

UNIFORM CONTINUITY OF THE SOLUTION MAP FOR NONLINEAR WAVE EQUATION IN REISSNER-NORDSTRÖM METRIC

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Abstract

In this paper we study the properties of the solutions to the Cauchy problem

$$(1) \quad (u_{tt} - \Delta u)_{g_s} = f(u) + g(|x|), \quad t \in [0, 1], x \in \mathcal{R}^3,$$

$$(2) \quad u(1, x) = u_0 \in \dot{H}^1(\mathcal{R}^3), \quad u_t(1, x) = u_1 \in L^2(\mathcal{R}^3),$$

where g_s is the Reissner-Nordström metric (see [2]); $f \in \mathcal{C}^1(\mathcal{R}^1)$, $f(0) = 0$, $a|u| \leq f'(u) \leq b|u|$, $g \in \mathcal{C}(\mathcal{R}^+)$, $g(|x|) \geq 0$, $g(|x|) = 0$ for $|x| \geq r_1$, a and b are positive constants, $r_1 > 0$ is suitable chosen.

When $g(r) \equiv 0$ we prove that the Cauchy problem (1), (2) has a nontrivial solution $u(t, r)$ in the form $u(t, r) = v(t)\omega(r) \in \mathcal{C}((0, 1]\dot{H}^1(\mathcal{R}^+))$, where $r = |x|$, and the solution map is not uniformly continuous.

When $g(r) \neq 0$ we prove that the Cauchy problem (1), (2) has a nontrivial solution $u(t, r)$ in the form $u(t, r) = v(t)\omega(r) \in \mathcal{C}((0, 1]\dot{H}^1(\mathcal{R}^+))$, where $r = |x|$, and the solution map is not uniformly continuous.

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1. Introduction

In this paper we study the properties of the solutions to the Cauchy problem

$$(1) \quad (u_{tt} - \Delta u)_{g_s} = f(u) + g(|x|), \quad t \in [0, 1], x \in \mathcal{R}^3,$$

$$(2) \quad u(1, x) = u_0 \in \dot{H}^1(\mathcal{R}^3), \quad u_t(1, x) = u_1 \in L^2(\mathcal{R}^3),$$

where g_s is the Reissner-Nordström metric (see [2])

$$g_s = \frac{r^2 - Kr + Q^2}{r^2} dt^2 - \frac{r^2}{r^2 - Kr + Q^2} dr^2 - r^2 d\phi^2 - r^2 \sin^2 \phi d\theta^2,$$

K and Q are positive constants, $f \in \mathcal{C}^1(\mathcal{R}^1)$, $f(0) = 0$, $a|u| \leq f'(u) \leq b|u|$, $g \in \mathcal{C}(\mathcal{R}^+)$, $g(|x|) \geq 0$, $g(|x|) = 0$ for $|x| \geq r_1$, a and b are positive constants, $r_1 > 0$ is suitable chosen.

The Cauchy problem (1), (2) we may rewrite in the form

$$(1) \quad \frac{r^2}{r^2 - Kr + Q^2} u_{tt} - \frac{1}{r^2} \partial_r((r^2 - Kr + Q^2)u_r) - \frac{1}{r^2 \sin \phi} \partial_\phi(\sin \phi u_\phi) - \frac{1}{r^2 \sin^2 \phi} u_{\theta\theta} = f(u) + g(r),$$

$$(2) \quad u(1, r, \phi, \theta) = u_0 \in \dot{H}^1(\mathcal{R}^+ \times [0, 2\pi] \times [0, \pi]), u_t(1, r, \phi, \theta) = u_1 \in L^2(\mathcal{R}^+ \times [0, 2\pi] \times [0, \pi]).$$

When g_s is the Minkowski metric; $u_0, u_1 \in \mathcal{C}_0^\infty(\mathcal{R}^3)$ in [5](see and [1], section 6.3) is proved that there exists $T > 0$ and a unique local solution $u \in \mathcal{C}^2([0, T] \times \mathcal{R}^3)$ for the Cauchy problem

$$(u_{tt} - \Delta u)_{g_s} = f(u), \quad f \in \mathcal{C}^2(\mathcal{R}), \quad t \in [0, T], x \in \mathcal{R}^3,$$

$$u|_{t=0} = u_0, u_t|_{t=0} = u_1,$$

for which

$$\sup_{t < T, x \in \mathcal{R}^3} |u(t, x)| = \infty.$$

When g_s is the Minkowski metric, $1 \leq p < 5$ and initial data are in $\mathcal{C}_0^\infty(\mathcal{R}^3)$, in [5](see and [1], section 6.3) is proved that the initial value problem

$$(u_{tt} - \Delta u)_{g_s} = u|u|^{p-1}, \quad t \in [0, T], x \in \mathcal{R}^3,$$

$$u|_{t=0} = u_0, u_t|_{t=0} = u_1,$$

admits a global smooth solution.

When g_s is the Minkowski metric and initial data are in $\mathcal{C}_0^\infty(\mathcal{R}^3)$, in [4](see and [1], section 6.3) is proved that there exists a number $\epsilon_0 > 0$ such that for any data $(u_0, u_1) \in \mathcal{C}_0^\infty(\mathcal{R}^3)$ with $E(u(0)) < \epsilon_0$, the initial value problem

$$(u_{tt} - \Delta u)_{g_s} = u^5, \quad t \in [0, T], x \in \mathcal{R}^3,$$

$$u|_{t=0} = u_0, u_t|_{t=0} = u_1,$$

admits a global smooth solution.

When g_s is the Minkowski metric in [6] is proved that the Cauchy problem

$$(u_{tt} - \Delta u)_{g_s} = f(u), \quad t \in [0, 1], x \in \mathcal{R}^3,$$

$$u(1, x) = u_0, \quad u_t(1, x) = u_1,$$

has global solution. Here $f \in \mathcal{C}^2(\mathcal{R})$, $f(0) = f'(0) = f''(0) = 0$,

$$|f''(u) - f''(v)| \leq B|u - v|^{q_1}$$

for $|u| \leq 1$, $|v| \leq 1$, $B > 0$, $\sqrt{2} - 1 < q_1 \leq 1$, $u_0 \in C^5_0(\mathcal{R}^3)$, $u_1 \in C^4_0(\mathcal{R}^3)$, $u_0(x) = u_1(x) = 0$ for $|x - x_0| > \rho$, x_0 and ρ are suitable chosen.

