f_n(t)\in F(t,x_t^n)$, a.e. $t\in[0,T], n\ge 1$ are such that $x^n\to x^0$, $f_n\underset{w}{\to}f_0.$, then $f_0(t)\in F(t,x_t^0)$ a.e. $t\in[0,T]$ (see [4]).

Also from the assumption $(H)-(ii)$, the function $(-A)^\alpha h$ is continuous. Since the family $\{e^{At}\}_{t\ge 0}$ is an analytic semigroup [11], the operator function $s\to Ae^{A(t-s)}$ is continuous in the uniform operator topology on $[0,t)$ which from the estimate

$$\begin{aligned}\left\|(-A)e^{A(t-s)}h(s,x_s)\right\| &= \left\|(-A)^{1-\alpha}e^{A(t-s)}(-A)^\alpha)h(s,x_s)\right\|\\ &\le \frac{C_{1-\alpha}}{(t-s)^{1-\alpha}}(d_1\|x_s(0)\|+d_2)\\ &\le \frac{C_{1-\alpha}}{(t-s)^{1-\alpha}}(d_1\|x\|_1+d_2)\end{aligned}$$

and the Bochner's theorem implies that $Ae^{A(t-s)}h(s,x_s)$ is integrable on $[0,t)$.

Now we shall prove our main result.

Theorem 3.1. *Let the assumptions* $(F1)-(F4)$ *and* (H) *be satisfied. If*

$$\max\left\{\theta\left\|(-A)^{-\alpha}\right\|, d_1\left\|(-A)^{-\alpha}\right\|\right\} < 1$$

then the solution set of the problem (1.1)-(1.2) is a nonempty compact subset of the space $C([-r,T],E)$.

Proof. In the space $C([-r,T],E]$, we define the operator $\Gamma : C([-r,T],E] \to P(C([-r,T],E])$ in the following way:

$$\begin{array}{rcl}\Gamma(x)(t) &=& \{y \in C([-r,T],E] : y(t) = \varphi(t),\ t \in [-r,0] \quad \text{and} \\ y(t) &=& S(f)(t) + h(t,x_t) + \displaystyle\int_0^t Ae^{A(t-s)}h(s,x_s)ds;\ for\ t\ \in [0,T]\ \Big\}\end{array}$$

where $f \in sel_F(x)$, and the operator $S : L^1([0,T],E) \to C([0,T],E)$ is defined by

$$S(f)(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}f(s)ds,\ t \in [0,T]$$

where $x_0 = \varphi(0) - h(0,\varphi)$.

Remark 3.1. *It is clear that the operator* Γ *is well defined, and the fixed points of* Γ *are mild solutions to (1.1)-(1.2)*

The proof will be given in four steps.

Step 1. The multivalued operator Γ is closed.

The operator Γ can be written in the form $\Gamma = \sum_1^3 \Gamma_i$ where the operators $\Gamma_i, i = 1,2,3$ are defined as follows: the multivalued operator $\Gamma_1 : C([-r,T],E) \to P(C([-r,T],E))$ by

$$\Gamma_1 x(t) = \left\{\begin{array}{ll} y(t) = \varphi(t) - h(t,\varphi), & t \in [-r,0] \\ y(t) = Sf(t), & t \in [0,T] \end{array}\right\}$$

where $f \in Sel_F(x)$, the operator $\Gamma_2 : C([-r,T],E) \to C([-r,T],E)$ by

$$\Gamma_2 x(t) = \left\{\begin{array}{ll} h(t,\varphi), & t \in [-r,0] \\ h(t,x_t), & t \in [0,T] \end{array}\right.$$

and the operator $\Gamma_3 : C([-r,T],E) \to C([-r,T],E)$ by

$$\Gamma_3 x(t) = \left\{\begin{array}{ll} 0, & t \in [-r,0] \\ \int_0^t Ae^{A(t-s)}h(s,x_s)ds, & t \in [0,T]. \end{array}\right.$$

