Electronic Journal of Qualitative Theory of Differential Equations
2008, No. **4**, 1-12; **http://www.math.u-szeged.hu/ejqtde/**

EXISTENCE AND UNIQUENESS OF A SOLUTION TO A PARTIAL INTEGRO-DIFFERENTIAL EQUATION BY THE METHOD OF LINES

D. BAHUGUNA & J. DABAS
DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY, KANPUR - 208 016, INDIA.

Abstract. In this work we consider a partial integro-differential equation. We reformulate it a functional integro-differential equation in a suitable Hilbert space. We apply the method of lines to establish the existence and uniqueness of a strong solution.

1. Introduction

In the present analysis we are concerned with the following partial integro-differential equation,

$$\begin{cases} \frac{\partial^2 w(x,t)}{\partial t^2} = \frac{\partial^2 w(x,t)}{\partial x^2} + q(x)\frac{\partial w(x,t)}{\partial t} + f_1(x,t,w(x,t),\frac{\partial w(x,t)}{\partial t}) \\ \qquad + \int_0^t k(t,s)\left[\frac{\partial^2 w(x,s)}{\partial x^2} + f_2(x,s,w(x,s),\frac{\partial w(x,s)}{\partial s})\right] ds, \\ (x,t) \in (0,1)\times(0,T],\ 0<T<\infty, \\ w(x,0) = g_1(x), \quad \frac{\partial w}{\partial t}(x,0) = g_2(x), \quad x\in(0,1), \\ w(0,t) = 0 = w(1,t),\ t\in[0,T], \end{cases} \tag{1.1}$$

where the unknown function $w : [0,1]\times[0,T] \to \mathbb{C}$, the coefficient function of the damping term $\frac{\partial w}{\partial t}$, $q : [0,1] \to \mathbb{C}$ satisfies certain integrability conditions, stated later, $f_i : (0,1)\times[0,T]\times\mathbb{C}^2 \to \mathbb{C}$, $g_i : (0,1) \to \mathbb{C}$, $i = 1,2$; and $k : [0,T]^2 \to \mathbb{C}$ are given functions satisfying certain required conditions.

In the next section we transform (1.1) as a Cauchy problem for the following functional integro-differential equation in the product Hilbert space $\mathcal{H} := H_0^1(0,1)\times L^2(0,1)$,

$$\begin{cases} \frac{du}{dt} - Au = \int_0^t k(t,s)Au(s)ds + F(t,u_t), \\ u_0 = \phi, \end{cases} \tag{1.2}$$

where $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ is shown to be the infinitesimal generator of a contraction semigroup in $\mathcal{H}$ and the nonlinear function $F : [0,T]\times\mathcal{C}_0 \to \mathcal{H}$. Here the space $\mathcal{C}_t := C([-T,t];\mathcal{H})$, $t\in[0,T]$, is the Banach space of all continuous functions from $[-T,t]$ into $\mathcal{H}$ endowed with the norm

$$\|\chi\|_{\mathcal{C}_t} := \sup_{-T\le s\le t} \|\chi(s)\|_{\mathcal{H}}, \quad \chi\in\mathcal{C}_t,$$

2000 *Mathematics Subject Classification.* 34K30, 34G20, 47H06.
Key words and phrases. method of lines, integro-differential equation, semigroups, contractions, strong solution.

$\|.\|_{\mathcal{H}}$ being the norm in $\mathcal{H}$ given by

$$\|u\|^2_{\mathcal{H}} := \|u_1\|^2_{H^1_0} + \|u_2\|^2_{L^2}, \quad u = (u_1, u_2) \in H^1_0(0,1) \times L^2(0,1),$$

$$\|u_1\|^2_{H^1_0} = \int_0^1 \left[|u'(x)|^2 + |u(x)|^2\right] dx,$$

$$\|u_2\|^2_{L^2} = \int_0^1 |u(x)|^2 dx.$$

The space $\mathcal{C}_0$ is called the "history space" or the phase space. We also show that F verifies a Lipschitz condition under certain assumptions on the functions f_1 and f_2.

Our aim is to apply Rothe's method to (1.2) involving delays to establish the existence and uniqueness of a strong solution which in turn will guarantee the well-posedness of (1.1). The delay differential equations arise in the study of various population dynamical models [9]. The method of lines is a powerful tool for proving the existence and uniqueness of solutions to evolution equations. This method is oriented towards the numerical approximations. For instance, we refer to Rektorys [11] for a rich illustration of the method applied to various interesting physical problems. Until today, the application of method of lines includes only those nonlinear differential and Volterra integro-differential equations (VIDEs) in which bounded, though nonlinear, operators appear inside the integrals, see Kacur [6, 7], Rektorys [11], Bahuguna and Raghavendra [2] and Bahuguna, Pani and Raghavendra [5]. In the present study we extend the application of the method of lines to a class of nonlinear VIDEs in which differential operators occur inside the integrals and hence are unbounded. Motivation for considering such problems arises from the theory of wave propagation under the influence of damping, see Bahuguna [1], and Bahuguna and Shukla [4] and references cited therein.