When g_s is the Reissner - Nordström metric, $n = 3$, $p > 1$, $q \geq 1$, $\gamma \in (0, 1)$ are fixed constants, $f \in C^1(\mathcal{R}^1)$, $f(0) = 0$, $a|u| \leq f'(u) \leq b|u|$, $g \in C(\mathcal{R}^+)$, $g(|x|) \geq 0$, $g(|x|) = 0$ for $|x| \geq r_1$, a and b are positive constants, $r_1 > 0$ is suitable chosen, in [7] is proved that the initial value problem (1), (2), has nontrivial solution $u \in C((0, 1] \dot{B}^{\gamma}_{p,q}(\mathcal{R}^+))$ in the form

$$u(t, r) = \begin{cases} v(t)\omega(r) & \text{for } r \leq r_1, \quad t \in [0, 1], \\ 0 & \text{for } r \geq r_1, \quad t \in [0, 1], \end{cases}$$

where $r = |x|$, for which $\lim_{t \rightarrow 0} \|u\|_{\dot{B}^{\gamma}_{p,q}(\mathcal{R}^+)} = \infty$.

In this paper we will prove that the Cauchy problem (1), (2) has nontrivial solution $u = u(t, r) \in C((0, 1] \dot{H}^1(\mathcal{R}^+))$ and the solution map is not uniformly continuous. When we say that the solution map $(u_0, u_1, g) \rightarrow u(t, r)$ is uniformly continuous we understand: *for every positive constant ϵ there exist positive constants δ and R such that for any two solutions u, v of the Cauchy problem (1), (2), with right hands $g = g_1, g = g_2$ of (1), so that*

$$(2') \quad E(1, u - v) \leq \delta, \quad \|g_1\|_{L^2(\mathcal{R}^+)} \leq R, \quad \|g_2\|_{L^2(\mathcal{R}^+)} \leq R, \quad \|g_1 - g_2\|_{L^2(\mathcal{R}^+)} \leq \delta,$$

the following inequality holds

$$(2'') \quad E(t, u - v) \leq \epsilon \quad \text{for } \forall t \in [0, 1],$$

where

$$E(t, u) := \|\partial_t u(t, \cdot)\|_{L^2(\mathcal{R}^+)}^2 + \left\| \frac{\partial}{\partial r} u(t, \cdot) \right\|_{L^2(\mathcal{R}^+)}^2.$$

Our main results are

Theorem 1.1. *Let K, Q are positive constants for which*

$$\begin{cases} K^2 > 4Q^2, & \frac{1}{1-K+Q^2} \geq 1, \\ 1 - K + Q^2 > 0 \text{ is enough small such that } & \frac{K - \sqrt{K^2 - 4Q^2}}{2} - 2\sqrt{1 - K + Q^2} > 0. \end{cases}$$

Let also $g \equiv 0$, $f \in C^1(\mathcal{R}^1)$, $f(0) = 0$, $a|u| \leq f'(u) \leq b|u|$, a and b are positive constants. Then the homogeneous Cauchy problem (1), (2) has nontrivial solution $u(t, r) = v(t)\omega(r) \in C((0, 1] \dot{H}^1(\mathcal{R}^+))$. Also there exists $t_0 \in [0, 1)$ for which exists constant $\epsilon > 0$ such that for every positive constant δ exist solutions u, v of (1), (2), so that

$$E(1, u - v) \leq \delta,$$

and

$$E(t_0, u - v) \geq \epsilon.$$

Theorem 1.2. *Let K, Q are positive constants for which*

$$\begin{cases} K^2 > 4Q^2, & \frac{1}{1-K+Q^2} \geq 1, \\ 1 - K + Q^2 > 0 \text{ is enough small such that } & \frac{K - \sqrt{K^2 - 4Q^2}}{2} - 2\sqrt{1 - K + Q^2} > 0. \end{cases}$$

Let also $g \neq 0, g \in \mathcal{C}(\mathcal{R}^+), g(r) \geq 0$ for $r \geq 0, g(r) = 0$ for $r \geq r_1, f \in \mathcal{C}^1(\mathcal{R}^1), f(0) = 0, a|u| \leq f'(u) \leq b|u|, a$ and b are positive constants. Then the nonhomogeneous Cauchy problem (1), (2) has nontrivial solution $u(t, r) = v(t)\omega(r) \in \mathcal{C}((0, 1] \dot{H}^1(\mathcal{R}^+))$. Also there exists $t_0 \in [0, 1)$ for which exists constant $\epsilon > 0$ such that for every pair positive constants δ and R exist solutions u, v of (1), (2), with right hands $g = g_1, g = g_2$ of (1), so that

$$E(1, u - v) \leq \delta, \quad \|g_1\|_{L^2(\mathcal{R}^+)} \leq R, \quad \|g_2\|_{L^2(\mathcal{R}^+)} \leq R, \quad \|g_1 - g_2\|_{L^2(\mathcal{R}^+)} \leq \delta,$$

and

$$E(t_0, u - v) \geq \epsilon.$$

The paper is organized as follows. In section 2 we prove theorem 1.1. In section 3 we prove theorem 1.2.

2. Proof of theorem 1.1.

For fixed $q \geq 1$ and $\gamma \in (0, 1)$ we put

$$C = \left(\frac{q\gamma 2^{q\gamma}}{2^{q\gamma} - 1} \right)^{\frac{1}{q}}$$

For fixed $p > 1, q \geq 1, \gamma \in (0, 1)$ and $g \in \mathcal{C}(\mathcal{R}^+), g(r) \geq 0$ for $r \geq 0$ we suppose that

the constants $A > 0$, $Q > 0$, $a > 0$, $b > 0$, $B > 0$, $K > 0$, $1 < \beta < \alpha$ satisfy the conditions