Let $\{x^n\}_{n=1}^{\infty}, \{z^n\}_{n=1}^{\infty}, x^n \to x^0, z^n \in \Gamma((x^n), n \geq 1$, and $z^n \to z^0$.
Let $\{f_n\}_{n=1}^{\infty} \subset L^1([0,T],E)$ an arbitrary sequence such that, for $n \geq 1$

$$f_n(t) \in F(t,x_t^n), a.e.\ t \in [0,T],$$

and

$$z^n = \begin{cases} \varphi(t), & t \in [-r, 0], \\ S(f_n)(t) + h(t, x_t^n) + \int_0^t Ae^{A(t-s)} h(s, x_s^n) ds, & t \in [0, T]. \end{cases}$$

The operator S satisfies the properties $\tilde{S}1$ and $\tilde{S}2$ of the Lemma 2.2, since e^{At} is a strongly continuous operator (see [1]). Hypothesis $(F3)$ implies that $\{f_n\}_{n=1}^{\infty}$ is integrably bounded, hypothesis $(F4)$ implies that $\chi(\{f_n(t)\}_{n=1}^{\infty} \leq k(t)\chi(\{x^n(t)\}_{n=1}^{\infty} = 0$ for a.e. $t \in [0, T]$, thus $(F3)$ and $(F4)$ implies that$\{f_n\}_{n=1}^{\infty}$ is semicompact sequence. Consequently $\{f_n\}_{n=1}^{\infty}$ is weakly compact in $L^1([0, T], E)$. So we can assume without loss of generality, that $f_n \underset{w}{\rightarrow} f_0$.

Lemma 2.2 implies that $Sf_n \to Sf_0$ in $C([0, T], E)$ and by using the fact that the operator sel_F is closed, we get $f_0 \in sel_F(x^0)$. Consequently

$$\begin{aligned} z_1^0(t) &= \begin{cases} y(t) = \varphi(t) - h(t, \varphi), & t \in [-r, 0], \\ y(t) = Sf_n(t), & t \in [0, T], \end{cases} \\ &\underset{n\to\infty}{\rightarrow} z_1^0(t) = \begin{cases} y(t) = \varphi(t) - h(t, \varphi), & t \in [-r, 0], \\ y(t) = Sf_0(t), & t \in [0, T], \end{cases} \end{aligned} \tag{3.1}$$

in the space $C([-r, T], E)$, with $f_0 \in sel_F(x^0)$.

On the other hand, we have the inequalities:

$$\begin{aligned} \left\|\Gamma_2 x^0(t) - \Gamma_2 x^n(t)\right\| &= \left\|h(t, x_t^n) - h(t, x_t^0)\right\| \\ &\leq \left\|(-A)^{-\alpha}\right\| \left\|(-A)^{\alpha} h(t, x_t^n) - (-A)^{\alpha} h(t, x_t^0)\right\| \\ &\leq \theta \left\|(-A)^{-\alpha}\right\| \left\|x_t^0(0) - x_t^n(0)\right\| \\ &\leq \theta \left\|(-A)^{-\alpha}\right\| \left\|x^0 - x^n\right\|_1 \end{aligned}$$

for any $t \in [0, T]$.

For $t \in [-r, 0]$ we have:

$$\left\|\Gamma_2 x^0(t) - \Gamma_2 x^n(t)\right\| = \|h(t, \varphi) - h(t, \varphi)\| = 0.$$

Then

$$\left\|\Gamma_2 x^0 - \Gamma_2 x^n\right\|_1 \leq \theta \left\|(-A)^{-\alpha}\right\| \left\|x^0 - x^n\right\|_1 \tag{3.2}$$

Using hypothesis $(H) - (ii)$ and the estimate in the family $\left\{Ae^{At}\right\}_{t>0}$we have:

$$\begin{aligned} &\left\|\int_0^t \left[Ae^{(t-s)} h(s, x_s^n) - Ae^{(t-s)} h(s, x_s^0)\right] ds\right\| \\ \leq &\int_0^t \left\|Ae^{(t-s)} h(s, x_s^n) - Ae^{(t-s)} h(s, x_s^0)\right\| ds \\ \leq &\ \theta \left\|(-A)^{-\alpha}\right\| \left\|x^0 - x^n\right\|_1 \int_0^t \left\|(-A)^{1-\alpha} e^{A(t-s)}\right\| ds \\ \leq &\ \theta \left\|(-A)^{-\alpha}\right\| \left\|x^0 - x^n\right\|_1 \int_0^t \tfrac{C_{1-\alpha}}{(t-s)^{1-\alpha}} ds \end{aligned}$$