2. Reformulation and Main Result

In order to reformulate (1.1) as (1.2) we choose the following settings. We consider the complex Hilbert space $L^2(0,1)$ of all square integrable complex-valued functions on $(0,1)$ with the norm $\|.\|_{L^2}$ introduced earlier. We shall also be dealing with the Sobolev spaces $H^d(0,1)$ and $H^d_0(0,1)$ for $d = 1, 2, \cdots$ (cf. Pazy [10] or Engel and Nagel [8] for definitions and details).

We assume that $q : [0,1] \to \mathbb{C}$ is measurable and satisfies the following conditions.

(C1) There exist constants $\gamma \geq 0$ and $\delta > 0$ such that

$$|\text{Im } q(x)| \leq \gamma \text{ Re } q(x), \quad \text{Re } q(x) \leq -\delta, \quad \text{a.e. } x \in [0,1],$$

where Re $q(x)$ and Im $q(x)$ denote the real and imaginary parts of $q(x)$.

(C2) For every $0 < \epsilon < 1/2$, $q \in L^2[\epsilon, 1-\epsilon]$ and the map $x \mapsto x(1-x)q^2(x)$ is in $L^1[0,1]$.

We define

$$D(P) = \{u \in L^2(0,1) : q(.)u(.) \in L^2(0,1)\}, \; (Pu)(x) = q(x)u(x), \tag{2.1}$$

$$D(A_0) = H^2(0,1) \times (D(P) \cap H^1_0(0,1)), \quad A_0 = \begin{pmatrix} 0 & I \\ \frac{\partial^2}{\partial x^2} & P \end{pmatrix}. \tag{2.2}$$

Under the conditions (C1) and (C2) on q, it follows that the closure A, of A_0 is given by

$$D(A) = H_0^2(0,1) \times H_0^1(0,1), \tag{2.3}$$

$$A = \begin{pmatrix} 0 & I \\ -C^*C & Q \end{pmatrix}, \tag{2.4}$$

where $Q = \overline{{C^*}^{-1} P C^{-1}}$, is a bounded linear operator on $L^2(0,1)$, is the infinitesimal generator of a C_0-semigroup of contractions in $\mathcal{H}$ (cf. Engel and Nagel [8], page 381). Here, for any operator L, $\overline{L}$ represents the closure of L.

With these operators introduced and for $g_1 \in H_0^1(0,1)$ and $g_2 \in L^2(0,1)$, we rewrite (1.1) as

$$\begin{cases} \frac{du}{dt} - Au = F_1(t,u(t)) + \int_0^t k(t,s)[Bu(s) + F_2(s,u(s))]ds, \\ u(0) = (g_1, g_2), \end{cases} \tag{2.5}$$

where

$$D(B) = D(A), \tag{2.6}$$

$$B = \begin{pmatrix} 0 & 0 \\ -C^*C & 0 \end{pmatrix}, \tag{2.7}$$

and, for $i = 1, 2$, $F_i : [0,T] \times \mathcal{H} \to \mathcal{H}$ are given by

$$F_i(t,(u_1,u_2))(x) = (0, f_i(x,t,u_1(x,t),u_2(x,t))), \quad u = (u_1,u_2) \in \mathcal{H}.$$

We put $t - s = -\eta$ in the integral term in (2.5) to obtain

$$\int_0^t k(t,s)F_2(s,u(s))ds = \int_{-t}^0 k(t,t+\eta)F_2(t+\eta, u(t+\eta))d\eta \tag{2.8}$$

$$= \int_{-t}^0 k(t,t+\eta)F_2(t+\eta, u_t(\eta))d\eta. \tag{2.9}$$

Thus, if for any $u \in \mathcal{C}_T := C([0,T];\mathcal{H})$ and $t \in [0,T]$, we let $u_t \in \mathcal{C}_0 := C([-T,0];\mathcal{H})$ given by $u_t(\eta) = u(t+\eta)$, $\eta \in [-t,0]$ and $u_t(\eta) = u(0)$ for $\eta \in [-T,-t]$, we may rewrite (2.5) as (1.2) where $F : [0,T] \times \mathcal{C}_0 \to \mathcal{H}$, given by

$$F(t,\chi) = F_1(t,\chi(0)(t)) + \int_{-t}^0 k(t,t+\eta)[(B-A)\chi(\eta)(s) + F_2(s,\chi(\eta)(s))]ds, \ \chi \in \mathcal{C}_0,$$

and $\phi \in \mathcal{C}_0$ is given by

$$\phi(\eta) \equiv u(0) = (g_1, g_2), \quad \eta \in [-T,0].$$

We list here the properties of the linear operator A, the nonlinear map F and the kernel k.

(P1) The operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ is the infinitesimal generator of a C_0 semigroup $S(t)$ of contractions in $\mathcal{H}$.

(P2) The function $F : [0,T] \times \mathcal{C}_0 \to \mathcal{H}$ satisfies the Lipschitz condition, i.e., there exists a positive constant L_F such that

$$\|F(t_1,\chi_1) - F(t_2,\chi_2)\|_{\mathcal{H}} \le L_F[|t_1 - t_2| + \|\chi_1 - \chi_2\|_{\mathcal{C}_0}],$$

for $(t_i,\chi_i) \in [0,T] \times \mathcal{C}_0$, $i = 1,2$.