$$\begin{aligned}
 i1) & \frac{1}{1-K+Q^2} \left(\frac{1}{1-K+Q^2} \frac{2a}{A^2} + \frac{b}{AB} \right) \leq 1, \quad A > 1, \quad \frac{A^2}{a} > 1; \\
 i2) & \begin{cases} \frac{1}{\alpha^2(1-\alpha K+\alpha^2 Q^2)} \frac{a}{2A^4} - \frac{br_1^2}{AB} \geq 0, \\ \frac{a}{2A^6 \alpha^2 (1-\alpha K+\alpha^2 Q^2)} - \frac{2br_1^2}{A^2 B^2} \geq 0, \\ \frac{a}{2A^4(1-\alpha K+\alpha^2 Q^2)} \left(\frac{1}{\beta} - \frac{1}{\alpha} \right) - \frac{4ar_1}{A^3 B(1-K+Q^2)^2} \geq 0, \\ \left(\frac{1}{\beta} - \frac{1}{\alpha} \right)^2 \frac{1}{(1-\alpha K+\alpha^2 Q^2)^2} \frac{a}{4A^6} - r_1^2 \frac{2b}{(1-K+Q^2)A^2 B^2} - r_1^2 \frac{1}{1-K+Q^2} \max_{r \in [0, r_1]} g(r) \geq \frac{1}{A^2}, \\ \left(\frac{1}{\beta} - \frac{1}{\alpha} \right)^2 \frac{1}{(1-\alpha K+\alpha^2 Q^2)^2} \frac{a}{2A^4} - \frac{2ar_1^4}{AB(1-K+Q^2)} > 0, \end{cases} \\
 i3) & C \left(\frac{1}{(1-K+Q^2)^2} \frac{2a}{A^2} + \frac{2b}{AB(1-K+Q^2)} + \frac{1}{A^2} \right) \frac{2^{2-\gamma}}{(q(1-\gamma))^{\frac{1}{q}}} + C \frac{2^{1-\gamma}}{A^2(1-K+Q^2)^2(q(1-\gamma))^{\frac{1}{q}}} < 1, \\
 i4) & \begin{cases} \left(\frac{1}{\alpha^2} \frac{1}{1-\alpha K+\alpha^2 Q^2} \frac{a}{4A^6} - \frac{1}{\beta^2} \frac{a}{4A^4} \right) > 0, \\ \left(\frac{1}{\alpha^2} \frac{1}{1-\alpha K+\alpha^2 Q^2} \frac{a}{2A^6} - \frac{1}{\beta^2} \frac{a}{2A^4} \right) > 0, \\ K^2 > 4Q^2, \quad A \geq \frac{1}{1-K+Q^2} > 1, \\ 1 > \frac{2Q^2}{K} > \frac{K-\sqrt{K^2-4Q^2}}{2}, \\ 1-K+Q^2 > 0 \text{ is enough small such that} \\ 1 > \frac{K-\sqrt{K^2-4Q^2}}{2} - 2\sqrt{1-K+Q^2} > 0, \\ \frac{K-\sqrt{K^2-4Q^2}-2(1-K+Q^2)}{2} < \beta < \alpha \leq 3, \\ \frac{a}{4A^6 \alpha^2 (1-\alpha K+\alpha^2 Q^2)} - \frac{2br_1^2}{A^2 B^2} - r_1^2 \max_{r \in [0, r_1]} g(r) \geq 0, \end{cases} \\
 i5) & \frac{1}{1-K+Q^2} \left(\frac{2}{AB} \left(\frac{1}{1-K+Q^2} \frac{2a}{A^2} + \frac{b}{AB} \right) + r_1^2 \max_{s \in [0, r_1]} g(s) \right) \leq \frac{2}{AB}; \\
 i7) & \max_{s \in [0, r_1]} g(s) \leq \left(\frac{1}{\beta} - \frac{1}{\alpha} \right)^2 \frac{1}{A^4},
 \end{aligned}$$

where

$$r_1 = \frac{K - \sqrt{K^2 - 4Q^2}}{2} - \sqrt{1 - K + Q^2}.$$

Example. Let $0 < \epsilon \ll \frac{1}{3}$ is enough small,

$$\begin{aligned}
 A &= \frac{1}{\epsilon^4}, \quad B = \frac{1}{\epsilon}, \quad p = \frac{3}{2}, \quad q = \frac{3}{2}, \quad \gamma = \frac{1}{3}, \quad \alpha = 3, \quad \frac{1}{\beta} = \frac{K - \sqrt{K^2 - 4Q^2}}{2} - 2\sqrt{1 - K + Q^2}, \\
 g(r) &= \begin{cases} \epsilon^{11}(r - r_1)^2 & \text{for } r \leq r_1, \\ 0 & \text{for } r \geq r_1, \end{cases} \\
 K &= \frac{4}{3} + \frac{1}{6}\epsilon^{20} - \frac{3}{2}\epsilon^2, \\
 Q^2 &= \frac{1}{3} + \frac{1}{6}\epsilon^{20} - \frac{1}{2}\epsilon^2, \\
 a &= \epsilon^4, \quad b = \epsilon^3.
 \end{aligned}$$

Then

$$\begin{aligned}
 1 - \alpha K + \alpha^2 Q^2 &= 1 - 3K + 9Q^2 = \epsilon^{20}, \\
 1 - K + Q^2 &= \epsilon^2 \bullet.
 \end{aligned}$$

When $g(r) \equiv 0$ we put

$$(1') \quad u_0 := v(1)\omega(r) =$$

$$= \begin{cases} \int_r^{r_1} \frac{1}{\tau^2 - K\tau + Q^2} \int_\tau^{r_1} \left(\frac{s^4}{s^2 - Ks + Q^2} v''(1)\omega(s) - s^2 f(v(1)\omega(s)) \right) ds d\tau & \text{for } r \leq r_1, \\ 0 & \text{for } r \geq r_1, \end{cases}$$

and $u_1 \equiv 0$. Here $v(t)$ is fixed function which satisfies the conditions

$$\begin{aligned} (H1) \quad & v(t) \in \mathcal{C}^3[0, \infty), \quad v(t) > 0 \quad \text{for } \forall t \in [0, 1]; \\ (H2) \quad & v''(t) > 0 \quad \text{for } \forall t \in [0, 1], \quad v'(1) = v'''(1) = 0, \quad v(1) \neq 0; \\ (H3) \quad & \begin{cases} \min_{t \in [0, 1]} \frac{v''(t)}{v(t)} \geq \frac{a}{2A^4}, & \max_{t \in [0, 1]} \frac{v''(t)}{v(t)} \leq \frac{2a}{A^2}; \\ v''(t) - \frac{a}{2A^4}v(t) \geq 0 & \text{for } t \in [0, 1]. \end{cases} \end{aligned}$$

Bellow we will prove that the equation (1') has unique nontrivial solution $\omega(r)$ for which $\omega(r) \in \mathcal{C}^2[0, r_1]$, $\omega(r) \in \dot{H}^1[0, r_1]$, $|\omega(r)| \leq \frac{2}{AB}$ for $r \in [0, r_1]$, $\omega(r) \geq \frac{1}{A^2}$ for $r \in \left[\frac{1}{\alpha}, \frac{1}{\beta}\right]$, $\omega(r_1) = \omega'(r_1) = \omega''(r_1) = 0$.

Example.

There exists function $v(t)$ for which (H1)-(H3) are hold. Really, let us consider the function

$$(3) \quad v(t) = \frac{(t-1)^2 + \frac{4A^4}{a} - 1}{\frac{A^3}{a}},$$

where the constants A and a satisfy the conditions $A > 1$, $\frac{A^2}{a} > 1$. Then

- 1) $v(t) \in \mathcal{C}^3[0, \infty)$ and $v(t) > 0$ for all $t \in [0, 1]$, i.e. (H1) is hold.
- 2)

$$\begin{aligned} v'(t) &= \frac{2(t-1)}{\frac{A^3}{a}}, \quad v'(1) = 0, \\ v''(t) &= \frac{2}{\frac{A^3}{a}} \geq 0 \quad \forall t \in [0, 1], \\ v'''(t) &= 0, \quad v'''(1) = 0, \end{aligned}$$

consequently (H2) is hold. On the other hand we have

$$\frac{v''(t)}{v(t)} = \frac{2}{(t-1)^2 + \frac{4A^4}{a} - 1}.$$

From here

$$\begin{aligned} \min_{t \in [0, 1]} \frac{v''(t)}{v(t)} &\geq \frac{a}{2A^4}, \\ \max_{t \in [0, 1]} \frac{v''(t)}{v(t)} &\leq \frac{2a}{A^2}, \\ v''(t) - \frac{a}{2A^4}v(t) &= \frac{1}{\frac{2A^7}{a^2}}(2-t)t, \end{aligned}$$

i.e. (H3) is hold.