$$\leq \ \theta \left\| (-A)^{-\alpha} \right\| \frac{C_{1-\alpha} T^{\alpha}}{\alpha} \left\| x^0 - x^n \right\|_1$$

for any $t \in [0, T]$. Then

$$\left\| \Gamma_3 x^0 - \Gamma_3 x^n \right\|_1 \leq \theta \left\| (-A)^{-\alpha} \right\| \frac{C_{1-\alpha} T^{\alpha}}{\alpha} \left\| x^0 - x^n \right\|_1 . \tag{3.3}$$

From the inequalities (3.1)-(3.3) follows immediately that $z^n \to z^0$ with

$$z^0(t) = \left\{ \begin{array}{ll} \varphi(t), & t \in [-r, 0] \\ S(f_0)(t) + h(t, x_t^0) + \int_0^t A e^{A(t-s)} h(s, x_s^0) ds, & t \in [0, T] \end{array} \right\}$$

where , $f_0 \in sel_F(x^0)$ and $z^0 \in \Gamma(x^0)$ and hence Γ is closed.

Now in the space $C([-r, T], E)$ we consider the measure of noncompactness Θ defined in the following way: for every bounded subset $\Omega \subset C([-r, T], E])$

$$\Theta(\Omega) = (\chi(\Omega[-r, 0]), \Psi(\Omega), mod_c \Omega) ,$$

where

$$\Psi(\Omega) = \sup_{t \in [0,T]} \left(e^{-Lt} \chi(\Omega(t)) \right) ,$$

and $mod_c \Omega$ is the module of equicontinuity of the set $\Omega \subset C([-r, T], E)$ given by:

$$mod_c \Omega = \limsup_{\delta \to 0} \max_{x \in \Omega |t_1 - t_2| \leq \delta} \| x(t_1) - x(t_2) \|$$

and $L > 0$ is chosen so that

$$\begin{aligned}
\sup_{t \in [0,T]} M \int_0^t e^{-L(t-s)} k(s) ds &\leq q_1 < 1 \\
\sup_{t \in [0,T]} M \int_0^t e^{-L(t-s)} \beta(s) ds &\leq q_2 < 1 \\
\sup_{t \in [0,T]} d_1 \int_0^t \frac{e^{-L(t-s)}}{(t-s)^{1-\alpha}} c_{1-\alpha} ds &\leq q_3 < 1 \\
\sup_{t \in [0,T]} d_2 \int_0^t \frac{e^{-L(t-s)}}{(t-s)^{1-\alpha}} c_{1-\alpha} ds &\leq q_4 < 1 \\
\sup_{t \in [0,T]} \omega \int_0^t \frac{C_{1-\alpha}}{(t-s)^{1-\alpha}} e^{-L(t-S)} ds &\leq q_5 < 1
\end{aligned}$$

where M is the constant from the estimation in the family of $\left\{ e^{At} \right\}_{t \geq 0}$, the constants d_1, d_2 from $(H)-(i)$, the function β from the hypothesis $(F3)$, and the function κ from the hypothesis $(F4)$.

From the Arzelá-Ascoli theorem, the measure Θ give a nonsingular and regular measure of noncompactness in $C([-r, T], E])$.

Remark 3.2. *If $\xi \in L^1([0,T],E)$, it is clear that*

$$\sup_{t\in[0,T]}\int_0^t e^{-L(t-s)}\xi(s)ds \underset{L\to+\infty}{\to} 0.$$

Step 2. The miltioperator Γ is Θ condensing on every bounded subset of $C([-r,T],E)$.

Let $\Omega \subset C([-r,T],E)$ be a bounded subset such that

$$\Theta(\Gamma(\Omega)) \geq \Theta(\Omega), \tag{3.4}$$

where the inequality is taking in the sense of the order in $\mathbb{R}^3$ induced by the positive cone $\mathbb{R}^3_+$. We will show that (3.3) implies that Ω is relatively compact in $C([-r,T],E)$.

