

VIABILITY PROBLEM WITH PERTURBATION IN HILBERT SPACE

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ABSTRACT. This paper deals with the existence result of viable solutions of the differential inclusion

$$\begin{aligned}\dot{x}(t) &\in f(t, x(t)) + F(x(t)) \\ x(t) &\in K \text{ on } [0, T],\end{aligned}$$

where K is a locally compact subset in separable Hilbert space H , $(f(s, \cdot))_s$ is an equicontinuous family of measurable functions with respect to s and F is an upper semi-continuous set-valued mapping with compact values contained in the Clarke subdifferential $\partial_c V(x)$ of an uniformly regular function V .

Key words: Regularity, upper semi-continuous, equicontinuous perturbation, Clarke subdifferential.

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1. INTRODUCTION

Existence result of local solution for differential inclusion with upper semi-continuous and cyclically monotone right hand-side whose values in finite-dimensional space, was first established by Bressan, Cellina and Colombo (see [6]). The authors exploited rich properties of subdifferential of convex lower semi-continuous function; in order to overcome the weakly convergence of derivatives of approximate solutions, they used the basic relation (see [7])

$$\frac{d}{dt}V(x(t)) = \|\dot{x}(t)\|^2.$$

Later, Ancona, Cellina and Colombo (see [1]), under the same hypotheses as the above paper, extend this result to the perturbed problem

$$\dot{x}(t) \in f(t, x(t)) + F(x(t))$$

where $f(\cdot, \cdot)$ is a Carathéodory function.

This program of research was pursued by a series of works. In the first one (see [9]), Truong proved a viability result for similar problem, where the perturbation f is replaced by a globally continuous set-valued mapping G with values in finite-dimensional space. This result was extended by Bounkhel (see [4]) for a similar

problem, where F is not cyclically monotone but contained in the Clarke subdifferential of locally Lipschitz uniformly regular function. However under very strong assumptions namely, the space of states is finite-dimensional and the following tangential condition

$$\left(G(t, x) + F(x)\right) \subset T_K(x)$$

where $T_K(x)$ is the contingent cone at x to K .

Recently, Morchadi and Sajid (see [8]) proved an exact viability version of the work of Ancona and Colombo assuming the same hypotheses and the following tangential condition

$\forall(t, x) \in \mathbb{R} \times K, \exists v \in F(x)$ such that

$$\liminf_{h \rightarrow 0^+} \frac{1}{h} d_K \left(x + hv + \int_t^{t+h} f(s, x) ds \right) = 0. \quad (1.1)$$

Remark that in all the above works, the convexity assumption of V and/or the finite-dimensional hypothesis of the space of states were widely used in the proof.

This paper is devoted to establish a local solution of the problem

$$\begin{aligned} \dot{x}(t) &\in f(t, x(t)) + F(x(t)), & F(x(t)) &\subset \partial_c V(x(t)) \\ x(t) &\in K \subset H, \end{aligned}$$

where K is a locally compact subset of a separable Hilbert space H , F is an upper semi-continuous multifunction, $\partial_c V$ denotes the Clarke subdifferential of a locally Lipschitz function V and the set $\{f(s, \cdot) : s \in \mathbb{R}\}$ is equicontinuous, where for each $x \in K$, $s \mapsto f(s, x)$ is measurable and the same tangential condition (1.1). One case deserves mentioning: when f is globally continuous, the condition (1.1) is weaker than the following

$$\left(f(t, x) + F(x)\right) \cap T_K(x) \neq \emptyset.$$

To remove the convexity assumption of V and the finite-dimensional hypothesis of H , we rely on some properties of Clarke subdifferential of uniformly regular function and the local compactness of K .

2. PRELIMINARIES AND STATEMENT OF THE MAIN RESULT

Let H be a real separable Hilbert space with the norm $\|\cdot\|$ and the scalar product $\langle \cdot, \cdot \rangle$. For $x \in H$ and $r > 0$ let $B(x, r)$ be the open ball centered at x with radius r and $\bar{B}(x, r)$ be its closure and put $B = B(0, 1)$.

Let us recall the definition of the Clarke subdifferential and the concept of regularity that will be used in the sequel.

Definition 2.1. Let $V : H \rightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous function and x be any point where V is finite. The Clarke subdifferential of V at x is defined by

$$\partial_c V(x) := \{y \in H : \langle y, h \rangle \leq V^\uparrow(x, h), \text{ for all } h \in H\},$$

where $V^\uparrow(x, h)$ is the generalized Rockafellar directional derivative given by

$$V^\uparrow(x, h) := \limsup_{x' \rightarrow x, V(x') \rightarrow V(x), t \rightarrow 0} \inf_{h' \rightarrow h} \frac{V(x' + th') - V(x')}{t}.$$

Definition 2.2. Let $V : H \rightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous function and let $U \subset \text{Dom}V$ be a nonempty open subset. We will say that V is uniformly regular over U if there exists a positive number β such that for all $x \in U$ and for all $\xi \in \partial_p V(x)$ one has

$$\langle \xi, x' - x \rangle \leq V(x') - V(x) + \beta \|x' - x\|^2 \quad \text{for all } x' \in U.$$

$\partial_p V(x)$ denotes the proximal subdifferential of V at x which is the set of all $y \in H$ for which there exist $\delta, \sigma > 0$ such that for all $x' \in x + \delta \bar{B}$

$$\langle y, x' - x \rangle \leq V(x') - V(x) + \sigma \|x' - x\|^2.$$

We say that V is uniformly regular over closed set S if there exists an open set U containing S such that V is uniformly regular over U . For more details on the concept of regularity, we refer the reader to [4].

