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Positive Solutions for Nonlinear Semipositone *n*th-Order Boundary Value Problems

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Abstract

In this paper, we investigate the existence of positive solutions for a class of nonlinear semipositone *n*th-order boundary value problems. Our approach relies on the Krasnosel'skii fixed point theorem. The result of this paper complement and extend previously known result.

Keywords: Boundary value problem; Positive solution; Semipositone; Fixed point.

1. Introduction

The BVPs for nonlinear higher-order differential equations arise in a variety of areas of applied mathematics, physics, and variational problems of control theory. Many authors have discussed the existence of solutions of higher-order BVPs, see for example [1-4, 6-7, 9-10], and the references therein.

In this paper, we study the existence of positive solutions of *n*th-order boundary value problem

$$\begin{cases} u^{(n)}(t) + \lambda f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = h\left(\int_0^1 u(t) d\zeta(t)\right), \quad u'(0) = 0, \cdots, u^{(n-2)}(0) = 0, \quad u(1) = g\left(\int_0^1 u(t) d\theta(t)\right), \end{cases} \tag{1.1}$$

where $n \geq 3$, λ is a positive parameter, $\int_0^1 u(t) d\zeta(t)$, $\int_0^1 u(t) d\theta(t)$ denote the Riemann-Stieltjes integrals, we assume that

(B_1) ζ, θ are increasing nonconstant functions defined on $[0, 1]$ with $\zeta(0) = \theta(0) = 0$;

(B_2) there exists $M > 0$ such that $f : [0, 1] \times [0, +\infty) \to [-M, +\infty)$ is continuous;

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(B_3) $h, g : [0, +\infty) \to [0, +\infty)$ are continuous and nondecreasing.

We note that our nonlinearity f may take negative values. Problems of this type are referred to as semipositone problems in the literature. To the best of the authors knowledge, no one has studied the existence of positive solutions for BVP (1.1).

In special case, the boundary value problems (1.1) reduces a four-point or three-point boundary value problems by applying the following well-known property of the Riemann-Stieltjes integral.

Lemma 1.1 [5]. *Assume that*

(1) $u(t)$ *is a bounded function value on* $[a, b]$*, i.e., there exist* c*,* $C \in \mathbb{R}$ *such that* $c \le u(t) \le C$*,* $\forall t \in [a, b]$*;*

(2) $\zeta(t)$*,* $\theta(t)$ *are increasing on* $[a, b]$*;*

(3) *The Riemann-Stieltjes integrals* $\int_a^b u(t)d\theta(t)$ *and* $\int_a^b u(t)d\zeta(t)$ *exist.*

Then there exist $v_1, v_2 \in \mathbb{R}$ *with* $c \le v_1, v_2 \le C$ *such that*

$$\int_a^b u(t)d\theta(t) = v_1(\theta(b) - \theta(a)), \quad \int_a^b u(t)d\zeta(t) = v_2(\zeta(b) - \zeta(a)).$$

For any continuous solution $u(t)$ of (1.1), by Lemma 1.1, there exist ξ, $\eta \in (0, 1)$ such that

$$\int_0^1 u(t)d\zeta(t) = u(\xi)(\zeta(1) - \zeta(0)) = u(\xi)\zeta(1),$$

$$\int_0^1 u(t)d\theta(t) = u(\eta)(\theta(1) - \theta(0)) = u(\eta)\theta(1).$$

If $h(t) = g(t) = t$, $t \in [0, +\infty)$, we take $\alpha = \zeta(1)$, $\beta = \theta(1)$, then BVP (1.1) can be rewritten as the following nth-order four-point boundary value problem

$$\begin{cases} u^{(n)}(t) + \lambda f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = \alpha u(\xi), \quad u'(0) = 0, \cdots, u^{(n-2)}(0) = 0, \quad u(1) = \beta u(\eta), \end{cases} \tag{1.2}$$

If $h(t) = 0$, $t \in [0, +\infty)$ and $g(t) = t$, $t \in [0, +\infty)$, we take $\beta = \theta(1)$, then BVP (1.1) reduces to the following nth-order three-point boundary value problem

$$\begin{cases} u^{(n)}(t) + \lambda f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = 0, \quad u'(0) = 0, \cdots, u^{(n-2)}(0) = 0, \quad u(1) = \beta u(\eta), \end{cases} \tag{1.3}$$

For the case in which $\lambda = 1$ and $f(t, u(t)) = a(t)f(u(t))$. Eloe and Ahmad [6] established the existence of one positive solution for BVPs (1.3) by applying the fixed point theorem on cones duo to Krasnosel'skii and Guo.