2.1. Local existence of nontrivial solutions of homogeneous Cauchy problem (1), (2)

Let $v(t)$ is fixed function which satisfies the hypothesis (H1) – (H3).

In this section we will prove that the homogeneous Cauchy problem (1), (2) has non-trivial solution in the form

$$u(t, r) = \begin{cases} v(t)\omega(r) & \text{for } r \leq r_1, \quad t \in [0, 1], \\ 0 & \text{for } r \geq r_1, \quad t \in [0, 1]. \end{cases}$$

Let us consider the integral equation

$$(\star) \quad u(t, r) = \begin{cases} \int_r^{r_1} \frac{1}{\tau^2 - K\tau + Q^2} \int_\tau^{r_1} \left(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} u(t, s) - s^2 f(u(t, s)) \right) ds d\tau, & 0 \leq r \leq r_1, \quad t \in [0, 1], \\ 0 & \text{for } r \geq r_1, \quad t \in [0, 1], \end{cases}$$

where $u(t, r) = v(t)\omega(r)$.

Theorem 2.1. *Let $v(t)$ is fixed function which satisfies the hypothesis (H1)-(H3). Let also $p > 1$, $q \in [1, \infty)$ and $\gamma \in (0, 1)$ are fixed and the positive constants $A, a, b, B, Q, K, \alpha > \beta > 1$ satisfy the conditions i1)-i6) and $f \in C^1(\mathcal{R}^1)$, $f(0) = 0$, $a|u| \leq f'(u) \leq b|u|$. Then the equation (\star) has unique nontrivial solution $u(t, r) = v(t)\omega(r)$ for which $w \in C^2[0, r_1]$, $u(t, r) = u_r(t, r) = u_{rr}(t, r) = 0$ for $r \geq r_1$, $u(t, r) \in C((0, 1] \dot{H}^1[0, r_1])$, for $r \in [\frac{1}{\alpha}, \frac{1}{\beta}]$ and $t \in [0, 1]$ $u(t, r) \geq \frac{1}{A^2}$, for $r \in [0, r_1]$ and $t \in [0, 1]$ $|u(t, r)| \leq \frac{2}{AB}$.*

Proof. In [7, p. 303-309, theorem 3.1] is proved that the equation (\star) has solution $u(t, r)$ in the form $u(t, r) = v(t)\omega(r)$ for which

$$\begin{aligned} & u(t, r) \in C([0, 1] \times [0, r_1]); \\ & u(t, r) = u_r(t, r) = u_{rr}(t, r) = 0 \quad \text{for } r \geq r_1 \quad \text{and } t \in [0, 1], \\ & u(t, r) \in C((0, 1] \dot{B}_{p,q}^\gamma[0, r_1]); \\ & \text{for } r \in [\frac{1}{\alpha}, \frac{1}{\beta}] \quad \text{and } t \in [0, 1] \quad u(t, r) \geq \frac{1}{A^2}; \\ & u(t, r) \geq 0 \quad \text{for } t \in [0, 1] \quad \text{and } r \in [\frac{1}{\alpha}, r_1]; \\ & \text{for } r \in [0, r_1] \quad \text{and } t \in [0, 1] \quad |u(t, r)| \leq \frac{2}{AB}. \end{aligned}$$

In [7] is used the following definition of the $\dot{B}_{p,q}^\gamma(M)$ -norm ($\gamma \in (0, 1)$, $p > 1$, $q \geq 1$) (see [3, p.94, def. 2], [1])

$$\|u\|_{\dot{B}_{p,q}^\gamma(M)} = \left(\int_0^2 h^{-1-q\gamma} \|\Delta_h u\|_{L^p(M)}^q dh \right)^{\frac{1}{q}},$$

where

$$\Delta_h u = u(x+h) - u(x).$$

Let $t \in [0, 1]$ is fixed. Then

$$\begin{aligned} & \left\| \frac{\partial}{\partial r} u \right\|_{L^2([0, \infty))}^2 = \\ & = \int_0^{r_1} \left(\frac{1}{r^2 - Kr + Q^2} \int_r^{r_1} \left(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} u(t, s) - s^2 f(u(t, s)) \right) ds d\tau \right)^2 dr \leq \end{aligned}$$

here we use that from i5) we have that $r_1 < 1$, $\frac{s^4}{s^2 - Ks + Q^2} \leq \frac{1}{1 - K + Q^2}$ for $s \in [0, r_1]$, $\frac{1}{r^2 - Kr + Q^2} \leq \frac{1}{1 - K + Q^2}$ for $r \in [0, r_1]$ (see [7, Remark, p.300])

$$\leq \int_0^{r_1} \left(\frac{1}{1 - K + Q^2} \int_r^{r_1} \left(\frac{1}{1 - K + Q^2} \max_{t \in [0, 1]} \frac{v''(t)}{v(t)} |u(t, s)| + s^2 |f(u(t, s))| \right) ds d\tau \right)^2 dr \leq$$

here we use that $f(0) = 0$, $|f(u)| \leq \frac{b}{2}|u|^2$,

$$\leq \int_0^{r_1} \left(\frac{1}{1 - K + Q^2} \int_r^{r_1} \left(\frac{1}{1 - K + Q^2} \max_{t \in [0, 1]} \frac{v''(t)}{v(t)} |u(t, s)| + s^2 \frac{b}{2} |u(t, s)|^2 \right) ds d\tau \right)^2 dr \leq$$

here we use that $|u(t, r)| \leq \frac{2}{AB}$ for $r \in [0, r_1]$, $t \in [0, 1]$,

$$\leq \int_0^{r_1} \left(\frac{1}{1 - K + Q^2} \int_r^{r_1} \left(\frac{1}{1 - K + Q^2} \max_{t \in [0, 1]} \frac{v''(t)}{v(t)} \frac{2}{AB} + r_1^2 \frac{b}{2} \frac{4}{A^2 B^2} \right) ds d\tau \right)^2 dr \leq$$

here we use that $\max_{t \in [0, 1]} \frac{v''(t)}{v(t)} \leq \frac{2a}{A^2}$,

$$\begin{aligned} &\leq \int_0^{r_1} \left(\frac{1}{1 - K + Q^2} \int_r^{r_1} \left(\frac{1}{1 - K + Q^2} \frac{2a}{A^2} \frac{2}{AB} + r_1^2 \frac{b}{2} \frac{4}{A^2 B^2} \right) ds d\tau \right)^2 dr \leq \\ &\leq r_1^3 \left(\frac{1}{1 - K + Q^2} \left(\frac{1}{1 - K + Q^2} \frac{4a}{A^3 B} + \frac{2r_1^2 b}{A^2 B^2} \right) \right)^2 < \infty. \end{aligned}$$