From the inequality (3.4) follows immediately that

$$\chi(\Omega([-r,0])) = 0. \tag{3.5}$$

Indeed, we have

$$\chi\left[(\Gamma\Omega)[-r,0]\right] = \chi\left\{\varphi(t), t\in[-r,0]\right\} = 0 \geq \chi(\Omega[-r,0]) \geq 0.$$

We give now an upper estimate for $\chi(\Gamma\Omega(t))$, for any $t\in[0,T]$.

Using $(F4)$, for $t\in[0,T]$, we have

$$\begin{aligned}
\chi(\{f(s), f\in sel_F(\Omega)\} &\leq \chi(F(s,\Omega_s)) \\
&\leq e^{Ls}k(s)e^{-Ls}\chi(\Omega_s(0)) \\
&\leq e^{Ls}k(s)e^{-Ls}\chi(\Omega(s)) \\
&\leq e^{Ls}k(s)\sup_{s\in[0,T]} e^{-Ls}\chi(\Omega(s)).
\end{aligned}$$

Then, from Remark 3.2 and Lemma 2.1 with $D = M$, we get

$$\begin{aligned}
\sup_{t\in[0,T]} e^{-lt}\chi(\{Sf(t)f\in sel_F(\Omega)\} &\leq M\int_0^t e^{-L(t-s)}k(s)ds \sup_{t\in[0,T]} e^{-Lt}\chi(\Omega(t)) \\
&\leq q_1 \sup_{t\in[0,T]} e^{-Lt}\chi(\Omega(t)). \\
&\leq q_1\Psi(\Omega) \qquad (3.6)
\end{aligned}$$

Since the measure χ is monotone, from $H_1-(iii)$, for $t\in[0,T]$ we get:

$$\begin{aligned}
e^{-Lt}\chi(h(t,\Omega_t) &\leq e^{-Lt}\chi\left((A)^{-\alpha}(-A)^{\alpha}h(t,\Omega_t)\right) \\
&\leq \left\|(-A)^{-\alpha}\right\| e^{-Lt}\chi(h((-A)^{\alpha}h(t,\Omega_t)) \\
&\leq \omega\left\|(-A)^{-\alpha}\right\| e^{-Lt}\chi(\Omega_t(0)) \\
&\leq \omega\left\|(-A)^{-\alpha}\right\| e^{-Lt}\chi(\Omega(t)) \\
&\leq \omega\left\|(-A)^{-\alpha}\right\| \sup_{t\in[0,T]} e^{-Lt}\chi(\Omega(t)) \\
&\leq \omega\left\|(-A)^{-\alpha}\right\| \Psi(\Omega).
\end{aligned}$$

Then

$$\sup_{t\in[0,T]} e^{-Lt}\chi(h(t,\Omega_t)) \le \omega \left\|(-A)^{-\alpha}\right\| \Psi(\Omega). \tag{3.7}$$

Let now $t \in [0,T]$ and $s \in [0,t]$. The function $G : s \to Ae^{A(t-s)}h(s,\Omega_s)$ is integrable and integrably bounded. Indeed for any $x \in \Omega$ we have:

$$\begin{aligned}
\left\|(-A)e^{A(t-s)}h(s,x_s)\right\| &= \left\|(-A)^{1-\alpha}e^{A(t-s)}(-A)^{\alpha})h(s,x_s)\right\| \\
&\le \frac{C_{1-\alpha}}{(t-s)^{1-\alpha}}(d_1 \|x_s(0)\| + d_2) \\
&\le \frac{C_{1-\alpha}}{(t-s)^{1-\alpha}}(d_1 \|x\|_1 + d_2).
\end{aligned}$$