(P3) The function $k : [0,T]^2 \to \mathbb{R}$ is continuous and there exists a positive constant L_k such that

$$|k(t_1,s) - k(t_2,s)| \le L_k |t_1 - t_2|, \quad t_1, t_2 \in [0,T].$$

We have the following main result.

Theorem 2.1. *Suppose that (P1)-(P3) are satisfied and $\phi \in \mathcal{C}_0$ is Lipschitz continuous. Then there exists a unique $u \in \mathcal{C}_T$, with $u_0 = \phi$, $u(t) \in D(A)$, a.e. $t \in [0,T]$, $u(t)$ is Lipschitz continuous on $[0,T]$ and satisfies (1.2) a.e. on $[0,T]$.*

3. Discretization Scheme and A Priori Estimates

To apply Rothe's method, We use the following procedure. For any positive integer n we consider a partition t_j^n defined by $t_j^n = jh;\ h = T/n,\ j = 0,1,2,\ldots,n$. Set $u_0^n = \phi(0)$ for all $n \in \mathbb{N}$. For $j = 1,2,\ldots,n$, we define $u_j^n \in D(A)$ the unique solutions of each of the equations

$$\frac{u_j^n - u_{j-1}^n}{h} - Au_j^n = F_j^n + h\sum_{i=0}^{j-1} k_{ji}^n \, Au_i^n, \tag{3.1}$$

Where $F_j^n = F(t_j^n, \tilde{u}_{j-1}^n)$ and $k_{ji}^n = k(t_j^n, t_i^n)$ $1 \le i \le j \le n$. and $\tilde{u}_0^n(t) = \phi(t)$ for $t \in [-T,0]$, $\tilde{u}_0^n(t) = \phi(0)$ for $t \in [0,T]$ and for $2 \le j \le n$,

$$\tilde{u}_j^n(\theta) = \begin{cases} \phi(t_j^n + \theta), & \theta \le -t_j^n, \\ u_{i-1}^n + (\theta - t_{j+1-i}^n)\delta u_i^n, & \theta \ge -t_j^n, \theta \in [-t_{j+1-i}^n, -t_{j-i}^n], 1 \le i \le j. \end{cases}$$

Now, the existence of a unique $u_j^n \in D(A)$ satisfying (3.1) follows from the m-dissipativity of A and by Theorems 1.4.2 and 1.4.3 in Pazy [10]. In order to ensure the existence of a unique solution $u_j^n \in D(A)$ of (3.1) we rewrite it as

$$u_j^n = (I - hA)^{-1}\left[u_{j-1}^n + hF_j^n + h^2\sum_{i=0}^{j-1} k_{ji}^n Au_i^n\right],$$

as $(I - hA)^{-1}$ exists for all $h > 0$. The existence of unique $u_j^n \in D(A)$ satisfying (3.1) is ensured.

Definition 3.1. *We define the Rothe sequence $\{U^n\} \subset C([-T,T];\mathcal{H})$ given by*

$$U^n(t) = \begin{cases} \phi(t), & t \in [-T,0] \\ u_{j-1}^n + (t - t_{j-1}^n)\delta u_j^n, & t \in [t_{j-1}^n, t_j^n],\ j = 1,2,\ldots,n. \end{cases}$$

and a sequence $\{X^n\}$ of step functions from $[-T,T]$ into $\mathcal{H}$ given by

$$X^n(t) = \phi(t), \quad t \in [-T,0] \quad X^n(t) = u_j^n, \quad t \in (t_{j-1}^n, t_j^n],\ j = 1,2,\ldots,n.$$

We prove the convergence of the sequence $\{U^n\}$ to the unique solution of the problem as $n \to \infty$ using some a priori estimates on $\{U^n\}$. For convenience, we shall denote by C a generic constant, i.e., KC, e^{KC}, etc., will be replace by C where K is a positive constant independent of j, h and n.

We shall use later the following lemma due to Sloan and Thomee [12].

Lemma 3.2. *Let $\{w_n\}$ be a sequence of nonnegative real numbers satisfying*

$$w_n \leq \alpha_n + \sum_{i=0}^{n-1} \beta_i w_i, \quad n > 0,$$

where $\{\alpha_n\}$ is a nondecreasing sequence of nonnegative real numbers and $\beta_n \geq 0$. Then

$$w_n \leq \alpha_n \exp\{\sum_{i=0}^{n-1} \beta_i\}, \quad n > 0.$$

Furthermore, we also require the following lemma for later use.

Lemma 3.3. *Let $C > 0$, $h > 0$ and let $\{\alpha_j\}_{j=1}^{n}$ be a sequence of nonnegative real numbers satisfying*

$$\alpha_j \leq (1 + Ch)\alpha_{j-1} + Ch^2 \sum_{i=1}^{j-1} \alpha_i + Ch, \quad 2 \leq j \leq n. \tag{3.2}$$

Then

$$\alpha_j \leq (1 + Ch)^j [\alpha_1 + jCh^2 \sum_{i=1}^{j-1} \alpha_i + jCh], \quad 2 \leq j \leq n.$$

Proof. From (3.2)

$$\begin{aligned} \alpha_{j-1} &\leq (1 + Ch)\alpha_{j-2} + Ch^2 \sum_{p=1}^{j-2} \alpha_p + Ch \\ &\leq (1 + Ch)\alpha_{j-2} + Ch^2 \sum_{p=1}^{j-1} \alpha_p + Ch. \end{aligned} \tag{3.3}$$