From here

$$\left\| \frac{\partial}{\partial r} u \right\|_{L^2([0, \infty))} < \infty$$

for every fixed $t \in (0, 1]$. Therefore $u(t, r) \in \mathcal{C}((0, 1] \dot{H}^1([0, \infty)))$. •

Let \tilde{u} is the solution from the theorem 2. 1, i.e \tilde{u} is the solution to the equation (★). From proposition 2.1([7]) \tilde{u} satisfies the equation (1). Then \tilde{u} is solution to the Cauchy problem (1), (2) with initial data

$$u_0 = \begin{cases} \int_r^{r_1} \frac{1}{\tau^2 - K\tau + Q^2} \int_\tau^{r_1} \left(\frac{s^4}{s^2 - Ks + Q^2} v''(1)\omega(s) - s^2 f(v(1)\omega(s)) \right) ds d\tau & \text{for } r \leq r_1, \\ 0 & \text{for } r \geq r_1, \end{cases}$$

$$u_1 = \begin{cases} \int_r^{r_1} \frac{1}{\tau^2 - K\tau + Q^2} \int_\tau^{r_1} \left(\frac{s^4}{s^2 - Ks + Q^2} v'''(1)\omega(s) - s^2 f'(u)v'(1)\omega(s) \right) ds d\tau = 0 & \text{for } r \leq r_1, \\ 0 & \text{for } r \geq r_1, \end{cases}$$

$$u_0 \in \dot{H}^1(\mathcal{R}^+), u_1 \in L^2(\mathcal{R}^+), \tilde{u} \in \mathcal{C}((0, 1] \dot{H}^1[0, r_1]).$$

2.2. Uniformly continuity of the solution map for the homogeneous Cauchy problem (1), (2)

Let $v(t)$ is same function as in Theorem 2.1.

Theorem 2.2. *Let $p > 1$, $q \geq 1$ and $\gamma \in (0, 1)$ are fixed and the positive constants $a, b, A, B, Q, K, 1 < \beta < \alpha$ satisfy the conditions i1)-i6). Let $f \in \mathcal{C}^1(\mathcal{R}^1)$, $f(0) = 0$,*

$a|u| \leq f'(u) \leq b|u|$. Then there exists $t_o \in [0, 1]$ for which there exists constant $\epsilon > 0$ such that for every positive constant δ exist solutions u_1 and u_2 so that

$$E(1, u_1 - u_2) \leq \delta$$

and

$$E(t_o, u_1 - u_2) \geq \epsilon.$$

Proof. Let us suppose that the solution map $(u_o, u_1, g) \longrightarrow u(t, r)$ is uniformly continuous.

Let

$$(4) \quad 0 < \epsilon < \left(\frac{1}{\beta} - \frac{1}{\alpha}\right)^3 \left(\frac{\alpha^2}{1 - \alpha K + \alpha^2 Q^2} \left(\frac{a}{2A^6 \alpha^2 (1 - \alpha L + \alpha^2 Q^2)} - \frac{2b}{\beta^2 A^2 B^2}\right)\right)^2.$$

Let also

$$u_1 = \tilde{u}, \quad u_2 = 0.$$

Then there exists positive constant δ such that

$$E(1, u_1 - u_2) \leq \delta,$$

and

$$E(t, u_1 - u_2) \leq \epsilon \quad \text{for } \forall t \in [0, 1].$$

From here, for $t \in [0, 1]$

$$\begin{aligned} \epsilon &\geq \left\| \frac{\partial}{\partial r} \tilde{u} \right\|_{L^2([0, \infty))}^2 = \\ &= \int_0^{r_1} \left(\frac{1}{r^2 - Kr + Q^2} \int_r^{r_1} \left(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} \tilde{u}(t, s) - s^2 f(\tilde{u}(t, s)) \right) ds d\tau \right)^2 dr \geq \\ &\geq \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \left(\frac{1}{r^2 - Kr + Q^2} \int_r^{r_1} \left(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} \tilde{u}(t, s) - s^2 f(\tilde{u}(t, s)) \right) ds d\tau \right)^2 dr \geq \end{aligned}$$

here we use that for $s \in \left[\frac{1}{\alpha}, r_1\right]$ and for $t \in [0, 1]$ we have that $\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} \tilde{u}(t, s) - s^2 f(\tilde{u}(t, s)) \geq 0$ (see [7, p. 305-306]) and $\frac{1}{r^2 - Kr + Q^2} \geq 0$ for $r \in [0, r_1]$,

$$\begin{aligned} &\geq \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \left(\frac{1}{r^2 - Kr + Q^2} \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \left(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} \tilde{u}(t, s) - s^2 f(\tilde{u}(t, s)) \right) ds d\tau \right)^2 dr \geq \\ &\geq \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \left(\frac{1}{r^2 - Kr + Q^2} \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \left(\frac{s^4}{s^2 - Ks + Q^2} \min_{t \in [0, 1]} \frac{v''(t)}{v(t)} \tilde{u}(t, s) - s^2 f(\tilde{u}(t, s)) \right) ds d\tau \right)^2 dr \geq \end{aligned}$$

from (H3) we have that $\min_{t \in [0, 1]} \frac{v''(t)}{v(t)} \geq \frac{a}{2A^4}$

$$\geq \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \left(\frac{1}{r^2 - Kr + Q^2} \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \left(\frac{s^4}{s^2 - Ks + Q^2} \frac{a}{2A^4} \tilde{u}(t, s) - s^2 f(\tilde{u}(t, s)) \right) ds d\tau \right)^2 dr \geq$$

here we use that $f(0) = 0$, $f(\tilde{u}) \leq \frac{b}{2}\tilde{u}^2$

$$\geq \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \left(\frac{1}{r^2 - Kr + Q^2} \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \left(\frac{s^4}{s^2 - Ks + Q^2} \frac{a}{2A^4} \tilde{u}(t, s) - s^2 \frac{b}{2} \tilde{u}^2(t, s) \right) ds d\tau \right)^2 dr \geq$$