Since

$$\begin{aligned}
\chi\left((-A)e^{A(t-s)}h(s,x_s)\right) &= \chi\left((-A)^{1-\alpha}e^{A(t-s)}(-A)^{\alpha}h(s,x_s)\right) \\
&\le \left\|(-A)^{1-\alpha}e^{A(t-s)}\right\| \chi\left((-A)^{\alpha}h(s,x_s)\right) \\
&\le \frac{C_{1-\alpha}}{(t-s)^{1-\alpha}}\omega\chi(\Omega_s(0)) \\
&\le \frac{C_{1-\alpha}}{(t-s)^{1-\alpha}}\omega\chi(\Omega(s)) \\
&\le \frac{\omega C_{1-\alpha}}{(t-s)^{1-\alpha}}e^{Ls}\sup_{s\in[0,T]} e^{-Ls}\chi(\Omega(s)) \\
&\le \frac{\omega C_{1-\alpha}}{(t-s)^{1-\alpha}}e^{LS}\Psi(\Omega),
\end{aligned}$$

using Lemma 2.1, we get for every $t \in [0,T]$

$$\begin{aligned}
e^{-lt}\int_0^t \chi\left((-A)e^{A(t-s)}h(s,x_s)\right)ds &\le \Psi(\Omega)\int_0^t \frac{\omega C_{1-\alpha}}{(t-s)^{1-\alpha}}e^{L(t-S)}ds \\
&\le \Psi(\Omega)\sup_{t\in[0,T]}\int_0^t \frac{\omega C_{1-\alpha}e^{-L(t-S)}}{(t-s)^{1-\alpha}}ds.
\end{aligned}$$

Therefore

$$\sup_{t\in[0,T]} e^{-lt}\int_0^t \chi\left((-A)e^{A(t-s)}h(s,x_s)\right)ds \le q_5\Psi(\Omega). \tag{3.8}$$

From the inequalities (3.6)-(3.8), remark 3.2 and the fact that $\|(-A)^{\alpha}\omega\| < 1$, it follows that:

$$\Psi(\Gamma(\Omega)) = \sup_{t\in[0,T]} e^{-Lt}\chi\left\{S(f)(t) + h(t,\Omega_t) + \int_0^t Ae^{A(t-s)}h(s,\Omega_s)ds\right\}$$

$$\begin{aligned} &\leq \Psi(\Omega\Big[\sup_{t\in[0,T]} M\int_0^t e^{-L(t-s)}k(s)ds + \|(-A)^\alpha\|\,\omega \\ &\quad +\omega\sup_{t\in[0,T]}\int_0^t \frac{\omega C_{1-\alpha}}{(t-s)^{1-\alpha}}e^{-L(t-S)}ds\Big] \\ &\leq \Psi(\Omega)\left[q_1+q_5+\omega\,\|(-A)^\alpha\|\right] \\ &< \Psi(\Omega). \end{aligned}$$

But the inequality (3.3) implies that $\Psi(\Gamma\Omega)\geq\Psi(\Omega)$ and consequently

$$\Psi(\Omega)=0. \tag{3.9}$$

We shall give now an upper estimate for $mod_c\Gamma\Omega$. We have shown that

$$\chi\left\{S(f)(t), f\in sel_F(x), x\in\Omega\right\}=0,$$

for any $t\in[0,T]$. From the conditions $(F3)$ and $(F4)$ follows that the sequence $\{f\in sel_F(x), x\in\Omega\}$ is semicompact in $L^1([0,T],E)$, and hence the set

$$\{y; y(t)=Sf(t), t\in[0,T], f\in sel_F(x), x\in\Omega\}$$

is relatively compact in $C([0,T],E)$ (see [1]). Therefore, the set

$$\begin{aligned} \Gamma_1\Omega &= \{y(t)=\varphi(t)-h(t,\varphi), t\in[-r,0]; \\ &\quad\; y(t)=Sf(t), t\in[0,T], f\in sel_F(x), x\in\Omega\} \end{aligned}$$

is relatively compact in $C([-r,T],E)$. Consequently:

$$mod_c\Gamma_1\Omega=0. \tag{3.10}$$

Now we will show that the set

$$\Gamma_3\Omega=\left\{\begin{array}{ll} y(t)=\int_0^t Ae^{A(t-s)}h(s,x_s)ds, & t\in[0,T] \\ y(t)=0, & t\in[-r,0], \end{array}\right\}$$

where $x\in\Omega$, is equicontinuous on $C([-r,T],E])$.