Motivated by [2] and [5], the purpose of this paper is to establish the existence of positive solutions for BVP (1.1) by using Krasnosel'skii fixed point theorem in cones.

The rest of the paper is organized as follows. In Section 2, we give some Lemmas. In Section 3, the main result of this paper for the existence of at least one positive solution of BVP (1.1) is established.

2. Lemmas

Lemma 2.1. *If* $n \geq 2$ *and* $y(t) \in C[0,1]$*, then the boundary value problem*

$$\begin{cases} u^{(n)}(t) + y(t) = 0, \quad 0 < t < 1, \\ u(0) = h\left(\int_0^1 u(t)d\zeta(t)\right), \quad u'(0) = 0, \cdots, u^{(n-2)}(0) = 0, \quad u(1) = g\left(\int_0^1 u(t)d\theta(t)\right), \end{cases} \tag{2.1}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s)y(s)ds + t^{n-1}g\left(\int_0^1 u(s)d\theta(s)\right) + (1-t^{n-1})h\left(\int_0^1 u(s)d\zeta(s)\right),$$

where

$$G(t,s) = \begin{cases} \dfrac{(1-s)^{n-1}t^{n-1} - (t-s)^{n-1}}{(n-1)!}, & 0 \leq s \leq t \leq 1, \\ \dfrac{(1-s)^{n-1}t^{n-1}}{(n-1)!}, & 0 \leq t \leq s \leq 1. \end{cases}$$

Proof. The proof follows by direct calculations, we omitted here.

Lemma 2.2. $G(t,s)$ *has the following properties*

(i) $0 \leq G(t,s) \leq k(s)$, $t, s \in [0,1]$*, where*

$$k(s) = \frac{s(1-s)^{n-1}}{(n-2)!};$$

(ii) $G(t,s) \geq \gamma(t)k(s)$, $t, s \in [0,1]$*, where*

$$\gamma(t) = \min\left\{\frac{t^{n-1}}{n-1}, \frac{(1-t)t^{n-2}}{n-1}\right\} = \begin{cases} \frac{t^{n-1}}{n-1}, & 0 \leq t \leq \frac{1}{2}, \\ \frac{(1-t)t^{n-2}}{n-1}, & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Proof. It is obvious that $G(t,s)$ is nonnegative. Moreover,

$$G(t,s) = \begin{cases} \dfrac{(t(1-s))^{n-1} - (t-s)^{n-1}}{(n-1)!}, & 0 \le s \le t \le 1, \\ \dfrac{(1-s)^{n-1} t^{n-1}}{(n-1)!}, & 0 \le t < s \le 1 \end{cases}$$

$$= \frac{1}{(n-1)!} \begin{cases} s(1-t)[(t(1-s))^{n-2} + (t(1-s))^{n-3}(t-s) + \cdots \\ +(t(1-s))(t-s)^{n-3} + (t-s)^{n-2}], & 0 \le s \le t \le 1, \\ (1-s)^{n-1} t^{n-1}, & 0 \le t < s \le 1 \end{cases}$$

$$\le \frac{1}{(n-1)!} \begin{cases} (n-1)s(1-s)^{n-1}, & 0 \le s \le t \le 1, \\ (1-s)^{n-1} s^{n-1}, & 0 \le t < s \le 1 \end{cases}$$

$$\le \frac{s(1-s)^{n-1}}{(n-2)!} = k(s), \quad t, s \in [0,1],$$

that is, (i) holds.