Putting in (3.2)

$$\alpha_j \leq (1 + Ch)^2 \alpha_{j-2} + Ch^2[1 + (1 + Ch)] \sum_{p=1}^{j-1} \alpha_p + Ch[1 + (1 + Ch)]. \tag{3.4}$$

By repeating the above process

$$\begin{aligned} \alpha_j &\leq (1 + Ch)^{(j-1)}\alpha_1 + Ch^2[1 + (1 + Ch) + \cdots + (1 + Ch)^{(j-1)}] \sum_{p=1}^{j-1} \alpha_p \\ &\quad + Ch[1 + (1 + Ch) + \cdots + (1 + Ch)^{(j-1)}] \\ &\leq (1 + Ch)^j [\alpha_1 + jCh^2 \sum_{p=1}^{j-1} \alpha_p + jCh]. \end{aligned} \tag{3.5}$$

This completes the proof of the lemma. □

Lemma 3.4. *There exists a constant C independent of j, h and n such that*

$$\|\delta u_j^n\|_{\mathcal{H}} \leq C.$$

Proof. From (3.1) for $j = 1$ we get

$$\delta u_1^n - hA\delta u_1^n = Au_0 + F_1^n + hk_{10}^n Au_0.$$

Theorem 1.4.2 [10] implies that

$$\|\delta u_1^n\|_{\mathcal{H}} \le \|Au_0 + F_1^n + hk_{10}^n Au_0\|_{\mathcal{H}} \le C.$$

Hence $\|Au_1^n\|_{\mathcal{H}} \le C$. Let $2 \le j \le n$. Subtracting (3.1) for $j-1$ from (3.1) for j, we get

$$\delta u_j^n - hA\delta u_j^n = \delta u_{j-1}^n + F_j^n - F_{j-1}^n + hk_{jj-1}^n Au_{j-1}^n + \sum_{i=0}^{j-2}[k_{ji}^n - k_{j-1i}^n]Au_i^n.$$

Applying Theorem 1.4.2 of [10] again, we get

$$\begin{aligned}\|\delta u_j^n\|_{\mathcal{H}} &\le \|\delta u_{j-1}^n\|_{\mathcal{H}} + \|F_j^n - F_{j-1}^n\|_{\mathcal{H}} + h|k_{jj-1}^n|\|Au_{j-1}^n\|_{\mathcal{H}} \\ &+ h\sum_{i=0}^{j-2}|k_{ji}^n - k_{j-1i}^n|\|Au_i^n\|_{\mathcal{H}}. \end{aligned} \tag{3.6}$$

Now using Lipschitz continuity of the function F

$$\begin{aligned}\|F_j^n - F_{j-1}^n\|_{\mathcal{H}} &= \|F(t_j^n, \tilde{u}_{j-1}^n) - F(t_{j-1}^n, \tilde{u}_{j-2}^n)\|_{\mathcal{H}} \\ &\le L_F(|t_j^n - t_{j-1}^n| + \|\tilde{u}_{j-1}^n - \tilde{u}_{j-2}^n\|_{\mathcal{C}_0}) \\ &\le Ch(1 + \max_{1\le p\le j-1}\|\delta u_p^n\|_{\mathcal{H}}). \end{aligned} \tag{3.7}$$

Then (3.6) becomes

$$\begin{aligned}\|\delta u_j^n\|_{\mathcal{H}} &\le \|\delta u_{j-1}^n\|_{\mathcal{H}} + Ch(1 + \max_{1\le p\le j-1}\|\delta u_p^n\|_{\mathcal{H}}) + Ch^2\sum_{i=0}^{j-1}\|Au_i^n\|_{\mathcal{H}} \\ &\le (1 + Ch)\max_{1\le p\le j-1}\|\delta u_p^n\|_{\mathcal{H}} + Ch^2\sum_{i=0}^{j-1}\|Au_i^n\|_{\mathcal{H}} + Ch. \end{aligned} \tag{3.8}$$

From (3.1), for $2 \le i \le j$, we have

$$\|Au_i^n\|_{\mathcal{H}} \le \|\delta u_i^n\|_{\mathcal{H}} + \|F_i^n\|_{\mathcal{H}} + Ch\sum_{p=0}^{i-1}\|Au_p^n\|_{\mathcal{H}}. \tag{3.9}$$

Again using Lipchitz continuity of F

$$\begin{aligned}\|F_i^n\|_{\mathcal{H}} &= \|F(t_i^n, \tilde{u}_{i-1}^n) - F(t_{i-1}^n, 0)\|_{\mathcal{H}} + \|F(t_{i-1}^n, 0)\|_{\mathcal{H}} \\ &\le L_F(|t_i^n - t_{i-1}^n| + \|\tilde{u}_{i-1}^n\|_{\mathcal{C}_0}) + \max_{0\le t\le T}\|F(t,0)\|_{\mathcal{H}} \\ &\le C(h + h\sum_{p=1}^{i-1}\|\delta\tilde{u}_p^n\|_{\mathcal{C}_0} + \|\tilde{u}_0\|_{\mathcal{C}_0}) + \max_{0\le t\le T}\|F(t,0)\|_{\mathcal{H}}. \end{aligned} \tag{3.10}$$