Let $0\leq t\leq t'\leq T$,and $x\in\Omega$. We have

$$\begin{aligned} &\left\|\int_0^{t'}\left[Ae^{A(t'-s)}h(s,x_s)-\int_0^t Ae^{A(t-s)}h(s,x_s)\right]ds\right\| \\ \leq\; &\left\|\left(e^{A(t'-t)}-I\right)\int_0^t Ae^{A(t-s)}h(s,x_s)ds\right\|+\left\|\int_t^{t'}Ae^{A(t'-s)}h(s,x_s)ds\right\| \\ \leq\; &\left\|\left(e^{A(t'-t)}-I\right)\int_0^t Ae^{A(t-s)}h(s,x_s)ds\right\| \\ &+C_{1-\alpha}(d_1\sup_{x\in\Omega}\|x\|_1+d_2)\frac{(t'-t)^\alpha}{\alpha} \end{aligned}$$

Since $\chi\left(\int_0^t Ae^{A(t-s)}h(s,\Omega_s)ds\right)=0$, for all $t\in[0,T]$ the first term on the right hand side converge to zero when $t'\to t$ uniformly on $x\in\Omega$. As consequence we have

$$mod_c\Gamma_3\Omega=0. \tag{3.11}$$

From the condition $(H1)-(ii)$ follows immediately that

$$mod_c\Gamma_2\Omega\le\left\|(-A)^{-\alpha}\right\|\theta mod_c\Omega. \tag{3.12}$$

Indeed for $-r\le t\le s\le 0$, and $x\in\Omega$, we have

$$\|\Gamma_2x(t)-\Gamma_2x(s)\|=\|h(t,\varphi)-h(s,\varphi)\|=0.$$

For $0\le t\le s\le T$, and $x\in\Omega$, we have

$$\begin{aligned}\|\Gamma_2x(t)-\Gamma_2x(s)\| &= \|h(t,x_t)-h(s,x_s)\|\le\\ &\le \left\|(-A)^{-\alpha}(-A)^{\alpha}h(t,x_t)-(-A)^{-\alpha}(-A)^{\alpha}h(s,x_s)\right\|\\ &\le \theta\left\|(-A)^{-\alpha}\right\|\|x_t(0)-x_s(0)\|\le\\ &\le \theta\left\|(-A)^{-\alpha}\right\|\|x(t)-x(s)\|.\end{aligned}$$

Since $mod_c\Gamma\Omega\le\sum_{i=1}^{3}mod_c\Gamma_i$ from inequalities (3.9)-(3.12), we get

$$mod_c\Gamma\Omega\le\left\|(-A)^{-\alpha}\right\|\theta mod_c\Omega. \tag{3.13}$$

Since $\|(-A)^{-\alpha}\|\,\theta<1$, from the last inequality and the inequality (3.4) follows

$$mod_c\Omega=0. \tag{3.14}$$

Finally from the inequalities (3.5), (3.9) and (3.14) we get

$$\Theta((\Omega))=(0,0,0).$$

Hence the subset Ω is relatively compact, concluding the proof of Step 2.

Now in the space $C([-r,T],E)$ we introduce the equivalent norm, given by

$$\|x\|_*=\sup_{t\in[-r,0]}\|x(t)\|+\sup_{t\in[0,T]}e^{-Lt}\|x(t)\|$$

where L is a positive constant.

Consider the ball

$$B_r(0)=\{x\in C([-r,T],E);\|x\|_*\}\le r$$

where r is a constant chosen so that

$$r\ge\frac{M\|x_0\|+\|\varphi\|+\|(-A)^{-\alpha}\|\,d_2+q_2+q_4}{1-[\|(-A)^{-\alpha}\|\,d_1+q_2+q_3]}$$

where $x_0=\varphi(0)-h(0,\varphi)$. Note that the last inequality implies

$$\left[d_1\left\|(-A)^{-\alpha}\right\|+q_2+q_3\right]r+M\|x_0\|+\|\varphi\|+\left\|(-A)^{-\alpha}\right\|d_2+q_2+q_4\le r.$$

Step 3. The multioperator Γ maps the ball $B_r(0)$ into itself.

Let $x \in B_r(0)$ and $y \in \Gamma(x)$ with

$$y(t) = e^{At}x_0 + \quad h(t, x_t) + \int_0^t e^{A(t-s)} f(s)ds +$$

$$+ \int_0^t Ae^{A(t-s)} h(s, x_s)ds, \quad t \in [0, T],$$

$$y(t) = \varphi(t), t \in [-r, 0],$$

where $f \in sel_F(x)$.