here we use that $\tilde{u}(t, s) \geq \frac{1}{A^2}$ for $t \in [0, 1]$ and $s \in \left[\frac{1}{\alpha}, \frac{1}{\beta}\right]$, $\tilde{u}^2(t, s) \leq \frac{4}{A^2 B^2}$ for $t \in [0, 1]$ and $s \in [0, r_1]$

$$\geq \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \left(\frac{1}{r^2 - Kr + Q^2} \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \left(\frac{s^4}{s^2 - Ks + Q^2} \frac{a}{2A^4} \frac{1}{A^2} - \frac{1}{\beta^2} \frac{b}{2} \frac{4}{A^2 B^2} \right) ds d\tau \right)^2 dr \geq$$

here we use that $\frac{s^2}{s^2 - Ks + Q^2}$ is increase function for $s \in [0, r_1]$. Therefore, for $s \in \left[\frac{1}{\alpha}, \frac{1}{\beta}\right]$ we have $\frac{s^4}{s^2 - Ks + Q^2} \geq \frac{1}{\alpha^2(1 - \alpha K + \alpha^2 Q^2)}$

$$\geq \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \left(\frac{1}{r^2 - Kr + Q^2} \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \left(\frac{1}{\alpha^2(1 - \alpha K + \alpha^2 Q^2)} \frac{a}{2A^6} - \frac{2b}{\beta^2 A^2 B^2} \right) ds d\tau \right)^2 dr \geq$$

here we use that for $r \in [0, r_1]$ the function $\frac{1}{r^2 - Kr + Q^2}$ is increase function. Therefore for $r \in \left[\frac{1}{\alpha}, \frac{1}{\beta}\right]$ we have $\frac{1}{r^2 - Kr + Q^2} \geq \frac{\alpha^2}{1 - \alpha K + \alpha^2 Q^2}$

$$\geq \left(\frac{1}{\beta} - \frac{1}{\alpha} \right)^3 \left(\frac{\alpha^2}{1 - \alpha K + \alpha^2 Q^2} \left(\frac{a}{2A^6 \alpha^2 (1 - \alpha K + \alpha^2 Q^2)} - \frac{2b}{\beta^2 A^2 B^2} \right) \right)^2$$

which is contradiction with (4). •

3. Proof of Theorem 1.2.

3.1. Local existence of nontrivial solutions for nonhomogenous Cauchy problem (1), (2)

Let $v(t)$ is fixed function which satisfies the conditions (H1), (H2) and (H4), where

$$(H4) \quad \min_{t \in [0, 1]} \frac{v''(t)}{v(t)} \geq \frac{a}{4A^4}, \quad \max_{t \in [0, 1]} \frac{v''(t)}{v(t)} \leq \frac{2a}{A^2}.$$

For instance, the function

$$v(t) = \frac{(t-1)^2 + \frac{8A^4}{a} - 1}{\frac{A^3}{a}},$$

satisfies the hypothesis (H1), (H2) and (H4).

Let us consider the equation

$$(\star') \quad u(t, r) = \begin{cases} \int_r^{r_1} \frac{1}{\tau^2 - K\tau + Q^2} \int_{\tau}^{r_1} \left(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} u(t, s) - s^2 f(u(t, s)) - s^2 g(s) \right) ds d\tau, & 0 \leq r \leq r_1, \\ 0 & \text{for } r \geq r_1, \end{cases}$$

$t \in [0, 1]$, where $u(t, r) = v(t)\omega(r)$.

Theorem 3.1. *Let $v(t)$ is fixed function which satisfies the hypothesis (H1), (H2), (H4). Let also $p > 1$, $q \in [1, \infty)$ and $\gamma \in (0, 1)$ are fixed and the positive constants $A, a, b, B, Q, K, \alpha > \beta > 1$ satisfy the conditions i1)-i7) and $f \in C^1(\mathcal{R}^1)$, $f(0) = 0$, $a|u| \leq f'(u) \leq b|u|$, $g \in \mathcal{C}(\mathcal{R}^+)$, $g(r) \geq 0$ for $\forall r \in \mathcal{R}^+$, $g(r) = 0$ for $r \geq r_1$. Then the equation (\star') has unique nontrivial solution $u(t, r) = v(t)\omega(r)$ for which $w \in C^2[0, r_1]$, $u(t, r) = u_r(t, r) = u_{rr}(t, r) = 0$ for $r \geq r_1$, $u(t, r) \in C((0, 1]\dot{H}^1[0, r_1])$, for $r \in [\frac{1}{\alpha}, \frac{1}{\beta}]$ and $t \in [0, 1]$ $u(t, r) \geq \frac{1}{A^2}$, for $r \in [0, r_1]$ and $t \in [0, 1]$ $|u(t, r)| \leq \frac{2}{AB}$.*

Proof. In [7, p. 313-316, theorem 4.1] is proved that the equation (\star) has solution $u(t, r)$ in the form $u(t, r) = v(t)\omega(r)$ for which

$$\begin{aligned} & u(t, r) \in \mathcal{C}([0, 1] \times [0, r_1]); \\ & u(t, r) = u_r(t, r) = u_{rr}(t, r) = 0 \quad \text{for } r \geq r_1 \quad \text{and } t \in [0, 1], \\ & u(t, r) \in \mathcal{C}((0, 1]\dot{B}_{p,q}^\gamma[0, r_1]); \\ & \text{for } r \in [\frac{1}{\alpha}, \frac{1}{\beta}] \quad \text{and } t \in [0, 1] \quad u(t, r) \geq \frac{1}{A^2}; \\ & u(t, r) \geq 0 \quad \text{for } t \in [0, 1] \quad \text{and } r \in [\frac{1}{\alpha}, r_1]; \\ & \text{for } r \in [0, r_1] \quad \text{and } t \in [0, 1] \quad |u(t, r)| \leq \frac{2}{AB}. \end{aligned}$$

Let $t \in [0, 1]$ is fixed. Then

$$\begin{aligned} & \left\| \frac{\partial}{\partial r} u \right\|_{L^2([0, \infty))}^2 = \\ & = \int_0^{r_1} \left(\frac{1}{r^2 - Kr + Q^2} \int_r^{r_1} \left(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} u(t, s) - s^2 (f(u(t, s)) + g(s)) \right) ds d\tau \right)^2 dr \leq \end{aligned}$$

here we use that from i5) we have that $r_1 < 1$, $\frac{s^4}{s^2 - Ks + Q^2} \leq \frac{1}{1 - K + Q^2}$ for $s \in [0, r_1]$, $\frac{1}{r^2 - Kr + Q^2} \leq \frac{1}{1 - K + Q^2}$ for $r \in [0, r_1]$ (see [7, Remark, p.300])

$$\leq \int_0^{r_1} \left(\frac{1}{1 - K + Q^2} \int_r^{r_1} \left(\frac{1}{1 - K + Q^2} \max_{t \in [0, 1]} \frac{v''(t)}{v(t)} |u(t, s)| + s^2 (|f(u(t, s))| + g(s)) \right) ds d\tau \right)^2 dr \leq$$