Proposition 2.3. [3, 4] Let $V : H \rightarrow \mathbb{R}$ be a locally Lipschitz function and S a nonempty closed set. If V is uniformly regular over S , then the following conditions holds:

- (a) The proximal subdifferential of V is closed over S , that is, for every $x_n \rightarrow x \in S$ with $x_n \in S$ and every $\xi_n \rightarrow \xi$ with $\xi_n \in \partial_p V(x_n)$ one has $\xi \in \partial_p V(x)$.
- (b) The proximal subdifferential of V coincides with the Clarke subdifferential of V for any point x .
- (c) The proximal subdifferential of V is upper hemicontinuous over S , that is, the support function $x \mapsto \sigma(v, \partial_p V(x))$ is u.s.c. over S for every $v \in H$.

Now let us state the main result.

Let $V : H \rightarrow \mathbb{R}$ be a locally Lipschitz function and β -uniformly regular over $K \subset H$. Assume that

- (H1) K is a nonempty locally compact subset in H ;
- (H2) $F : K \rightarrow 2^H$ is an upper semi-continuous set valued map with compact values satisfying

$$F(x) \subset \partial_c V(x) \quad \text{for all } x \in K;$$

- (H3) $f : \mathbb{R} \times H \rightarrow H$ is a function with the following properties:

- (1) For all $x \in H$, $t \mapsto f(t, x)$ is measurable,
- (2) The family $\{f(s, \cdot) : s \in \mathbb{R}\}$ is equicontinuous,
- (3) For all bounded subset S of H , there exists $M > 0$ such that

$$\|f(t, x)\| \leq M, \quad \forall (t, x) \in \mathbb{R} \times S;$$

- (H4) (Tangential condition) $\forall (t, x) \in \mathbb{R} \times K, \exists v \in F(x)$ such that

$$\liminf_{h \rightarrow 0^+} \frac{1}{h} d_K \left(x + hv + \int_t^{t+h} f(s, x) ds \right) = 0.$$

For any $x_0 \in K$, consider the problem:

$$\begin{cases} \dot{x}(t) \in f(t, x(t)) + F(x(t)) & \text{a.e;} \\ x(0) = x_0; \\ x(t) \in K. \end{cases} \quad (2.1)$$

Theorem 2.4. *If assumptions (H1)-(H4) are satisfied, then there exists $T > 0$ such that the problem (2.1) admits a solution on $[0, T]$.*

3. PROOF OF THE MAIN RESULT

Choose $r > 0$ such that $K_0 = K \cap (x_0 + r\bar{B})$ is compact and V is Lipschitz continuous on $x_0 + r\bar{B}$ with Lipschitz constant $\lambda > 0$. Then $\partial_c V(x) \subset \lambda\bar{B}$ for every $x \in K_0$. Let $M > 0$ such that

$$\|f(t, x)\| \leq M, \quad \forall (t, x) \in \mathbb{R} \times (x_0 + r\bar{B}). \quad (3.1)$$

Set

$$T = \frac{r}{2(\lambda + 1 + M)}. \quad (3.2)$$

In the sequel, we will use the following important Lemma. It will play a crucial role in the proof of the main result.

Lemma 3.1. *If assumptions (H1)-(H4) are satisfied, then for all $0 < \varepsilon < \inf(T, 1)$, there exists $\eta > 0$ ($\eta < \varepsilon$) such that*

$\forall (t, x) \in [0, T] \times K_0, \exists h_{t,x} \in [\eta, \varepsilon], u \in F(x) + \frac{1}{h_{t,x}} \int_t^{t+h_{t,x}} f(s, x) ds + \frac{\varepsilon}{T} B, y_{t,x} \in K_0$
and $v \in F(y_{t,x})$ such that

$$(x + h_{t,x}u) \in K \cap B\left(x + h_{t,x}v + \int_t^{t+h_{t,x}} f(s, x) ds, \lambda + M + 1\right).$$

Proof. Let $(t, x) \in [0, T] \times K_0$, be fixed, let $0 < \varepsilon < \inf(T, 1)$. Since F is *u.s.c* on x , then there exists $\delta_x > 0$ such that

$$F(y) \subset F(x) + \frac{\varepsilon}{2T} B, \quad \text{for all } y \in B(x, \delta_x).$$

Let $(s, y) \in [0, T] \times K_0$. By the tangential condition, there exists $v \in F(y)$ and $h_{s,y} \in]0, \varepsilon]$ such that

$$d_K\left(y + h_{s,y}v + \int_s^{s+h_{s,y}} f(\tau, y) d\tau\right) < h_{s,y} \frac{\varepsilon}{4T}.$$

Consider the subset

$$N(s, y) = \left\{ (t, z) \in \mathbb{R} \times H/d_K\left(z + h_{s,y}v + \int_t^{t+h_{s,y}} f(\tau, z) d\tau\right) < h_{s,y} \frac{\varepsilon}{4T} \right\}.$$