If $s = 0$ or $s = 1$, it is easy to know that (ii) holds. If $s \in (0,1)$ and $t \in [0,1]$, then we have

$$\frac{G(t,s)}{k(s)} = \begin{cases} \dfrac{(1-s)^{n-1} t^{n-1} - (t-s)^{n-1}}{(n-1)s(1-s)^{n-1}}, & s \le t, \\ \dfrac{(1-s)^{n-1} t^{n-1}}{(n-1)s(1-s)^{n-1}}, & t < s \end{cases}$$

$$= \begin{cases} \dfrac{s(1-t)[(t(1-s))^{n-2} + (t(1-s))^{n-3}(t-s) + \cdots + (t-s)^{n-2}]}{(n-1)s(1-s)^{n-1}}, & s \le t, \\ \dfrac{t^{n-1}}{(n-1)s}, & t < s \end{cases}$$

$$\ge \begin{cases} \dfrac{s(1-t)t^{n-2}(1-s)^{n-2}}{(n-1)s(1-s)^{n-1}}, & s \le t, \\ \dfrac{t^{n-1}}{(n-1)s}, & t < s \end{cases}$$

$$\ge \begin{cases} \dfrac{(1-t)t^{n-2}}{n-1}, & s \le t, \\ \dfrac{t^{n-1}}{n-1}, & t < s, \end{cases}$$

which implies that

$$\frac{G(t,s)}{k(s)} \ge \gamma(t), \quad \text{for } s \in (0,1) \text{ and } t \in [0,1].$$

Thus, (ii) holds.

Lemma 2.3 [8]. *Let $E = (E, \|\cdot\|)$ be a Banach space and let $K \subset E$ be a cone in E. Assume Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$ and let*

$A : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be a continuous and completely continuous. In addition, suppose either

(1) $\|Au\| \le \|u\|$, $u \in K \cap \partial\Omega_1$, *and* $\|Au\| \ge \|u\|$, $u \in K \cap \partial\Omega_2$, *or*
(2) $\|Au\| \ge \|u\|$, $u \in K \cap \partial\Omega_1$, *and* $\|Au\| \le \|u\|$, $u \in K \cap \partial\Omega_2$.

Then A has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Main result

Let $E = C[0,1]$ be the real Banach space with the maximum norm, and define the cone $P \subset E$ by

$$P = \{u \in E : u(t) \ge \gamma(t)\|u\|, \quad t \in [0,1]\},$$

where $\gamma(t)$ is as in Lemma 2.2.

For convenience, let

$$\Omega_1 = \{u \in E : \|u\| < r_1\}, \quad \Omega_2 = \{u \in E : \|u\| < r_2\},$$

$$p(t) = \int_0^1 G(t,s)ds = \frac{t^{n-1}(1-t)}{n!}, \quad u_0(t) = \lambda M p(t), \quad 0 \le t \le 1, \tag{3.1}$$

$$C_1 = \int_0^1 k(s)ds, \quad C_2 = \int_\tau^{1-\tau} k(s)ds,$$
$$\gamma^* = \max_{0\le t\le 1} \gamma(t) = \max\left\{\frac{1}{(n-1)2^{n-1}}, \frac{(n-1)^{n-3}}{n^{n-1}}\right\}. \tag{3.2}$$

Theorem 3.1. *Assume that* (B_1), (B_2) *and* (B_3) *hold. Suppose the following conditions are satisfied:*

(B_4) *there exist $\phi, \psi \in C([0,\infty),[0,\infty))$ are nondecreasing functions with $\phi(u), \psi(u) > 0$ for $u > 0$, such that*

$$f(t,u(t)) + M \le \psi(u), \ (t,u) \in [0,1] \times [0,\infty), \tag{3.3}$$
$$f(t,u(t)) + M \ge \phi(u), \ (t,u) \in [\tau, 1-\tau] \times [0,\infty) \ (\text{ we choose and fix } \tau \in (0, \tfrac{1}{2}) \text{ }), \tag{3.4}$$

(B_5) *there exist two positive numbers r_1, r_2 with $r_1 > \max\{r_2, g(r_1\theta(1)) + h(r_1\zeta(1))\}$ and $r_2 > \lambda M(n-1)/n!$ such that*

$$\frac{r_2}{\phi(\varepsilon r_2\tau^{n-1}/(n-1))} \le \lambda\gamma^* C_2, \quad \frac{r_1 - g(r_1\theta(1)) - h(r_1\zeta(1))}{\psi(r_1)} \ge \lambda C_1, \tag{3.5}$$

here $\varepsilon > 0$ *is any constant (choose and fix it) so that* $1 - \lambda M(n-1)(1-\tau)/r_2 n! \ge \varepsilon$ *(note* ε *exists since* $r_2 > \lambda M(n-1)/n! > \lambda M(n-1)(1-\tau)/n!$*) and* C_1, C_2, γ^* *are as in* (3.2)*, respectively.*