Then (3.9) becomes

$$\|Au_i^n\|_{\mathcal{H}} \le C(1 + \max_{1\le p\le i}\|\delta u_p^n\|_{\mathcal{H}}) + Ch + Ch\sum_{p=1}^{i-1}\|Au_p^n\|_{\mathcal{H}}. \tag{3.11}$$

Applying Lemma 3.2 in (3.11), we get

$$\|Au_i^n\|_{\mathcal{H}} \le Ce^{CT}(1 + \max_{1\le p\le i} \|\delta u_p^n\|_{\mathcal{H}}). \tag{3.12}$$

Using (3.12) in (3.8), we have

$$\max_{1\le p\le j} \|\delta u_p^n\|_{\mathcal{H}} \le (1+Ch) \max_{1\le p\le j-1} \|\delta u_p^n\|_{\mathcal{H}} + Ch^2 \sum_{p=1}^{j-1} \max_{1\le p\le i} \|\delta u_p^n\|_{\mathcal{H}} + Ch. \tag{3.13}$$

To use Lemma 3.3 in (3.13), we take $\alpha_j = \max_{1\le p\le j} \|\delta u_p^n\|_{\mathcal{H}}$ and the fact that $(1+Ch)^j \le e^{CT}$, $2 \le j \le l$ and $\alpha_1 \le C$ to get the estimate

$$\max_{1\le p\le j} \|\delta u_p^n\|_{\mathcal{H}} \le C + Ch \sum_{p=1}^{j-1} \max_{1\le p\le j-1} \|\delta u_p^n\|_{\mathcal{H}}. \tag{3.14}$$

Again we apply Lemma 3.2 to get the required estimate. This completes the proof of the lemma. □

Remark 3.5. *Each of the functions $\{U^n\}$ is Lipschitz continuous with uniform Lipschitz constant, i.e.,*

$$\|U^n(t) - U^n(s)\|_{\mathcal{H}} \le C|t-s|, \quad t, s \in [0,T].$$

Furthermore,

$$\|U^n(t) - X^n(t)\|_{\mathcal{H}} \le \frac{C}{n}.$$

Definition 3.6. *We define the sequence $\{F^n\}$ and $\{K^n\}$ of step functions $[0,T]$ into $\mathcal{H}$ by*

$$K^n(0) = 0, \quad K^n(t) = h\sum_{i=0}^{j-1} k_{ji}^n Au_i^n, \quad t \in (t_{j-1}^n, t_j^n], \tag{3.15}$$

$$F^n(0) = F(0,\phi), \quad F^n(t) = F(t_j^n, \tilde{u}_{j-1}^n), \quad t \in (t_{j-1}^n, t_j^n]. \tag{3.16}$$

Lemma 3.7. *Under the given assumptions we have*
(a) $\{K^n(t)\}$ *is uniformly bounded;*
(b) $\int_0^t AX^n(s)ds = u_0 - U^n(t) - \int_0^t K^n(s)ds - \int_0^t F^n(s)$;
(c) $\frac{d^-}{dt}U^n(t) - AX^n(t) = K^n(t) + F^n(t), \quad t \in (0,T]$,
where $\frac{d^-}{dt}$ *is the left-derivative.*

Proof. (a) This is a direct consequence of the estimates in (3.12) and (3.14).

(b) For $2 \le j \le n$ and $t \in (t_{j-1}^n, t_j^n]$, by Definition 3.1, we have

$$\begin{aligned}
\int_0^t AX^n(s)ds &= \sum_{i=1}^{j-1} \int_{t_{i-1}^n}^{t_i^n} AX^n(s)ds + \int_{t_{j-1}^n}^{t} AX^n(s)ds \\
&= \sum_{i=1}^{j-1} (u_i^n - u_{i-1}^n) + \frac{1}{h}(t - t_{j-1}^n)(u_j^n - u_{j-1}^n) \\
&\quad - h \sum_{i=1}^{j-1} \left[h \sum_{p=0}^{i-1} k_{ip}^n A u_p^n \right] - (t - t_{j-1}^n)[h \sum_{p=0}^{j-1} k_{ip}^n A u_p] \\
&\quad - h \sum_{i=0}^{j-1} F_i^n - (t - t_{j-1}^n) F_j^n \\
&= u_0 - U^n(t) - \int_0^t K^n(s)ds - \int_0^t F^n(s)ds
\end{aligned}$$

When $j = 1$, $t \in (0, t_1^n]$. we have

$$\begin{aligned}
\int_0^t AX^n(s)ds &= tAu_1^n \\
&= \frac{t}{h}(u_1^n - u_0^n) - thk_{10}^n A u_0 - tF_1^n \\
&= u_0 - U^n(t) - \int_0^t K^n(s)ds - \int_0^t F^n(s)ds.
\end{aligned}$$

(c) for $t \in (t_{j-1}^n, t_j^n]$,

$$AX^n(t) = Au_j^n \quad \text{and} \quad \frac{d^- u^n}{dt}(t) = \frac{1}{h}(u_j^n - u_{j-1}^n).$$

Therefore,

$$\begin{aligned}
\frac{d^- u^n}{dt}(t) - AX^n(t) &= \frac{1}{h}(u_j^n - u_{j-1}^n) - Au_j^n \\
&= h \sum_{i=0}^{j-1} k_{ji}^n A u_i^n + F_j^n \\
&= K^n(t) + F^n(t).
\end{aligned}$$

This completes the proof of the lemma. □

In the next lemma we prove the local uniform convergence of the Rothe sequence.