Using the conditions $(F3)$ and $(H) - (i)$, we have, for any $t \in [0, T]$

$$\begin{aligned} e^{-Lt} \|y(t)\| \leq\ & e^{-Lt} \left\|e^{At}x_0\right\| \\ & + e^{-Lt} \left\|(-A)^{-\alpha}\right\| \|(-A)^{\alpha} h(t, x_t)\| \\ & + e^{-Lt} \int_0^t \left\|e^{A(t-s)}\right\| \|f(s)\| ds + \\ & + e^{-Lt} \int_0^t \left\|Ae^{A(t-s)} h(s, x_s)\right\| ds \\ \leq\ & e^{-Lt} M \|x_0\| + \left\|(-A)^{-\alpha}\right\| e^{-Lt} \left[d_1 \|x_t(0)\| + d_2\right] \\ & + Me^{-Lt} \int_0^t \beta(s)(1 + \|x_s(0)\|)ds \\ & + e^{-Lt} \int_0^t \left\|(-A)^{1-\alpha} e^{A(t-s)}\right\| \left[d_1 \|x_s(0)\| + d_2\right] ds \\ \leq\ & e^{-Lt} M \|x_0\| + \left\|(-A)^{-\alpha}\right\| e^{-Lt} \left[d_1 \|x(t)\| + d_2\right] \\ & + e^{-Lt} M \int_0^t e^{Ls} e^{-Ls} \beta(s)(1 + \|x(s)\|)ds \\ & + e^{-Lt} \int_0^t \left\|(-A)^{1-\alpha} e^{A(t-s)}\right\| e^{Ls} e^{-Ls} \left[d_1 \|x(s)\| + d_2\right] ds \\ \leq\ & e^{-Lt} M \|x_0\| + e^{-Lt} \left\|(-A)^{-\alpha}\right\| d_2 + \left\|(-A)^{-\alpha}\right\| d_1 \sup_{t \in [0,T]} e^{-Lt} \|x(t)\| \\ & + M \int_0^t e^{-L(t-s)} \beta(s)ds + M \int_0^t e^{-L(t-s)} \beta(s)ds \sup_{t \in [0,T]} e^{-Lt} \|x(t)\| \\ & + d_1 \int_0^t \frac{e^{-L(t-s)} C_{1-\alpha}}{(t-s)^{1-\alpha}} ds \sup_{t \in [0,T]} e^{-Lt} \|x(t)\| \\ & + d_2 \int_0^t \frac{e^{-L(t-s)} C_{1-\alpha}}{(t-s)^{1-\alpha}} ds. \end{aligned}$$

Hence

$$\begin{aligned}\sup_{t\in[0,T]} e^{-Lt}\|y(t)\| &\leq \left[\left\|(-A)^{-\alpha}\right\| d_1 + q_2 + q_3\right] \sup_{t\in[0,T]} e^{-Lt}\|x(t)\| + \\ &\quad +M\|x_0\| + \left\|(-A)^{-\alpha}\right\| d_2 + q_2 + q_4 \\ &\leq \left[\left\|(-A)^{-\alpha}\right\| d_1 + q_2 + q_3\right] \|x\|_\star \\ &\quad +M\|x_0\| + \left\|(-A)^{-\alpha}\right\| d_2 + q_2 + q_4 \\ &\leq \left[\left\|(-A)^{-\alpha}\right\| d_1 + q_2 + q_3\right] r + M\|x_0\| \\ &\quad + \left\|(-A)^{-\alpha}\right\| d_2 + q_2 + q_4\end{aligned}$$

It results that:

$$\begin{aligned}\|y\|_* &= \sup_{t\in[-r,0]}\|y(t)\| + \sup_{t\in[0,T]} e^{-Lt}\|y(t)\| \\ &\leq \|\varphi\| + \sup_{t\in[0,T]} e^{-Lt}\|y(t)\| \\ &\leq \left[\left\|(-A)^{-\alpha}\right\| d_1 + q_2 + q_3\right] r + M\|x_0\| + \|\varphi\| \\ &\quad + \left\|(-A)^{-\alpha}\right\| d_2 + q_2 + q_4 \\ &\leq r\end{aligned}$$

According to Lemma 2.3, the problem (1.1)-(1.2) has at last one solution.