Since

$$\|f(\tau, z)\| \leq M, \quad \forall (\tau, z) \in \mathbb{R} \times \bar{B}(x_0, r),$$

then the dominated convergence theorem applied to the sequence $(\chi_{[t,t+h_{s,y}]}f(\cdot, \cdot))_t$ of functions shows that the function

$$(l, z) \mapsto z + h_{s,y}v + \int_l^{l+h_{s,y}} f(\tau, z)d\tau$$

is continuous. So that, the function

$$(l, z) \mapsto d_K \left(z + h_{s,y}v + \int_l^{l+h_{s,y}} f(\tau, z)d\tau \right)$$

is continuous and consequently $N(s, y)$ is open. Moreover, since (s, y) belongs to $N(s, y)$, there exists a ball $B((s, y), \eta_{s,y})$ of radius $\eta_{s,y} < \delta_x$ contained in $N(s, y)$, therefore, the compact subset $[0, T] \times K_0$ can be covered by q such balls $B((s_i, y_i), \eta_{s_i, y_i})$. For simplicity, we set

$$h_{s_i, y_i} := h_i \text{ and } \eta_i := \eta_{s_i, y_i}, \quad i = 1, \dots, q.$$

Put $\eta = \min\{h_i/1 \leq i \leq q\}$ and let $i \in \{1, \dots, q\}$ such that $(t, x) \in B((s_i, y_i), \eta_i)$, hence $(t, x) \in N(s_i, y_i)$. Then there exists $v_i \in F(y_i)$ such that

$$d_K \left(x + h_i v_i + \int_t^{t+h_i} f(\tau, x)d\tau \right) < h_i \frac{\varepsilon}{4T}.$$

Let $x_i \in K$ such that

$$\frac{1}{h_i} \left\| x_i - \left(x + h_i v_i + \int_t^{t+h_i} f(\tau, x)d\tau \right) \right\| \leq \frac{1}{h_i} d_K \left(x + h_i v_i + \int_t^{t+h_i} f(\tau, x)d\tau \right) + \frac{\varepsilon}{4T}.$$

Hence

$$\left\| \frac{x_i - x}{h_i} - \left(v_i + \frac{1}{h_i} \int_t^{t+h_i} f(\tau, x)d\tau \right) \right\| < \frac{\varepsilon}{2T}.$$

Set

$$u = \frac{x_i - x}{h_i},$$

then $x_i = x + h_i u \in K$ and

$$u \in \left(\frac{1}{h_i} \int_t^{t+h_i} f(\tau, x)d\tau + F(y_i) + \frac{\varepsilon}{2T}B \right).$$

Since $\|x - y_i\| < \eta_i < \delta_x$ we have

$$F(y_i) \subset F(x) + \frac{\varepsilon}{2T}B,$$

then

$$u \in \left(\frac{1}{h_i} \int_t^{t+h_i} f(\tau, x)d\tau + F(x) + \frac{\varepsilon}{T}B \right).$$

On the other hand, since $x \in K$, we have

$$\left\| x_i - \left(x + h_i v_i + \int_t^{t+h_i} f(\tau, x)d\tau \right) \right\| \leq d_K \left(x + h_i v_i + \int_t^{t+h_i} f(\tau, x)d\tau \right) + \frac{\varepsilon}{4T}$$

$$\begin{aligned} &\leq \left\| h_i v_i + \int_t^{t+h_i} f(\tau, x) d\tau \right\| + \frac{\varepsilon}{4T} \\ &\leq h_i(\lambda + M) + 1 < \lambda + M + 1. \end{aligned}$$

Thus $x_i \in B\left(x + h_i v_i + \int_t^{t+h_i} f(\tau, x) d\tau, \lambda + M + 1\right)$. □

Now, we are able to prove the main result. Our approach consists of constructing, in a first step, a sequence of approximate solutions and deduce, in a second step, from available estimates that a subsequence converges to a solution of (2.1).

Step 1. Approximate solutions. Let $x_0 \in K_0$ and $0 < \varepsilon < \inf(T, 1)$. By Lemma 3.1, there exist $\eta > 0$, $h_0 \in [\eta, \varepsilon]$, $u_0 \in \left(\frac{1}{h_0} \int_0^{h_0} f(s, x_0) ds + F(x_0) + \frac{\varepsilon}{T} B\right)$, $y_0 \in K_0$ and $v_0 \in F(y_0)$ such that

$$x_1 = x_0 + h_0 u_0 \in K \cap B\left(x_0 + h_0 v_0 + \int_0^{h_0} f(s, x_0) ds, \lambda + M + 1\right).$$

Then by (H2), (3.1) and (3.2), we have

$$\|x_1 - x_0\| = \|h_0 u_0\| \leq (\lambda + 1 + M)T < r$$

and thus $x_1 \in K_0$. Set $h_{-1} = 0$. By induction, for $q \geq 2$ and for every $p = 1, \dots, q - 1$, we construct the sequences $(h_p)_p \subset [\eta, \varepsilon]$, $((x_p)_p, (y_p)_p) \subset K_0 \times K_0$ and $((u_p)_p, (v_p)_p) \subset H \times H$ such that $\sum_{p=1}^{q-1} h_p \leq T$ and

$$\left\{ \begin{array}{l} x_p = x_{p-1} + h_{p-1} u_{p-1}; \\ x_p \in K \cap B\left(x_{p-1} + h_{p-1} v_{p-1} + \int_{\sum_{i=0}^{p-2} h_i}^{\sum_{i=0}^{p-1} h_i} f(s, x_{p-1}) ds, \lambda + M + 1\right); \\ u_p \in \left(\frac{1}{h_p} \int_{\sum_{i=0}^{p-1} h_i}^{\sum_{i=0}^p h_i} f(s, x_p) ds + F(x_p) + \frac{\varepsilon}{T} B\right); \\ v_p \in F(y_p). \end{array} \right.$$