here we use that $f(0) = 0$, $|f(u)| \leq \frac{b}{2}|u|^2$,

$$\leq \int_0^{r_1} \left(\frac{1}{1 - K + Q^2} \int_r^{r_1} \left(\frac{1}{1 - K + Q^2} \max_{t \in [0, 1]} \frac{v''(t)}{v(t)} |u(t, s)| + s^2 \left(\frac{b}{2} |u(t, s)|^2 + g(s) \right) \right) ds d\tau \right)^2 dr \leq$$

here we use that $|u(t, r)| \leq \frac{2}{AB}$ for $r \in [0, r_1]$, $t \in [0, 1]$, from i7) we have $\max_{r \in [0, r_1]} g(r) \leq \left(\frac{1}{\beta} - \frac{1}{\alpha} \right)^2 \frac{1}{A^4}$

$$\leq \int_0^{r_1} \left(\frac{1}{1 - K + Q^2} \int_r^{r_1} \left(\frac{1}{1 - K + Q^2} \max_{t \in [0, 1]} \frac{v''(t)}{v(t)} \frac{2}{AB} + r_1^2 \frac{b}{2} \frac{4}{A^2 B^2} + r_1^2 \left(\frac{1}{\beta} - \frac{1}{\alpha} \right)^2 \frac{1}{A^4} \right) ds d\tau \right)^2 dr \leq$$

here we use that $\max_{t \in [0,1]} \frac{v''(t)}{v(t)} \leq \frac{2a}{A^2}$,

$$\begin{aligned} &\leq \int_0^{r_1} \left(\frac{1}{1-K+Q^2} \int_r^{r_1} \left(\frac{1}{1-K+Q^2} \frac{2a}{A^2} \frac{2}{AB} + r_1^2 \frac{b}{2A^2B^2} + r_1^2 \left(\frac{1}{\beta} - \frac{1}{\alpha} \right)^2 \frac{1}{A^4} \right) dsd\tau \right)^2 dr \leq \\ &\leq r_1^3 \left(\frac{1}{1-K+Q^2} \left(\frac{1}{1-K+Q^2} \frac{4a}{A^3B} + \frac{2r_1^2b}{A^2B^2} + r_1^2 \left(\frac{1}{\beta} - \frac{1}{\alpha} \right)^2 \frac{1}{A^4} \right) \right)^2 < \infty. \end{aligned}$$

From here

$$\left\| \frac{\partial}{\partial r} u \right\|_{L^2([0,\infty))} < \infty$$

for every fixed $t \in (0, 1]$. Therefore $u(t, r) \in \mathcal{C}((0, 1] \dot{H}^1([0, \infty)))$. •

Let \bar{u} is the solution from the theorem 3. 1. , i.e \bar{u} is the solution to the equation (\star). From proposition 2.3[7] we have that \bar{u} satisfies the equation (1). Then \bar{u} is solution to the Cauchy problem (1), (2) with initial data

$$\bar{u}_0 = \begin{cases} \int_r^{r_1} \frac{1}{\tau^2-K\tau+Q^2} \int_\tau^{r_1} \left(\frac{s^4}{s^2-Ks+Q^2} v''(1)\omega(s) - s^2 f(v(1)\omega(s)) - s^2 g(s) \right) dsd\tau & \text{for } r \leq r_1, \\ 0 & \text{for } r \geq r_1, \end{cases}$$

$$\bar{u}_1 = \begin{cases} \int_r^{r_1} \frac{1}{\tau^2-K\tau+Q^2} \int_\tau^{r_1} \left(\frac{s^4}{s^2-Ks+Q^2} v'''(1)\omega(s) - s^2 f'(u)v'(1)\omega(s) \right) dsd\tau \equiv 0 & \text{for } r \leq r_1, \\ 0 & \text{for } r \geq r_1, \end{cases}$$

$$\bar{u}_0 \in \dot{H}^1(\mathcal{R}^+), \bar{u}_1 \in L^2(\mathcal{R}^+), \bar{u} \in \mathcal{C}((0, 1] \dot{H}^1[0, r_1]).$$

3.2. Uniformly continuity of the solution map for the nonhomogeneous Cauchy problem (1), (2)

Let $v(t)$ is same function as in Theorem 3.1.

Theorem 3.2. *Let $p > 1$, $q \geq 1$ and $\gamma \in (0, 1)$ are fixed and the positive constants $a, b, A, B, Q, K, 1 < \beta < \alpha$ satisfy the conditions i1)-i7). Let $f \in \mathcal{C}^1(\mathcal{R}^1)$, $f(0) = 0$, $a|u| \leq f'(u) \leq b|u|$, $g \in \mathcal{C}(\mathcal{R}^+)$, $g(r) \geq 0$ for $r \geq 0$, $g(r) = 0$ for $r \geq r_1$. Then there exists $t_o \in [0, 1)$ for which there exists constant $\epsilon > 0$ such that for every positive constants δ and R exist solutions u_1, u_2 of (1), (2) with right hands $g = g_1, g = g_2$ of (1), so that*

$$E(1, u_1 - u_2) \leq \delta, \quad \|g_1\|_{L^2(\mathcal{R}^+)} \leq R, \quad \|g_2\|_{L^2(\mathcal{R}^+)} \leq R, \quad \|g_1 - g_2\|_{L^2(\mathcal{R}^+)} \leq \delta,$$

and

$$E(t_o, u_1 - u_2) \geq \epsilon.$$

Proof. Let us suppose that the solution map $(u_o, u_1, g) \longrightarrow u(t, r)$ is uniformly continuous. Let

(5)

$$0 < \epsilon < \left(\frac{1}{\beta} - \frac{1}{\alpha} \right)^3 \left(\frac{\alpha^2}{1 - \alpha K + \alpha^2 Q^2} \left(\frac{a}{4A^6 \alpha^2 (1 - \alpha K + \alpha^2 Q^2)} - \frac{2b}{\beta^2 A^2 B^2} - r_1^2 \left(\frac{1}{\beta} - \frac{1}{\alpha} \right)^2 \frac{1}{A^4} \right) \right)^2.$$