Step 4. The solutions set is compact.

The solution set is a priori bounded. In fact, if x is a solution of the problem (1.1)-(1.2), then as above for $t \in [-r, T]$ we have

$$\begin{aligned}\|x(t)\| &\leq \frac{1}{1 - \|(-A)^{-\alpha}\| d_1}\Big[\|\varphi\| + M\|x_0\| \\ &\quad + \left(\left\|(-A)^{-\alpha}\right\| + C_{1-\alpha}\frac{T^\alpha}{\alpha}\right) d_2 \\ &\quad +M\|\beta\|_{L^1} + \int_0^t \left[M\beta(s) + \frac{d_1 C_{1-\alpha}}{(t-s)^{1-\alpha}}\right]\|x(s)\|\,ds\Big].\end{aligned}$$

Using Gronwall's type inequality, we get

$$\|x\|_1 \leq \xi e^\gamma$$

where

$$\begin{aligned}\xi &= \frac{1}{1 - \|(-A)^{-\alpha}\| d_1}\Big[\|\varphi\| + M\|x_0\| \\ &\quad + \left(\left\|(-A)^{-\alpha}\right\| + C_{1-\alpha}\frac{T^\alpha}{\alpha}\right) d_2 + M\|\beta\|_{L^1}\Big]\end{aligned}$$

and

$$\gamma = \frac{1}{1 - \|(-A)^{-\alpha}\| \, d_1} \left[M \, \|B\|_{L^1} + T^{\alpha} \frac{d_1 C_{1-\alpha}}{\alpha} \right].$$

To complete the proof it remains to apply Lemma 2.4.

References

[1] J.F. Couchouron and M. Kamenskii, A unified topological point of view for integro-diffrential inclusions and optimal control, J.Andres, L. Gorniewicz and P. Nistri eds., Lecture Notes in Nonlinear Anal. **2** (1998), 123-137.

[2] C. Gori, V. Obukhovskii, M. Ragni and P. Rubbioni, Existence and continuous dependence results for semilinear functional differential inclusions with infinite delay, Nonlinear Anal. **51** (2002), 765-782.

[3] E. Hernandez; A remark on neutral partial differential equations, Cadernos De Matematica,**04** (2003), 311-318.

[4] M. Kameskii, V. Obukhovskii and P. Zecca, Condensing multivalued maps and semilinear differential inclusions in Banach spaces, Berlin-New York, 2001.

[5] M. Kameskii, V. Obukhovskii and P. Zecca, Method of the solution sets for a quasilinear functional differential inclusion in a Banach space, Differ. Equ. Dyn. Syst. **4** (1996), 339-359.

[6] M. Kameskii and V. Obukhovskii, Condensing multioperators and periodic solutions of parabolic functional differential inclusions in Banach spaces, Nonlinear Anal. **20** (1993), 781-792.

[7] S.K. Ntouyas, Existence results for impulsive partial neutral functional differential inclusions, Electron. J. Differential Equations, Vol. 2005(2005), No. 30, pp. 1-11.

[8] V. V. Obukhovskii, Semilinear functional-differential inclusions in a Banach space and controlled parabolic systems, Soviet J. Automat. Inform. Sci. **24** (1991), 71-79.

[9] S. N. Papageorgiou, On multivalued evolutions equations and differential inclusions in Banach spaces, Comment. Math. Univ. San. Pauli **36** (1987), 21-39.

[10] S.N. Papageorgiou, On multivalued semilinear evolution equations, Boll. U.M.I. (**7**) 3-B (1990), 1-16.

[11] A. Pazy, Semigroups of linear Operators and Applications to Partial Differential Equations, Applied Mathematical Sciences, vol.**44**, Springer Verlag, New York,1983.

[12] B. N. Sadovskii, On a fixed point principle, Funct. Anal. Appl.**1** (1967), 74-76

(Received April 14, 2006)

University of Tiaret, BP 78, Algeria.
E-mail address: lahcene_guedda@yahoo.fr