Then BVP (1.1) *has at least one positive solution.*

Proof. Consider the following boundary value problems

$$\begin{cases} u^n(t) + \lambda f^*(t, u(t) - u_0(t)) = 0, \quad 0 < t < 1, \\ u(0) = h^*\left(\int_0^1 (u(t) - u_0(t))d\zeta(t)\right), \quad u'(0) = 0, \cdots, u^{(n-2)}(0) = 0, \\ u(1) = g^*\left(\int_0^1 (u(t) - u_0(t))d\theta(t)\right), \end{cases} \tag{3.6}$$

where $u_0(t)$ is as in (3.1),

$$f^*(t, u(t) - u_0(t)) = \begin{cases} f(t, u(t) - u_0(t)) + M, & u(t) - u_0(t) \ge 0, \\ f(t, 0) + M, & u(t) - u_0(t) < 0, \end{cases} \tag{3.7}$$

$$g^*\left(\int_0^1 (u(t) - u_0(t))d\theta(t)\right) = \begin{cases} g\left(\int_0^1 (u(t) - u_0(t))d\theta(t)\right), & u(t) - u_0(t) \ge 0, \\ 0, & u(t) - u_0(t) < 0, \end{cases} \tag{3.8}$$

and

$$h^*\left(\int_0^1 (u(t) - u_0(t))d\zeta(t)\right) = \begin{cases} h\left(\int_0^1 (u(t) - u_0(t))d\zeta(t)\right), & u(t) - u_0(t) \ge 0, \\ 0, & u(t) - u_0(t) < 0. \end{cases} \tag{3.9}$$

By Lemma 2.1, this problem is equivalent to the integral equation

$$u(t) = \lambda \int_0^1 G(t,s) f^*(s, u(s) - u_0(s))ds + t^{n-1} g^*\left(\int_0^1 (u(s) - u_0(s))d\theta(s)\right)$$
$$+(1 - t^{n-1})h^*\left(\int_0^1 (u(s) - u_0(s))d\zeta(s)\right)$$

We define the operator T as follows

$$(Tu)(t) = \lambda \int_0^1 G(t,s) f^*(s, u(s) - u_0(s))ds + t^{n-1} g^*\left(\int_0^1 (u(s) - u_0(s))d\theta(s)\right)$$
$$+(1 - t^{n-1})h^*\left(\int_0^1 (u(s) - u_0(s))d\zeta(s)\right).$$

Next, we claim $T(P) \subset P$. In fact, for each $u \in P$, by view of Lemma 2.2 (i) and Condition (B_3), we obtain

$$
\begin{aligned}
(Tu)(t) &= \lambda \int_0^1 G(t,s) f^*(s, u(s) - u_0(s))ds + t^{n-1} g^* \left(\int_0^1 (u(s) - u_0(s)) d\theta(s) \right) \\
&\quad + (1 - t^{n-1}) h^* \left(\int_0^1 (u(s) - u_0(s)) d\zeta(s) \right) \\
&\le \lambda \int_0^1 k(s) f^*(s, u(s) - u_0(s))ds + g^* \left(\int_0^1 (u(s) - u_0(s)) d\theta(s) \right) \\
&\quad + h^* \left(\int_0^1 (u(s) - u_0(s)) d\zeta(s) \right),
\end{aligned}
$$

which implies

$$
\begin{aligned}
\|Tu\| &\le \lambda \int_0^1 k(s) f^*(s, u(s) - u_0(s))ds + g^* \left(\int_0^1 (u(s) - u_0(s)) d\theta(s) \right) \\
&\quad + h^* \left(\int_0^1 (u(s) - u_0(s)) d\zeta(s) \right).
\end{aligned} \tag{3.10}
$$