Lemma 3.8. *There exist a subsequence $\{U^{n_p}\}$ of $\{U^n\}$ and a function $u : [0, T] \to D(A)$ such that $U^{n_p} \to u$ in $C([0, T]; \mathcal{H})$, and $AU^{n_p}(t) \rightharpoonup Au(t)$ uniformly in $\mathcal{H}$ as $p \to \infty$, where $\rightharpoonup$ denotes the weak convergence in $\mathcal{H}$. Furthermore, $Au(t)$ is weakly continuous on $[0, T]$.*

Proof. Since $\{U^n(t)\}$ and $\{AX^n(t)\}$ are uniformly bounded in the Hilbert space $\mathcal{H}$, there exist weakly convergent subsequences $\{U^{n_p}(t)\}$ and $\{AX^{n_p}(t)\}$ (we take the same indices without loss of generality otherwise we first take the subsequence

$\{U^{n_p}(t)\}$ of $\{U^n(t)\}$ and then take the subsequence $\{U^{n_{p_n}}(t)\}$ and $\{AX^{n_{p_n}}(t)\}$ of $\{U^{n_p}(t)\}$ and $\{AX^{n_p}(t)\}$, respectively). Thus, there exist functions $u, w : [0,T] \to \mathcal{H}$ such that $U^{n_p}(t) \rightharpoonup u(t)$ and $AX^{n_p}(t) \rightharpoonup w(t)$ as $p \to \infty$. Also, we have $X^{n_p}(t) \rightharpoonup u(t)$ as $p \to \infty$. Clearly, $\{X^{n_p}(t)\}$ and $\{AX^{n_p}(t)\}$ are uniformly bounded and $X^{n_p}(t) \rightharpoonup u(t)$ and $AX^{n_p}(t) \rightharpoonup w(t)$ as $p \to \infty$. Since $D(A) = H_0^2(0,1) \times H_0^1(0,1)$ is compactly embedded in $\mathcal{H} = H_0^1(0,1) \times L^2(0,1)$, it follows that $(I+A)^{-1} : \mathcal{H} \to \mathcal{H}$ is compact. The boundedness of $V^n = (I+A)U^n$ and the compactness of $(I+A)^{-1}$ imply that $U^n = (I+A)^{-1}V^n$ has a convergent subsequence. For convenience, we again denote this convergent subsequence by $\{U^{n_p}\}$. Thus, $U^{n_p}(t) \to u(t)$ as $p \to \infty$. Also, $X^{n_p}(t) \to u(t)$ as $p \to \infty$. By the maximal dissipativity of A, it follows that $u(t) \in D(A)$ and $AX^{n_p}(t) \rightharpoonup Au(t)$. Since U^{n_p} is Lipschitz continuous with uniform Lipschitz constant, it follows that $\{U^{n_p}\}$ is equi-continuous in $C([0,T];\mathcal{H})$ and $\{U^{n_p}(t)\}$ is relatively compact in $\mathcal{H}$. Hence by Ascoli-Arzela theorem, $U^{n_p} \to u$ as $p \to \infty$ in $C([0,T];\mathcal{H})$.

To show the weak continuity of $Au(t)$ in t, let $\{t_p\} \subset [0,T]$ such that $t_p \to t$ as $p \to \infty$, $t \in [0,T]$. Then $u(t_p) \to u(t)$ and since $\|Au(t_p)\|_{\mathcal{H}} \le C$, there exists a subsequence $\{Au(t_{k_p})\} \subset \{Au(t_p)\}$ such that $Au(t_{p_m}) \rightharpoonup z(t)$ as $m \to \infty$. Since $u(t_{p_m}) \to u(t)$ and $Au(t_{p_m}) \rightharpoonup z(t)$ as $m \to \infty$, it follows as above that $u(t) \in D(A)$ and $Au(t) = z(t)$. Hence $Au(t)$ is weakly continuous. This completes the proof of the lemma. □

Lemma 3.9. *$Au(t)$ is Bochner integrable on $[0,T]$.*

For a proof of this lemma we refer to Bahuguna and Raghavendra [2].

Lemma 3.10. *Let $\{K^n(t)\}$ be the sequence of functions defined by (3.15) and*

$$K(\phi)(t) = \int_0^t k(t,s)\phi(s)ds,$$

where $\phi : [0,T] \to H$ is Bochner integrable. We have

$$K^{n_p}(t) \rightharpoonup K(Au)(t),$$

uniformly on $[0,T]$ as $p \to \infty$.