Since $h_i \geq \eta > 0$ there exists an integer s such that

$$\sum_{i=0}^{s-1} h_i < T \leq \sum_{i=0}^s h_i.$$

Then we have constructed the sequences $(h_p)_p \subset [\eta, \varepsilon]$, $((x_p)_p, (y_p)_p) \subset K_0 \times K_0$ and $((u_p)_p, (v_p)_p) \subset H \times H$ such that for every $p = 1, \dots, s$, we have

$$\begin{aligned} &\text{(i) } x_p = x_{p-1} + h_{p-1} u_{p-1}; \\ &\text{(ii) } x_p \in K \cap B\left(x_{p-1} + h_{p-1} v_{p-1} + \int_{\sum_{i=0}^{p-2} h_i}^{\sum_{i=0}^{p-1} h_i} f(s, x_{p-1}) ds, \lambda + M + 1\right); \end{aligned}$$

$$(iii) \quad u_p \in F(x_p) + \frac{1}{h_p} \int_{\sum_{i=0}^{p-1} h_i}^{\sum_{i=0}^p h_i} f(s, x_p) ds + \frac{\varepsilon}{T} B;$$

$$(iv) \quad v_p \in F(y_p).$$

By induction, for all $p = 1, \dots, s$ we have

$$x_p = x_0 + \sum_{i=0}^{p-1} h_i u_i.$$

Moreover by (iii), (H2), (3.1), (3.2) and because $\sum_{i=0}^{p-1} h_i < T$, we have

$$\|x_p - x_0\| = \left\| \sum_{i=0}^{p-1} h_i u_i \right\| \leq \sum_{i=0}^{p-1} h_i \|u_i\| \leq \sum_{i=0}^{p-1} h_i (\lambda + 1 + M) < r, \quad (3.3)$$

hence $x_p \in K_0$.

For any nonzero integ k and for every integer $q = 0, \dots, s - 1$ denote by h_q^k a real associated to $\varepsilon = \frac{1}{k}$ and $x = x_q$ given by Lemma 3.1. Consider the sequence $(\tau_k^q)_k$ defined as the following

$$\begin{cases} \tau_k^0 = 0, \tau_k^{s+1} = T; \\ \tau_k^q = h_0^k + \dots + h_{q-1}^k \quad \text{if } 1 \leq q \leq s, \end{cases}$$

and define on $[0, T]$ the sequence of functions $(x_k(\cdot))_k$ by

$$\begin{cases} x_k(t) = x_{q-1} + (t - \tau_k^{q-1}) u_{q-1}, \quad \forall t \in [\tau_k^{q-1}, \tau_k^q]; \\ x_k(0) = x_0. \end{cases}$$

Step 2. Convergence of approximate solutions. By definition of $x_k(\cdot)$, for all $t \in [\tau_k^{q-1}, \tau_k^q]$ we have $\dot{x}_k(t) = u_{q-1}$. By (iii), (H2), (3.1), for a. e. $t \in [0, T]$, we have

$$\|\dot{x}_k(t)\| \leq \lambda + 1 + M.$$

On the other hand, by (ii), (iv), (H2), (3.1) and (3.3) we have

$$\begin{aligned} \|x_q\| &\leq \left\| x_q - (x_{q-1} + h_{q-1}^k v_{q-1} + \int_{\tau_k^{q-1}}^{\tau_k^q} f(s, x_{q-1}) ds) \right\| \\ &\quad + \left\| x_{q-1} + h_{q-1}^k v_{q-1} + \int_{\tau_k^{q-1}}^{\tau_k^q} f(s, x_{q-1}) ds \right\| \\ &\leq \lambda + M + 1 + \left\| x_0 - (x_0 - x_{q-1}) + h_{q-1}^k v_{q-1} + \int_{\tau_k^{q-1}}^{\tau_k^q} f(s, x_{q-1}) ds \right\| \\ &\leq \lambda + M + 1 + \|x_0\| + \|x_0 - x_{q-1}\| + h_{q-1}^k \|v_{q-1}\| + h_{q-1}^k M \\ &\leq \lambda + M + 1 + \|x_0\| + r + \lambda + M \\ &< 2(\lambda + M + 1) + \|x_0\| + r = R. \end{aligned}$$

Then $x_q \in K_0 \cap \bar{B}(0, R) = K_1$. By construction, for all $t \in [\tau_k^{q-1}, \tau_k^q]$ we have

$$x_k(t) = x_{q-1} + (t - \tau_k^{q-1}) u_{q-1} = x_{q-1} + \frac{(t - \tau_k^{q-1})}{h_{q-1}^k} (x_q - x_{q-1}).$$

Also since $0 \leq t - \tau_k^{q-1} \leq \tau_k^q - \tau_k^{q-1} = h_{q-1}^k$, we have

$$0 \leq \frac{(t - \tau_k^{q-1})}{h_{q-1}^k} \leq 1.$$

Then

$$\frac{(t - \tau_k^{q-1})}{h_{q-1}^k} (x_q - x_{q-1}) \in \bar{co}\{\{0\} \cup (K_1 - K_0)\},$$

hence $x_k(t) \in K_0 + \bar{co}\{\{0\} \cup (K_1 - K_0)\}$ which is compact. Therefore, we can select a subsequence, again denoted by $(x_k(\cdot))_k$ which converges uniformly to an absolutely continuous function $x(\cdot)$ on $[0, T]$, moreover $\dot{x}_k(\cdot)$ converges weakly to $\dot{x}(\cdot)$ in $L^2([0, T], H)$. The family of approximate solution $x_k(\cdot)$ satisfies the following property.