On the other hand, by (3.10) and Lemma 2.2 (ii), we have

$$
\begin{aligned}
(Tu)(t) &= \lambda \int_0^1 G(t,s) f^*(s, u(s) - u_0(s))ds + t^{n-1} g^* \left(\int_0^1 (u(s) - u_0(s)) d\theta(s) \right) \\
&\quad + (1 - t^{n-1}) h^* \left(\int_0^1 (u(s) - u_0(s)) d\zeta(s) \right) \\
&\ge \lambda \gamma(t) \int_0^1 k(s) f^*(s, u(s) - u_0(s))ds + \frac{t^{n-1}}{n-1} g^* \left(\int_0^1 (u(s) - u_0(s)) d\theta(s) \right) \\
&\quad + (1-t)(1 + t + \cdots + t^{n-3} + t^{n-2}) h^* \left(\int_0^1 (u(s) - u_0(s)) d\zeta(s) \right) \\
&\ge \lambda \gamma(t) \int_0^1 k(s) f^*(s, u(s) - u_0(s))ds + \gamma(t) g^* \left(\int_0^1 (u(s) - u_0(s)) d\theta(s) \right) \\
&\quad + (1-t) t^{n-2} h^* \left(\int_0^1 (u(s) - u_0(s)) d\zeta(s) \right) \\
&\ge \lambda \gamma(t) \int_0^1 k(s) f^*(s, u(s) - u_0(s))ds + \gamma(t) g^* \left(\int_0^1 (u(s) - u_0(s)) d\theta(s) \right) \\
&\quad + \frac{(1-t) t^{n-2}}{n-1} h^* \left(\int_0^1 (u(s) - u_0(s)) d\zeta(s) \right) \\
&\ge \gamma(t) \left\{ \lambda \int_0^1 k(s) f^*(s, u(s) - u_0(s))ds + g^* \left(\int_0^1 (u(s) - u_0(s)) d\theta(s) \right) \right.
\end{aligned}
$$

$$
\begin{aligned}
&+h^*\left(\int_0^1 (u(s)-u_0(s))d\zeta(s)\right)\Bigg\}\\
&\geq \gamma(t)\|Tu\|, \quad t\in[0,1].
\end{aligned}
$$

which implies that $T(P)\subset P$. Similar to the proof of Remark 2.1 [3], it is easy to prove that the operator $T : P\cap(\overline{\Omega}_1\setminus\Omega_2)\to P$ is continuous and compact.

Let $u\in P\cap\partial\Omega_2$, then $\|u\| = r_2$ and $u(t)\geq\gamma(t)\|u\| = \gamma(t)r_2$ for $t\in[0,1]$. Let ε,τ be as in the statement of Theorem 3.1. Thus, we have

$$
\begin{aligned}
u(t)-u_0(t) = u(t)-\lambda Mp(t) &\geq \frac{r_2t^{n-1}}{n-1}-\frac{\lambda Mt^{n-1}(1-t)}{n!}\\
&= \frac{r_2t^{n-1}}{n-1}\left(1-\frac{\lambda M(1-t)(n-1)}{n!r_2}\right)\geq \frac{r_2t^{n-1}}{n-1}\left(1-\frac{\lambda M(1-\tau)(n-1)}{n!r_2}\right)\\
&\geq \frac{\varepsilon r_2\tau^{n-1}}{n-1}, \quad t\in[\tau,\frac{1}{2}], \qquad (3.11)
\end{aligned}
$$

and we get for $t\in[\frac{1}{2},1-\tau]$ that

$$
\begin{aligned}
u(t)-u_0(t) = u(t)-\lambda Mp(t) &\geq \frac{r_2t^{n-2}(1-t)}{n-1}-\frac{\lambda Mt^{n-1}(1-t)}{n!}\\
&= \frac{r_2t^{n-2}(1-t)}{n-1}\left(1-\frac{\lambda M(n-1)t}{n!r_2}\right)\\
&\geq \frac{r_2t^{n-2}(1-t)}{n-1}\left(1-\frac{\lambda M(n-1)(1-\tau)}{n!r_2}\right)\\
&\geq \frac{\varepsilon r_2\tau^{n-1}}{n-1}. \qquad (3.12)
\end{aligned}
$$

Clearly, from (3.11) and (3.12), we obtain

$$
u(t)-u_0(t)\geq\frac{\varepsilon r_2\tau^{n-1}}{n-1}, \quad t\in[\tau,1-\tau],
$$

which yields

$$
\begin{aligned}
f^*(t,u(t)-u_0(t)) = f(t,u(t)-u_0(t))+M &\geq \phi(u(t)-u_0(t))\\
&\geq \phi\left(\frac{\varepsilon r_2\tau^{n-1}}{n-1}\right), \quad t\in[\tau,1-\tau]. \qquad (3.13)
\end{aligned}
$$