Proof. We first show that $K^{n_p}(t) - K(AX^{n_p})(t) \to 0$ uniformly on $[0,T]$ as $p \to \infty$. For $t \in (t_{j-1}^{n_p}, t_j^{n_p}]$, we have

$$\begin{aligned} K^{n_p}(t) - K(AX^{n_p})(t) &= h\sum_{i=0}^{j-1} k_{ji}^{n_p} Au_i^{n_p} - \int_0^t k(t,s)AX^{n_p}(s)\,ds \\ &= \sum_{i=1}^{j-1}\left[\int_{t_{i-1}^{n_p}}^{t_i^{n_p}} [k_{ji}^{n_p} - k(t,s)]\,ds\right] Au_i^{n_p} \\ &\quad + hk(t_j^{n_p}, t_0^{n_p})Au_0^{n_p} - \Big[\int_{t_{j-1}^{n_p}}^{t} k(t,s)\,ds\Big] Au_j^{n_p}. \end{aligned}$$

Since $\|Au_j^{n_p}\|_{\mathcal{H}} \le C$, and $k : [0,T] \to \mathbb{R}$ Lipschitz continuous imply that the last two terms on the right hand side tend to zero strongly and uniformly on $[0,T]$ as

$p \to \infty$. we have

$$\|K^{n_p}(t) - K(AX^{n_p})(t)\|_{\mathcal{H}} \quad \leq \quad C[\sum_{i=0}^{j-2} \int_{t_i^{n_p}}^{t_{i+1}^{n_p}} |k_{ji}^{n_p} - k(t,s)|ds.$$

Now, since k satisfies (K3) hence $k(t,s)$ is uniformly continuous in t as well as in s on $[0,T]$. Hence for each $\epsilon > 0$ we can choose n sufficiently large such that for $|t_1 - t_2| + |s_1 - s_2| < h = \frac{T}{n}, t_i, s_i \in [0,T]$, $i = 1, 2$, we have

$$|k(t_1, s_1) - k(t_2, s_2)| < \frac{\epsilon}{CT}.$$

Then for sufficiently large n, we have

$$\|K^{n_p}(t) - K(AX^{n_p})(t)\|_{\mathcal{H}} \leq \frac{\epsilon}{CT} Cjh < \epsilon,$$

Which show that $K^{n_p}(t) - K(AX^{n_p})(t) \to 0$ as $p \to \infty$, uniformly on $[0,T]$. Now we show that $K(X^{n_p})(t) \rightharpoonup \int_0^t k(t,s)Au(s)\, ds$ uniformly as $p \to \infty$. For any $v \in \mathcal{H}$, We note that $\langle Au(t), v\rangle$ is continuous hence we may write

$$\langle \int_0^t k(t,s)Au(s)\, ds, v\rangle = \int_0^t k(t,s)\langle Au(s), v\rangle\, ds.$$

Now, for any $v \in \mathcal{H}$,

$$\begin{aligned} \langle K(X^{n_p})(t), v\rangle &= \langle \int_0^t k(t,s)AX^{n_p}(s)ds, v\rangle \\ &= \sum_{i=0}^{j-2} \int_{t_i^{n_p}}^{t_{i+1}^{n_p}} k(t,s)\langle Au_{i+1}^{n_p}, v\rangle ds \\ &\quad + \int_{t_{j-1}^{n_p}}^{t} k(t,s)\langle Au_j^{n_p}, v\rangle ds \to \int_0^t k(t,s)\langle Au(s), v\rangle\, ds, \end{aligned}$$

as $p \to \infty$. This completes the proof of the lemma. □

Proof of the Theorem 2.1.

Proof. From Lemma 3.7, for each $v \in \mathcal{H}$, and $t \in (0,T]$ we have

$$\langle U^{n_p}(t), v\rangle \quad = \quad \langle u_0, v\rangle + \int_0^t \langle AX^{n_p}(s), v\rangle ds + \int_0^t \langle K^{n_p}(s) + F^{n_p}(s), v\rangle\, ds.$$

Passing to the limit as $p \to \infty$ using bounded convergence theorem and Lemmas 3.8 and 3.10, we obtain

$$\langle u(t), v\rangle = \langle u_0, v\rangle + \int_0^t \langle Au(s), v\rangle ds + \int_0^t \langle K(u)(s) + F(s, u_s), v\rangle\, ds. \qquad (3.17)$$

The integrands in (3.17) are continuous on $[0,T]$ for each fixed $v \in \mathcal{H}$ and hence $\langle u(t), v\rangle$ is continuously differentiable. The boundedness of $Au(t)$ on $[0,T]$ implies that $K(u)(t)$ is Lipschitz continuous. Making use of Lemma 3.10 and continuity of

$K(u)(t)$ and $F(t, u_t)$ on $[0, T]$ in (3.17) we obtain the existence of a strong derivative on $u(t)$ almost everywhere on $[0, T]$ and

$$\frac{du(t)}{dt} - Au(t) = F(t, u_t) + \int_0^t k(t, s)Au(s)\, ds, \quad \text{a.e. } t \in [0, T],\ u_0 = \phi. \tag{3.18}$$