Proposition 3.2. *For every $t \in [0, T]$, there exists $q \in \{1, \dots, s+1\}$ such that*

$$\lim_{k \rightarrow +\infty} d_{grF} \left((x_k(t), \dot{x}_k(t) - \frac{1}{h_{q-1}^k} \int_{\tau_k^{q-1}}^{\tau_k^q} f(s, x_{q-1}) ds) \right) = 0.$$

Proof. Let $t \in [0, T]$, there exists $q \in \{1, \dots, s+1\}$ such that $t \in [\tau_k^{q-1}, \tau_k^q]$ and $\lim_{k \rightarrow +\infty} \tau_k^{q-1} = t$. Since

$$\dot{x}_k(t) - \frac{1}{h_{q-1}^k} \int_{\tau_k^{q-1}}^{\tau_k^q} f(s, x_{q-1}) ds \in F(x_{q-1}) + \frac{1}{kT} B, \quad (3.4)$$

we have

$$d_{grF} \left((x_k(t), \dot{x}_k(t) - \frac{1}{h_{q-1}^k} \int_{\tau_k^{q-1}}^{\tau_k^q} f(s, x_{q-1}) ds) \right) \leq \|x_k(t) - x_k(\tau_k^{q-1})\| + \frac{1}{kT},$$

hence

$$\lim_{k \rightarrow +\infty} d_{grF} \left((x_k(t), \dot{x}_k(t) - \frac{1}{h_{q-1}^k} \int_{\tau_k^{q-1}}^{\tau_k^q} f(s, x_{q-1}) ds) \right) = 0. \quad \square$$

Claim 3.3.

$$\lim_{k \rightarrow +\infty} \frac{1}{h_{q-1}^k} \int_{\tau_k^{q-1}}^{\tau_k^q} f(s, x_{q-1}) ds = f(t, x(t)).$$

Proof. Fix any $t \in [0, T]$, there exists $q \in \{1, \dots, s+1\}$ such that $t \in [\tau_k^{q-1}, \tau_k^q]$, $\lim_{k \rightarrow +\infty} \tau_k^{q-1} = \lim_{k \rightarrow +\infty} \tau_k^q = t$ and $\lim_{k \rightarrow +\infty} x_k(\tau_k^{q-1}) = x(t)$. Put

$$G(t, y) = \int_0^t f(s, y) ds.$$

Note that the function G is differentiable on t and

$$\frac{dG}{dt}(t, y) = f(t, y).$$

We have

$$\begin{aligned} & \left\| \frac{G(\tau_k^q, x(t)) - G(\tau_k^{q-1}, x(t))}{\tau_k^q - \tau_k^{q-1}} - f(t, x(t)) \right\| \\ & \leq \left\| \frac{G(\tau_k^q, x(t)) - G(\tau_k^{q-1}, x(t))}{\tau_k^q - \tau_k^{q-1}} - \frac{G(t, x(t)) - G(\tau_k^{q-1}, x(t))}{t - \tau_k^{q-1}} \right\| \\ & \quad + \left\| \frac{G(t, x(t)) - G(\tau_k^{q-1}, x(t))}{t - \tau_k^{q-1}} - f(t, x(t)) \right\|. \end{aligned}$$

On the other hand

$$\begin{aligned} & \left\| \frac{G(\tau_k^q, x(t)) - G(\tau_k^{q-1}, x(t))}{\tau_k^q - \tau_k^{q-1}} - \frac{G(t, x(t)) - G(\tau_k^{q-1}, x(t))}{t - \tau_k^{q-1}} \right\| \\ & = \left\| \frac{\tau_k^q - t}{\tau_k^q - \tau_k^{q-1}} \left(\frac{G(\tau_k^q, x(t)) - G(t, x(t))}{\tau_k^q - t} - \frac{G(\tau_k^{q-1}, x(t)) - G(t, x(t))}{\tau_k^{q-1} - t} \right) \right\| \\ & \leq \left\| \frac{G(\tau_k^q, x(t)) - G(t, x(t))}{\tau_k^q - t} - f(t, x) \right\| \\ & \quad + \left\| \frac{G(\tau_k^{q-1}, x(t)) - G(t, x(t))}{\tau_k^{q-1} - t} - f(t, x(t)) \right\|. \end{aligned}$$

Hence

$$\begin{aligned} & \left\| \frac{G(\tau_k^q, x(t)) - G(\tau_k^{q-1}, x(t))}{\tau_k^q - \tau_k^{q-1}} - f(t, x(t)) \right\| \\ & \leq \left\| \frac{G(\tau_k^q, x(t)) - G(t, x(t))}{\tau_k^q - t} - f(t, x(t)) \right\| \\ & \quad + 2 \left\| \frac{G(t, x(t)) - G(\tau_k^{q-1}, x(t))}{t - \tau_k^{q-1}} - f(t, x(t)) \right\|. \end{aligned}$$