Let also

$$u_1 = \bar{u}, \quad u_2 = 0, \quad g_2 \equiv 0, \quad g_1 \equiv g,$$

where g is the function from theorem 3.1. Then there exist positive constants δ and R such that

$$\|g_1\|_{L^2(\mathcal{R}^+)} \leq R, \quad \|g_2\|_{L^2(\mathcal{R}^+)} \leq R, \quad \|g_1 - g_2\|_{L^2(\mathcal{R}^+)} \leq \delta, \\ E(1, u_1 - u_2) \leq \delta,$$

and

$$E(t, u_1 - u_2) \leq \epsilon \quad \text{for } \forall t \in [0, 1].$$

From here, for $t \in [0, 1]$

$$\epsilon \geq \left\| \frac{\partial}{\partial r} \bar{u} \right\|_{L^2([0, \infty))}^2 = \\ = \int_0^{r_1} \left(\frac{1}{r^2 - Kr + Q^2} \int_r^{r_1} \left(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} \bar{u}(t, s) - s^2 (f(\bar{u}(t, s)) + g(s)) \right) ds d\tau \right)^2 dr \geq \\ \geq \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \left(\frac{1}{r^2 - Kr + Q^2} \int_r^{r_1} \left(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} \bar{u}(t, s) - s^2 (f(\bar{u}(t, s)) + g(s)) \right) ds d\tau \right)^2 dr \geq$$

here we use that for $r \in \left[\frac{1}{\alpha}, r_1 \right]$ we have

$$\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} \bar{u}(t, s) - s^2 f(\bar{u}(t, s)) - s^2 g(s) \geq \\ \geq \frac{a}{4A^6 \alpha^2 (1 - \alpha K + \alpha^2 Q^2)} - \frac{2br_1^2}{A^2 B^2} - r_1^2 \max_{r \in [0, r_1]} g(r) \geq 0$$

(see i5)) and for $r \in [0, r_1]$ we have $\frac{1}{r^2 - Kr + Q^2} > 0$

$$\geq \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \left(\frac{1}{r^2 - Kr + Q^2} \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \left(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} \bar{u}(t, s) - s^2 (f(\bar{u}(t, s)) + g(s)) \right) ds d\tau \right)^2 dr \geq \\ \geq \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \left(\frac{1}{r^2 - Kr + Q^2} \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \left(\frac{s^4}{s^2 - Ks + Q^2} \min_{t \in [0, 1]} \frac{v''(t)}{v(t)} \bar{u}(t, s) - s^2 (f(\bar{u}(t, s)) + g(s)) \right) ds d\tau \right)^2 dr \geq$$

from (H4) we have that $\min_{t \in [0, 1]} \frac{v''(t)}{v(t)} \geq \frac{a}{4A^4}$

$$\geq \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \left(\frac{1}{r^2 - Kr + Q^2} \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \left(\frac{s^4}{s^2 - Ks + Q^2} \frac{a}{4A^4} \bar{u}(t, s) - s^2 (f(\bar{u}(t, s)) + g(s)) \right) ds d\tau \right)^2 dr \geq$$

here we use that $f(0) = 0$, $f(\bar{u}) \leq \frac{b}{2} \bar{u}^2$

$$\geq \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \left(\frac{1}{r^2 - Kr + Q^2} \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \left(\frac{s^4}{s^2 - Ks + Q^2} \frac{a}{4A^4} \bar{u}(t, s) - s^2 \frac{b}{2} \bar{u}^2(t, s) - s^2 g(s) \right) ds d\tau \right)^2 dr \geq$$

here we use that $\bar{u}(t, s) \geq \frac{1}{A^2}$ for $t \in [0, 1]$ and $s \in \left[\frac{1}{\alpha}, \frac{1}{\beta} \right]$, $\bar{u}^2(t, s) \leq \frac{4}{A^2 B^2}$ for $t \in [0, 1]$ and $s \in [0, r_1]$, $\max_{r \in [0, r_1]} g(r) \leq \left(\frac{1}{\beta} - \frac{1}{\alpha} \right)^2 \frac{1}{A^4}$,

$$\geq \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \left(\frac{1}{r^2 - Kr + Q^2} \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \left(\frac{s^4}{s^2 - Ks + Q^2} \frac{a}{4A^4} \frac{1}{A^2} - \frac{1}{\beta^2} \frac{b}{2} \frac{4}{A^2 B^2} - r_1^2 \left(\frac{1}{\beta} - \frac{1}{\alpha} \right)^2 \frac{1}{A^4} \right) ds d\tau \right)^2 dr \geq$$

here we use that $\frac{s^2}{s^2-Ks+Q^2}$ is increase function for $s \in [0, r_1]$. Therefore, for $s \in \left[\frac{1}{\alpha}, \frac{1}{\beta}\right]$ we have $\frac{s^4}{s^2-Ks+Q^2} \geq \frac{1}{\alpha^2(1-\alpha K+\alpha^2 Q^2)}$

$$\geq \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \left(\frac{1}{r^2-Kr+Q^2} \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \left(\frac{1}{\alpha^2(1-\alpha K+\alpha^2 Q^2)} \frac{a}{4A^6} - \frac{2b}{\beta^2 A^2 B^2} - r_1^2 \left(\frac{1}{\beta} - \frac{1}{\alpha} \right)^2 \frac{1}{A^4} \right) ds d\tau \right)^2 dr \geq$$

here we use that for $r \in [0, r_1]$ the function $\frac{1}{r^2-Kr+Q^2}$ is increase function. Therefore for $r \in \left[\frac{1}{\alpha}, \frac{1}{\beta}\right]$ we have $\frac{1}{r^2-Kr+Q^2} \geq \frac{\alpha^2}{1-\alpha K+\alpha^2 Q^2}$

$$\geq \left(\frac{1}{\beta} - \frac{1}{\alpha} \right)^3 \left(\frac{\alpha^2}{1-\alpha K+\alpha^2 Q^2} \left(\frac{a}{4A^6 \alpha^2 (1-\alpha K+\alpha^2 Q^2)} - \frac{2b}{\beta^2 A^2 B^2} - r_1^2 \left(\frac{1}{\beta} - \frac{1}{\alpha} \right)^2 \frac{1}{A^4} \right) \right)^2$$

which is contradiction with (5). •

References

- [1] Shatah, J., M. Struwe. Geometric wave equation. Courant lecture notes in mathematics 2(1998).
- [2] Taylor, M. Partial differential equations III. Springer-Verlag, 1996.
- [3] Runst, T., W. Sickel. Sobolev spaces of fractional order, Nemytskij operators and nonlinear partial differential equations, 1996, New York.
- [4] Rauch, J. The u^5 - Klein - Gordon equation, Brezis. and Lions. eds., Pilman Research Notes in Math. 53, pp. 335-364.
- [5] Jörgens, K. Das Anfangswertproblem im Grossen für eine Klasse nicht-linear Wellengleichungen, Math. Z., 77, 1961, pp. 295-308.
- [6] John, F. Blow up of solutions of nonlinear wave equations in three space dimensions, Manuscripta Math. 28, 1979, 235-268.
- [7] Georgiev, S. Blow - up of the solutions of nonlinear wave equation in Reissner - Nordström metric, Dynamics of PDE, Vol. 3, No. 4, 295-329, 2006.

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