By (3.5), (3.13) and Lemma 2.2 (ii), we get

$$\begin{aligned}\|Tu\| &= \max_{0\le t\le 1}\left\{\lambda\int_0^1 G(t,s)f^*(s,u(s)-u_0(s))ds + t^{n-1}g^*\left(\int_0^1 (u(s)-u_0(s))d\theta(s)\right)\right.\\ &\left.+(1-t^{n-1})h^*\left(\int_0^1 (u(s)-u_0(s))d\zeta(s)\right)\right\}\\ &\ge \lambda\max_{0\le t\le 1}\int_0^1 G(t,s)f^*(s,u(s)-u_0(s))ds\\ &\ge \lambda\max_{0\le t\le 1}\gamma(t)\int_\tau^{1-\tau} k(s)f^*(s,u(s)-u_0(s))ds\\ &\ge \lambda\max_{0\le t\le 1}\gamma(t)\phi\left(\frac{\varepsilon r_2\tau^{n-1}}{n-1}\right)\int_\tau^{1-\tau} k(s)ds\\ &= \lambda\gamma^*\phi\left(\frac{\varepsilon r_2\tau^{n-1}}{n-1}\right)C_2\\ &\ge r_2 = \|u\|.\end{aligned}$$

Thus,

$$\|Tu\| \ge \|u\|, \quad u\in P\cap\partial\Omega_2. \tag{3.14}$$

On the other hand, let $u\in P\cap\partial\Omega_1$, so $\|u\| = r_1$, and $u(t)\ge\gamma(t)r_1$ for $t\in[0,1]$. Thus, in view of (3.3) and ψ, g are nondecreasing, for each $u\in P\cap\partial\Omega_1$, we have

$$\begin{aligned}f^*(t,u(t)-u_0(t)) &= f(t,u(t)-u_0(t)) + M \le \psi(u(t)-u_0(t))\\ &\le \psi(u(t)) \le \varphi(r_1), \quad u(t)-u_0(t)\ge 0,\end{aligned}$$

$$f^*(t,u(t)-u_0(t)) = f(t,0)+M \le \psi(0) \le \psi(u(t)) \le \psi(r_1), \quad u(t)-u_0(t) < 0,$$

$$\begin{aligned}g^*\left(\int_0^1 (u(t)-u_0(t))d\theta(t)\right) &= g\left(\int_0^1 (u(t)-u_0(t))d\theta(t)\right) \le g\left(\int_0^1 u(t)d\theta(t)\right)\\ &\le g\left(\int_0^1 r_1 d\theta(t)\right) = g(r_1\theta(1)), \quad u(t)-u_0(t)\ge 0,\end{aligned}$$

and

$$g^*\left(\int_0^1 (u(t)-u_0(t))d\theta(t)\right) = 0 \le g(r_1\theta(1)), \quad u(t)-u_0(t) < 0,$$

which implies

$$f^*(t,u(t)-u_0(t)) \le \psi(r_1), \quad g^*\left(\int_0^1 (u(t)-u_0(t))d\theta(t)\right) \le g(r_1\theta(1)). \tag{3.15}$$

Similarly, we get

$$h^*\left(\int_0^1 (u(t)-u_0(t))d\zeta(t)\right) \le h(r_1\zeta(1)). \tag{3.16}$$

Form (3.5), (3.15) and (3.16) and Lemma 2.2 (i), we obtain for $u \in P \cap \Omega_1$ that

$$\begin{aligned}
\|Tu\| &= \max_{0\le t\le 1}\left\{\lambda\int_0^1 G(t,s)f^*(s,u(s)-u_0(s))ds + t^{n-1}g^*\left(\int_0^1 (u(s)-u_0(s))d\theta(s)\right)\right.\\
&\quad \left. +(1-t^{n-1})h^*\left(\int_0^1 (u(s)-u_0(s))d\zeta(s)\right)\right\}\\
&\le \lambda\int_0^1 k(s)f^*(s,u(s)-u_0(s))ds + g^*\left(\int_0^1 (u(s)-u_0(s))d\theta(s)\right)\\
&\quad +h^*\left(\int_0^1 (u(s)-u_0(s))d\zeta(s)\right)\\
&\le \lambda C_1\psi(r_1) + g(r_1\theta(1)) + h(r_1\zeta(1))\\
&\le r_1 = \|u\|.
\end{aligned}$$