As

$$\lim_{k \rightarrow +\infty} \frac{G(\tau_k^q, x(t)) - G(t, x(t))}{\tau_k^q - t} = \frac{dG}{dt}(t, x(t)) = f(t, x(t))$$

and

$$\lim_{k \rightarrow +\infty} \frac{G(\tau_k^{q-1}, x(t)) - G(t, x(t))}{\tau_k^{q-1} - t} = \frac{dG}{dt}(t, x(t)) = f(t, x(t)),$$

we have

$$\lim_{k \rightarrow +\infty} \frac{G(\tau_k^q, x(t)) - G(\tau_k^{q-1}, x(t))}{\tau_k^q - \tau_k^{q-1}} = f(t, x(t)). \quad (3.5)$$

Put

$$\rho_k = \left\| \frac{G(\tau_k^q, x(t)) - G(\tau_k^{q-1}, x(t))}{\tau_k^q - \tau_k^{q-1}} - f(t, x(t)) \right\|.$$

On the other hand we have

$$\begin{aligned}
 & \left\| \frac{1}{h_{q-1}^k} \int_{\tau_k^{q-1}}^{\tau_k^q} f(s, x_k(\tau_k^{q-1})) ds - f(t, x(t)) \right\| \\
 &= \left\| \frac{G(\tau_k^q, x_k(\tau_k^{q-1})) - G(\tau_k^{q-1}, x_k(\tau_k^{q-1}))}{\tau_k^q - \tau_k^{q-1}} - f(t, x(t)) \right\| \\
 &\leq \left\| \frac{G(\tau_k^q, x_k(\tau_k^{q-1})) - G(\tau_k^{q-1}, x_k(\tau_k^{q-1}))}{\tau_k^q - \tau_k^{q-1}} - \frac{G(\tau_k^q, x(t)) - G(\tau_k^{q-1}, x(t))}{\tau_k^q - \tau_k^{q-1}} \right\| + \rho_k \\
 &= \left\| \frac{1}{\tau_k^q - \tau_k^{q-1}} \int_{\tau_k^{q-1}}^{\tau_k^q} (f(s, x_k(\tau_k^{q-1})) - f(s, x(t))) ds \right\| + \rho_k.
 \end{aligned}$$

Since the family $\{f(s, \cdot) : s \in \mathbb{R}\}$ is equicontinuous, then there exists k_0 such that

$$\|f(s, x_k(\tau_k^{q-1})) - f(s, x(t))\| \leq \frac{1}{k} \text{ for all } k \geq k_0 \text{ and for all } s \in \mathbb{R},$$

consequently we have for $k \geq k_0$

$$\left\| \frac{G(\tau_k^q, x_k(\tau_k^{q-1})) - G(\tau_k^{q-1}, x_k(\tau_k^{q-1}))}{\tau_k^q - \tau_k^{q-1}} - f(t, x(t)) \right\| \leq \frac{1}{k} + \rho_k.$$

By (3.5), the last term converges to 0. This completes the proof of the Claim. \square

The function $x(\cdot)$ has the following property

Proposition 3.4. *For all $t \in [0, T]$, we have $\dot{x}(t) - f(t, x(t)) \in \partial_c V(x(t))$.*

Proof. The weak convergence of $\dot{x}_k(\cdot)$ to $\dot{x}(\cdot)$ in $L^2([0, T], H)$ and the Mazur's Lemma entail

$$\dot{x}(t) \in \bigcap_k \bar{co}\{\dot{x}_m(t) : m \geq k\}, \text{ for a.e. on } [0, T].$$

Fix any $t \in [0, T]$, there exists $q \in \{1, \dots, s+1\}$ such that $t \in [\tau_k^{q-1}, \tau_k^q]$ and $\lim_{k \rightarrow +\infty} \tau_k^{q-1} = t$. Then for all $y \in H$

$$\langle y, \dot{x}(t) \rangle \leq \inf_m \sup_{k \geq m} \langle y, \dot{x}_k(t) \rangle.$$

Since $F(x) \subset \partial_c V(x)$, then by (3.4), one has

$$\dot{x}_k(t) \in \partial_c V(x_{q-1}) + \frac{1}{h_{q-1}^k} \int_{\tau_k^{q-1}}^{\tau_k^q} f(s, x_{q-1}) ds + \frac{1}{kT} B.$$

Thus for all m

$$\langle y, \dot{x}(t) \rangle \leq \sup_{k \geq m} \sigma \left(y, \partial_c V(x_{q-1}) + \frac{1}{h_{q-1}^k} \int_{\tau_k^{q-1}}^{\tau_k^q} f(s, x_{q-1}) ds + \frac{1}{kT} B \right),$$

from which we deduce that

$$\langle y, \dot{x}(t) \rangle \leq \limsup_{k \rightarrow +\infty} \sigma \left(y, \partial_c V(x_{q-1}) + \frac{1}{h_{q-1}^k} \int_{\tau_k^{q-1}}^{\tau_k^q} f(s, x_{q-1}) ds + \frac{1}{kT} B \right).$$

By Proposition 2.3, the function $x \mapsto \sigma(y, \partial_c V(x))$ is *u.s.c* and hence we get

$$\langle y, \dot{x}(t) \rangle \leq \sigma(y, \partial_c V(x(t)) + f(t, x(t))).$$

So, the convexity and the closedness of the set $\partial_c V(x(t))$ ensure

$$\dot{x}(t) - f(t, x(t)) \in \partial_c V(x(t)).$$

□

Proposition 3.5. *The application $x(\cdot)$ is a solution of the problem (2.1).*

