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AN EXISTENCE RESULT FOR NEUTRAL FUNCTIONAL DIFFERENTIAL INCLUSIONS IN A BANACH SPACE

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ABSTRACT. In this paper we prove the existence of mild solutions for semilinear neutral functional differential inclusions with unbounded linear part generating a noncompact semigroup in a Banach space. This work generalizes the result given in [4].

1. INTRODUCTION

Semilinear neutral functional differential inclusion has been the object of many studies by many researchers in the recent years. We only mention the works of some authors ([1], [2], [6]). The method which consists in defining an integral multioperator for which fixed points set coincides with the solutions set of differential inclusion has been often applied to existence problems.

Our aim in this paper is to give an existence result for initial value problems for first order semilinear neutral functional differential inclusions in a separable Banach space E of the form:

$$\frac{d}{dt}\left[x(t) - h(t, x_t)\right] \in Ax(t) + F(t, x_t), \qquad t \in [0, T], \tag{1.1}$$

$$x(t) = \varphi(t), \ t \in [-r, 0], \tag{1.2}$$

where $A : D(A) \subset E \to E$ is the infinitesimal generator of an uniformly bounded analytic semigroup of linear operators $\{S(t)\}_{t\geq 0}$ on a separable Banach space E; the multimap $F : [0, T] \times C([-r, 0], E) \to P(E)$ and $h : [0, T] \times C([-r, 0], E) \to E$, are given functions, $0 < r < \infty, \varphi \in C([-r, 0], E)$, where $P(E)$ denotes the class of all nonempty subsets of E, and $C([-r, 0], E)$ denotes the space of continuous functions from $[-r, 0]$ to E.

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For any continuous function x defined on $[-r, T]$ and any $t \in [0, T]$, we denote by x_t the element of $C([-r, 0], E)$ defined by

$$x_t(\theta) = x(t+\theta), \theta \in [-r, 0].$$

For any $u \in C([-r, 0], E)$ the norm $\|.\|_{C([-r,0],E)}$ of u is defined by

$$\|u\|_{C([-r,0],E)} = \sup\{\|u(s)\| : s \in [-r, 0]\}.$$

The function $x_t(.)$ represents the history of the state from time $t-r$, up the present time t.

In [8] using topological methods of multivalued analysis, existence results for semilinear inclusions with unbounded linear part generating a noncompact semigroup in Banach space were given. In this paper, using the method of fractional power of closed operators theory and by giving a special measure of noncompactness, we extend this line of attack to the problem (1.1)-(1.2). More precisely in section 3 we give the measure of noncompactness for which the integral multioperator is condensing, this will allow us to give an existence result for the problem (1.1)-(1.2), and by using the properties of fixed points set of condensing operators we deduce that the mild solutions set is compact.

2. Preliminaries

Along this work, E will be a separable Banach space provided with norm $\|.\|$, $A : D(A) \subset E \to E$ is the infinitesimal generator of an uniformly bounded analytic semigroup of linear operators $\{S(t)\}_{t\geq 0}$ in E. We will assume that $0 \in \rho(A)$ and that $\|S(t)\| \leq M$ for all $t \in [0, T]$. Under these conditions it is possible to define the fractional power $(-A)^\alpha, 0 < \alpha \leq 1$, as closed linear operator on its domain $D(-A)^\alpha$. Furthermore, $D(-A)^\alpha$ is dense in E and the function $\|x\|_\alpha = \|(-A)^\alpha x\|$ defines a norm in $D(-A)^\alpha$. If X_α is the space $D(-A)^\alpha$ endowed with the norm $\|.\|_\alpha$, then X_α is a Banach space and there exists $c_\alpha > 0$ such that $\|(-A)^\alpha S(t)\| \leq \frac{c_\alpha}{t^\alpha}$, for $t > 0$. Also the inclusion $X_\alpha \hookrightarrow X_\beta$ for $0 < \beta \leq \alpha \leq 1$ is continuous.

For additional details respect of fractional power of a linear operator and semigroup theory, we refer the reader to [11] and [16] ..