Hence,

$$\|Tu\| \le \|u\|, \quad u \in P\cap\partial\Omega_1. \tag{3.17}$$

Therefore, from (3.14), (3.17) and Lemma 2.3, it follows that T has at least one fixed point $u^* \in P\cap(\overline{\Omega}_1\setminus\Omega_2)$ with $r_2 \le \|u^*\| \le r_1$. This implies that the BVP (3.6) has at least one solution $u^* \in P\cap(\overline{\Omega}_1\setminus\Omega_2)$ satisfying $r_2 \le \|u^*\| \le r_1$.

Let $u_*(t) = u^*(t) - u_0(t)$ for $0 < t < 1$, then $r_2 \le \|u_* + u_0\| \le r_1$. This together with $r_2 > \frac{\lambda M(n-1)}{n!}$ yields

$$\begin{aligned}
u_*(t) &= [u_*(t)+u_0(t)] - u_0(t) \ge \gamma(t)\|u_*(t)+u_0(t)\| - \lambda Mp(t)\\
&= \begin{cases} \frac{t^{n-1}}{n-1}\|u_*(t)+u_0(t)\| - \frac{\lambda Mt^{n-1}(1-t)}{n!}, & 0<t\le\frac12,\\ \frac{t^{n-2}(1-t)}{n-1}\|u_*(t)+u_0(t)\| - \frac{\lambda Mt^{n-1}(1-t)}{n!}, & \frac12\le t<1,\end{cases}\\
&\ge \begin{cases} \frac{r_2t^{n-1}}{n-1} - \frac{\lambda Mt^{n-1}}{n!}, & 0<t\le\frac12,\\ \frac{r_2t^{n-2}(1-t)}{n-1} - \frac{\lambda Mt^{n-1}(1-t)}{n!}, & \frac12\le t<1,\end{cases}\\
&= \begin{cases} \frac{t^{n-1}}{n-1}\left(r_2 - \frac{\lambda M(n-1)}{n!}\right), & 0<t\le\frac12,\\ \frac{t^{n-2}(1-t)}{n-1}\left(r_2 - \frac{\lambda M(n-1)}{n!}\right), & \frac12\le t<1\end{cases}
\end{aligned}$$

which implies

$$u_*(t) > 0, \qquad \text{for } t \in (0,1). \tag{3.18}$$

Then, by (3.7)-(3.9) and (3.18), we get

$$\begin{aligned}
u_*(t) + u_0(t) = u^*(t) = (Tu^*)(t) \\
&= \int_0^1 G(t,s) f^*(s, u^*(s) - u_0(s))ds + t^{n-1} g^*\left(\int_0^1 (u^*(s) - u_0(s)) d\theta(s)\right) \\
&\quad + (1 - t^{n-1}) h^*\left(\int_0^1 (u^*(s) - u_0(s)) d\zeta(s)\right) \\
&= \int_0^1 G(t,s) f^*(t, u_*(s))ds + u_0(t) + t^{n-1} g^*\left(\int_0^1 u_*(s) d\theta(s)\right) \\
&\quad + (1 - t^{n-1}) h^*\left(\int_0^1 u_*(s) d\zeta(s)\right). \\
&= \int_0^1 G(t,s) f(t, u_*(s))ds + u_0(t) + t^{n-1} g\left(\int_0^1 u_*(s) d\theta(s)\right) \\
&\quad + (1 - t^{n-1}) h\left(\int_0^1 u_*(s) d\zeta(s)\right).
\end{aligned}$$

It follows that

$$\begin{aligned}
u_*(t) = \int_0^1 G(t,s) f(t, u_*(s))ds + t^{n-1} g\left(\int_0^1 u_*(s) d\theta(s)\right) \\
+ (1 - t^{n-1}) h\left(\int_0^1 u_*(s) d\zeta(s)\right), \quad 0 < t < 1.
\end{aligned} \tag{3.19}$$

Thus, by (3.18) and (3.19), we assert that u_* is a positive solution of the BVP (1.1). The proof is complete.

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