Proof. As $x(\cdot)$ is an absolutely continuous function and V is uniformly regular locally Lipschitz function over K (hence directionally regular over K (see [5])), by Theorem 2 in Valadier [10, 11] and by Proposition 3.4, we obtain

$$\frac{d}{dt} V(x(t)) = \langle \dot{x}(t), \dot{x}(t) - f(t, x(t)) \rangle \text{ a. e. on } [0, T],$$

therefore,

$$V(x(T)) - V(x_0) = \int_0^T \|\dot{x}(s)\|^2 ds - \int_0^T \langle \dot{x}(s), f(s, x(s)) \rangle ds. \quad (3.6)$$

On the other hand, by construction, for all $q = 1, \dots, s+1$, we have

$$\dot{x}_k(t) - \frac{1}{h_{q-1}^k} \int_{\tau_k^{q-1}}^{\tau_k^q} f(s, x_{q-1}) ds \in \partial_c V(x_{q-1}) + \frac{1}{kT} B.$$

Let b_q such that

$$\dot{x}_k(t) - \frac{1}{h_{q-1}^k} \int_{\tau_k^{q-1}}^{\tau_k^q} f(s, x_{q-1}) ds + \frac{1}{kT} b_q \in \partial_c V(x_{q-1}).$$

Since V is β -uniformly regular over K , we have

$$\begin{aligned} V(x_k(\tau_k^q)) - V(x_k(\tau_k^{q-1})) &\geq \langle x_k(\tau_k^q) - x_k(\tau_k^{q-1}), \dot{x}_k(t) \\ &\quad - \frac{1}{h_{q-1}^k} \int_{\tau_k^{q-1}}^{\tau_k^q} f(s, x_{q-1}) ds + \frac{1}{kT} b_q \rangle \\ &\quad - \beta \left\| x_k(\tau_k^q) - x_k(\tau_k^{q-1}) \right\|^2 \\ &= \langle \int_{\tau_k^{q-1}}^{\tau_k^q} \dot{x}_k(s) ds, \dot{x}_k(t) \\ &\quad - \frac{1}{h_{q-1}^k} \int_{\tau_k^{q-1}}^{\tau_k^q} f(s, x_{q-1}) ds + \frac{1}{kT} b_q \rangle \\ &\quad - \beta \left\| x_k(\tau_k^q) - x_k(\tau_k^{q-1}) \right\|^2 \\ &= \int_{\tau_k^{q-1}}^{\tau_k^q} \langle \dot{x}_k(s), \dot{x}_k(s) \rangle ds \\ &\quad - \int_{\tau_k^{q-1}}^{\tau_k^q} \langle \dot{x}_k(s), \frac{1}{h_{q-1}^k} \int_{\tau_k^{q-1}}^{\tau_k^q} f(\tau, x_{q-1}) d\tau \rangle ds \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{kT} \int_{\tau_k^{q-1}}^{\tau_k^q} \langle \dot{x}_k(s), b_q \rangle ds \\
& - \beta \left\| x_k(\tau_k^q) - x_k(\tau_k^{q-1}) \right\|^2.
\end{aligned}$$

By adding, we obtain

$$\begin{aligned}
V(x_k(T)) - V(x_0) & \geq \int_0^T \|\dot{x}_k(s)\|^2 ds \\
& - \sum_{q=1}^{s+1} \int_{\tau_k^{q-1}}^{\tau_k^q} \langle \dot{x}_k(s), \frac{1}{h_{q-1}^k} \int_{\tau_k^{q-1}}^{\tau_k^q} f(\tau, x_{q-1}) d\tau \rangle ds \\
& + \frac{1}{kT} \sum_{q=1}^{s+1} \int_{\tau_k^{q-1}}^{\tau_k^q} \langle \dot{x}_k(s), b_q \rangle ds \\
& - \sum_{q=1}^{s+1} \beta \left\| x_k(\tau_k^q) - x_k(\tau_k^{q-1}) \right\|^2.
\end{aligned} \tag{3.7}$$

Claim 3.6.

$$\lim_{k \rightarrow +\infty} \sum_{q=1}^{s+1} \int_{\tau_k^{q-1}}^{\tau_k^q} \langle \dot{x}_k(s), \frac{1}{h_{q-1}^k} \int_{\tau_k^{q-1}}^{\tau_k^q} f(\tau, x_{q-1}) d\tau \rangle ds = \int_0^T \langle \dot{x}(s), f(s, x(s)) \rangle ds.$$