Let X be a Banach space and $(Y, \geq)$ a partially ordered set. A function $\Psi : P(X) \to Y$ is called a measure of noncompactness in X if $\Psi(\Omega) = \Psi(\bar{co}\Omega)$ for every $\Omega \in P(X)$, where $\bar{co}\Omega$ denote the closed convex hull of Ω.

A measure of noncompactness Ψ is called:

(i) nonsingular if $\Psi(\{a\} \cup \Omega) = \Psi(\Omega)$ for every $a \in X$, $\Omega \in P(X)$;

(ii) monotone if $\Omega_0, \Omega_1 \in P(X)$ and $\Omega_0 \subseteq \Omega_1$ imply $\Psi(\Omega_0) \leq \Psi(\Omega_1)$;

(iii) real if $Y = [0, \infty]$ with the natural ordering, and $\Psi(\Omega) < +\infty$ for every bounded set $\Omega \in P(X)$.

If Y is a cone in a Banach space we will say that the measure of noncompactness Ψ is regular if $\Psi(\Omega) = 0$ is equivalent to the relative compactness of Ω.

One of most important examples of measure of noncompactness possessing all these properties, is the Hausdorff measure of noncompactness

$$\chi(\Omega) = \inf\{\varepsilon > 0; \Omega \text{ has a finite } \varepsilon\text{-net in } X\}$$

Let $K(X)$ denotes the class of compact subsets of X, $Kv(X)$ denotes the class of compact convex subsets of X, and (Q, d) a metric space.

A multimap $G : Z \to K(X)$ is called $\Psi-$condensing if for every bounded set $\Omega \subset E$, that is not relatively compact we have $\Psi(G(\Omega)) \ngeq \Psi(\Omega)$, where $Z \subset X$.

A multivalued map $G : X \to K(Q)$ is u.s.c at a point $x \in X$, if for every $\varepsilon > 0$ there exists neighborhood $V(x)$ such that $G(x') \subset W_\varepsilon(G(x))$, for every $x' \in V(x)$. Here by $W_\varepsilon(A)$ we denote the ε-neighborhood of a set A, i.e., $W_\varepsilon(A) = \{y \in Y : d(y, A) < \varepsilon\}$, where $d(y, A) = \inf_{x \in A} d(x, y)$.

A multimap $G : X \to P(Q)$ is said to be quasicompact if its restriction to every compact subset $A \subset X$ is compact.

A multifunction $\mathcal{F} : [0, T] \to K(X)$ is said to be strongly measurable if there exists a sequence $\{\mathcal{F}_n\}_{n=1}^{\infty}$ of step multifunctions such that $Haus\,(\mathcal{F}(t), \mathcal{F}_n(t)) \to 0$ as $n \to \infty$ for $\mu - a.e.\ t \in [0, T]$ where μ denotes a Lebesgue measure on $[0, T]$ and $Haus$ is the Hausdorff metric on $K(X)$. Every strongly measurable multifunction $\mathcal{F}$ admits a strongly measurable selection $g : [0, T] \to X$, i.e., $g(t) \in \mathcal{F}(t)$ for a.e. $t \in [0, T]$.

Let $L^1([0, T], X)$ denotes the space of all Bochner summable functions

A multifunction $\mathcal{F} : [0, T] \to K(X)$ is said to be

(i) integrable provided it has a summable selection $g \in L^1([0, T], X)$,

(ii) integrably bounded if there exists a summable function $q(.) \in L^1([0, T], X)$ such that $\|\mathcal{F}(t)\| = \sup\{\|y\| : y \in \mathcal{F}(t)\} \leq q(t)$ for a.e. $t \in [0, T]$.

A sequence $\{f_n\}_{n=1}^{\infty} \subset L^1([0, T], X)$ is semicompact if

(i) it is integrably bounded: $\|f_n(t)\| \leq q(t)$ for a.e. $t \in [0, T]$ and for every $n \geq 1$, where $q(.) \in L^1([0, T], \mathbb{R}^+)$

(ii) the set $\{f_n(t)\}_{n=1}^{\infty}$is relatively compact for almost every $t \in [0,T]$.

Any semicompact sequence in $L^1([0,T],X)$ is weakly compact in $L^1([0,T],X)$.