This show that $u(t)$ ia a strong solution to (1.2) since $u(0) = \phi$ and $u(t)$ is absolutely (in fact Lipschitz) continuous on $[0, T]$ satisfying (1.2) a.e on $[0, T]$. Now we prove the uniqueness. Since k is a real valued Lipschitz continuous function on $[0, T]$, it is differentiable a.e. on $[0, T]$ and its derivative is essentially bounded on $[0, T]$. Let u_1 and u_2 be two solutions of (1.2) and let $u = u_1 - u_2$. Then

$$\begin{aligned} u(t) &= \int_0^t T(t-s)[F(s, (u_1)_s) - F(s, (u_2)_s) \\ &\quad + \int_0^s k(s, \tau)Au(\tau)\, d\tau]\, ds \\ &= \int_0^t T(t-s)[F(s, (u_1)_s) - F(s, (u_2)_s)]ds \\ &\quad + \int_0^t \left(\int_0^s k(s, \tau)T(t-s)Au(\tau))\, d\tau \right) ds \\ &= \int_0^t T(t-s)[F(s, (u_1)_s) - F(s, (u_2)_s)]ds \\ &\quad + \int_0^t \left(\int_\tau^t k(s, \tau)T(t-s)Au(\tau))\, ds \right) d\tau \\ &= \int_0^t T(t-s)[F(s, (u_1)_s) - F(s, (u_2)_s)]ds \\ &\quad + \int_0^t \left(\int_0^{t-\tau} k(t-\eta, \tau)T(\eta)Au(\tau))\, d\eta \right) d\tau. \end{aligned} \tag{3.19}$$

Since $u(\tau) \in D(A)$ for $\tau \in [0, T]$, we have $T(\eta)Au(\tau) = \frac{\partial}{\partial \eta}(T(\eta)u(\tau))$ (cf. Theorem 1.2.4 in Pazy). Thus, we have

$$\begin{aligned} u(t) &= \int_0^t T(t-s)[f(s, (u_1)_s) - f(s, (u_2)_s)]ds \\ &\quad + \int_0^t \left(\int_0^{t-\tau} k(t-\eta, \tau)\frac{\partial}{\partial \eta}(T(\eta)u(\tau))\, d\eta \right) d\tau \\ &= \int_0^t T(t-s)[f(s, (u_1)_s) - f(s, (u_2)_s)]ds \\ &\quad + \int_0^t k(\tau, \tau)T(t-\tau)u(\tau)d\tau - \int_0^t k(t, \tau)u(\tau)d\tau \\ &\quad + \int_0^t \left(\int_0^{t-\tau} k'(t-\eta, \tau)T(\eta)u(\tau)d\eta \right) d\tau. \end{aligned} \tag{3.20}$$

Now taking the norm and using the fact that $\|T(t)\|_{\mathcal{H}} \leq 1$, we have

$$\max_{0\leq r\leq t} \|u(r)\|_{\mathcal{H}} \leq C \int_0^t \max_{0\leq r\leq s} \|u(r)\|_{\mathcal{H}}\, ds.$$

Gronwall's inequality implies that $u(t) \equiv 0$. This completes the proof of the theorem. □

Acknowledgements:
The authors would like to thank the referee for his valuable suggestions which has helped us to improve the original manuscript.

References

[1] Bahuguna, D., Rothe's method to strongly damped wave equations, *Acta. Applicandae Mathematicae.* **38** (1995), 185–196.

[2] Bahuguna, D. and Raghavendra, V., Rothe's method to parabolic integro-differntial equations via abstract integro-differential equations, *Appl. Anal.* **33** (1989), 3-4, 153–167.

[3] Bahuguna, D. and Raghavendra, V., Application of Rothe's method to nonlinear integro-differential equations in Hilbert spaces, *Nonlinear Analysis, TMA.* **23** (1994), no. 1, 75-81.

[4] Bahuguna, D. and Shukla, R., Approximations of solutions to second order semilinear integrodifferential equations.*Numer. Funct. Anal. Optim.* **24** (2003), no. 3-4, 365–390.

[5] Bahuguna, D. Pani, A.K. and Raghavendra, V., Rothe's method to semilinear hyperbolic integro-differential equations, *Journal of Applied Mathematics and Stochatic Analysis.* **3** (1990), no. 4, 245-252.

[6] Kacur, J., Application of Rothe's method to perturbed linear hyperbolic equations and variantional inequalities, *Czechoslovak Mathematical Journal.* **34** (1984),(109), 92-106.

[7] Kacur, J., Method of Rothe in evolution equations, *Lecture Notes in Mathematics Springer, Berlin.* **1192** (1985), 23-34.

[8] Klaus, J.E and Nagel, R., *One-Parameter Semigroups for Linear Evolution Equations*, Springer Verlag (2000).

[9] Kuang, Y., *Delay Differential Equations with Applications in Population Dynamics,* Academic Press, Inc. v-191 (1993).

[10] Pazy, A., *Semi-groups of Linear Operators and Applications to Partial Differential Equations*, Springer Verlag (1983).

[11] Rektorys, K., *The Method of Discretization in Time and Partial Differential Equations*, Dordrecht-Boston-London, Reidel (1982).

[12] Sloan, I.H. and Thomee, V., Time discretization of an integro-differential equation of parabolic type, *SIAM J. Numer. Anal.* **23** (1986), 1052-1061.

(Received July 5, 2007)

Dhirendra Bahuguna: Department of Mathematics Indian Institute of Technology, Kanpur - 208 016, India.
E-mail address: dhiren@iitk.ac.in

Jaydev Dabas: Department of Mathematics Indian Institute of Technology, Kanpur - 208 016, India.
E-mail address: jaydev@iitk.ac.in