Proof. We have

$$\begin{aligned}
& \left\| \sum_{q=1}^{s+1} \int_{\tau_k^{q-1}}^{\tau_k^q} \langle \dot{x}_k(s), \frac{1}{h_{q-1}^k} \int_{\tau_k^{q-1}}^{\tau_k^q} f(\tau, x_{q-1}) d\tau \rangle ds - \int_0^T \langle \dot{x}(s), f(s, x(s)) \rangle ds \right\| \\
& = \left\| \sum_{q=1}^{s+1} \int_{\tau_k^{q-1}}^{\tau_k^q} (\langle \dot{x}_k(s), \frac{1}{h_{q-1}^k} \int_{\tau_k^{q-1}}^{\tau_k^q} f(\tau, x_{q-1}) d\tau \rangle - \langle \dot{x}(s), f(s, x(s)) \rangle) ds \right\| \\
& \leq \left\| \sum_{q=1}^{s+1} \int_{\tau_k^{q-1}}^{\tau_k^q} (\langle \dot{x}_k(s), \frac{1}{h_{q-1}^k} \int_{\tau_k^{q-1}}^{\tau_k^q} f(\tau, x_{q-1}) d\tau \rangle - \langle \dot{x}_k(s), f(s, x(s)) \rangle) ds \right\| \\
& \quad + \left\| \sum_{q=1}^{s+1} \int_{\tau_k^{q-1}}^{\tau_k^q} (\langle \dot{x}_k(s), f(s, x(s)) \rangle - \langle \dot{x}(s), f(s, x(s)) \rangle) ds \right\| \\
& \leq \sum_{q=1}^{s+1} \int_{\tau_k^{q-1}}^{\tau_k^q} \left\| \langle \dot{x}_k(s), \frac{1}{h_{q-1}^k} \int_{\tau_k^{q-1}}^{\tau_k^q} f(\tau, x_{q-1}) d\tau \rangle - \langle \dot{x}_k(s), f(s, x(s)) \rangle \right\| ds \\
& \quad + \left\| \int_0^T (\langle \dot{x}_k(s), f(s, x(s)) \rangle - \langle \dot{x}(s), f(s, x(s)) \rangle) ds \right\|.
\end{aligned}$$

Since

$$\|\dot{x}_k(t)\| \leq \lambda + M + 1, \quad \lim_{k \rightarrow +\infty} \frac{1}{h_{q-1}^k} \int_{\tau_k^{q-1}}^{\tau_k^q} f(s, x_{q-1}) ds = f(t, x(t))$$

and $\dot{x}_k(\cdot)$ converges weakly to $\dot{x}(\cdot)$, the last term converges to 0. This completes the proof of the Claim. \square

Claim 3.7.

$$\lim_{k \rightarrow +\infty} \sum_{q=1}^{s+1} \beta \left\| x_k(\tau_k^q) - x_k(\tau_k^{q-1}) \right\|^2 = 0.$$

Proof. By construction we have

$$\begin{aligned} \|x_k(\tau_k^q) - x_k(\tau_k^{q-1})\| &= \|(\tau_k^q - \tau_k^{q-1})u_{q-1}\| \\ &\leq (\tau_k^q - \tau_k^{q-1}) \|u_{q-1}\| \\ &\leq (\tau_k^q - \tau_k^{q-1})(\lambda + 1 + M). \end{aligned}$$

Hence

$$\begin{aligned} \|x_k(\tau_k^q) - x_k(\tau_k^{q-1})\|^2 &\leq (\tau_k^q - \tau_k^{q-1})^2(\lambda + 1 + M)^2 \\ &\leq (\tau_k^q - \tau_k^{q-1})h_{q-1}^k(\lambda + 1 + M)^2 \\ &\leq (\tau_k^q - \tau_k^{q-1})\frac{1}{k}(\lambda + 1 + M)^2. \end{aligned}$$

Then

$$\sum_{q=1}^{s+1} \beta \|x_k(\tau_k^q) - x_k(\tau_k^{q-1})\|^2 \leq \frac{\beta T(\lambda + 1 + M)^2}{k},$$

hence

$$\lim_{k \rightarrow +\infty} \sum_{q=1}^{s+1} \beta \|x_k(\tau_k^q) - x_k(\tau_k^{q-1})\|^2 = 0.$$

□

Note that

$$\lim_{k \rightarrow +\infty} \frac{1}{kT} \sum_{q=1}^{s+1} \int_{\tau_k^{q-1}}^{\tau_k^q} \langle \dot{x}_k(s), b_q \rangle ds = 0.$$

By passing to the limit for $k \rightarrow \infty$ in (3.7) and using the continuity of the function V on the ball $B(x_0, r)$, we obtain

$$V(x(T)) - V(x_0) \geq \limsup_{k \rightarrow +\infty} \int_0^T \|\dot{x}_k(s)\|^2 ds - \int_0^T \langle \dot{x}(s), f(s, x(s)) \rangle ds.$$

Moreover, by (3.6), we have

$$\|\dot{x}\|_2^2 \geq \limsup_{k \rightarrow +\infty} \|\dot{x}_k\|_2^2$$

and by the weak *l.s.c* of the norm ensures

$$\|\dot{x}\|_2^2 \leq \liminf_{k \rightarrow +\infty} \|\dot{x}_k\|_2^2.$$

Hence we get

$$\|\dot{x}\|_2^2 = \lim_{k \rightarrow +\infty} \|\dot{x}_k\|_2^2.$$

Finally, there exists a subsequence of $(\dot{x}_k(\cdot))_k$ (still denoted $(\dot{x}_k(\cdot))_k$) converges pointwisely to $\dot{x}(\cdot)$. In view of Proposition (3.2), we conclude that

$$d_{grF}((x(t), \dot{x}(t) - f(t, x(t)))) = 0$$

and as F has a closed graph, we obtain

$$\dot{x}(t) \in f(t, x(t)) + F(x(t)) \text{ a.e on } [0, T].$$

Now, let $t \in [0, T]$, there exists $q \in \{1, \dots, s + 1\}$ such that $t \in [\tau_k^{q-1}, \tau_k^q]$ and $\lim_{k \rightarrow +\infty} \tau_k^{q-1} = t$. Since

$$\lim_{k \rightarrow +\infty} \|x(t) - x_k(\tau_k^{q-1})\| = 0,$$

$x_k(\tau_k^{q-1}) \in K_0$ and K_0 is closed we obtain $x(t) \in K_0 \subset K$. The proof is complete.

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