For all this definitions see for example [8].

In the sequel, $C([-r,T],E)$ denotes the space of continuous functions from $[-r,T]$ to E endowed with the supremum norm. For any $x \in C([-r,T],E)$,

$$\|x\|_{C([-r,T],E)} = \sup\{\|x(t)\| : t \in [-r,T]\}.$$

In section 3 we establish an existence result to the problem (1.1)-(1.2) using the following well known results. (See [8]).

Lemma 1. *Let E be a Banach space and $\phi : E \to E$ a bounded linear operator. Then for every bounded subset $\Omega \subset E$*

$$\chi(\phi(\Omega)) \leq \|\phi\|\,\chi(\Omega).$$

Lemma 2. *Let E be a separable Banach space and $G : [0,T] \to P(E)$ an integrable, integrably bounded multifunction such that*

$$\chi\left(G(t)\right) \leq q(t)$$

for a.e. $t \in [0,T]$ where $q \in L^1_+([0,T])$. Then for all $t \in [0,T]$

$$\int_0^t \chi\left(G(s)\right) ds \leq \int_0^t q(s)ds.$$

Lemma 3. *Let E be a separable Banach space and J an operator*

$$J : L^1([0,T],E) \to C([0,T],E)$$

which satisfies the following conditions:

J$_1$) *There exists $D > 0$ such that*

$$\|Jf(t) - J(g)(t)\| \leq D\int_0^t \|f(s) - g(s)\|\,ds,\ \ 0 \leq t \leq T$$

for every $f, g \in L^1([0,T],E)$.

J$_2$) *For any compact $K \subset E$ and sequence $\{f_n\}_{n=1}^{\infty} \subset L^1([0,T],E)$ such that $\{f_n(t)\}_{n=1}^{\infty} \subset K$ for a.e. $t \in [0,T]$ the weak convergence $f_n \underset{w}{\to} f_0$ implies $J(f_n) \to J(f_0)$.*

Then,

(i) If the sequence of functions $\{f_n\}_{n=1}^{\infty} \subset L^1([0,T],E)$ is such that $\|f_n(t)\| \leq \pi(t)$ a.e $t \in [0,T]$, for all $n = 1,2,\ldots$, and $\chi(\{f_n\}_{n=1}^{\infty}) \leq \zeta(t)$ a.e. $t \in [0,T]$, where $\pi, \zeta \in L^1_+([0,T]$, then

$$\chi(\ \{J(f_n)(t)\}_{n=1}^{\infty}) \leq D\int_0^t \zeta(s)ds.$$

for all $t \in [0,T]$.

(ii) For every semicompact sequence $\{f_n\}_{n=1}^{\infty} \subset L^1([0,T];E)$ the sequence $\{J(f_n)\}_{n=1}^{\infty}$ is relatively compact in $C([0,T];E)$, and; moreover, if $f_n \underset{w}{\to} f_0$ then $J(f_n) \rightarrow> J(\ f_0)$.

An example of this operator is the Cauchy operator $J : L^1([0,T],E) \to C([0,T],E)$ defined for every $f \in L^1([0,T],E)$ by

$$J(f)(t) = \int_0^t S(t-s)f(s)ds,$$

where $\{S(t)\}_{t\geq 0}$ is a C_0−semigroup in E (see [3]).

Lemma 4. *([8]). If G is a convex closed subset of a Banach space E, and $\Gamma : G \to Kv(G)$ is closed Θ condensing, where Θ is nonsingular measure of noncompactness defined on subsets of G, then $Fix\Gamma \neq \emptyset$.*

Lemma 5. *([8]). Let Z be a closed subset of a Banach space E and $F : Z \to K(E)$ a closed multimap, which is α-condensing on every bounded subset of Z, where α is a monotone measure of noncompactness. If the fixed points set $FixF$ is bounded, then it is compact.*

3. Existence Result

Let us define what we mean by a mild solution of the problem (1.1)-(1.2).

Definition 1. *A function $x \in C([-r,T],E)$ is said to be a mild solution of the problem (1.1)-(1.2) if the function $s \to AS(t-s)h(s,x_s)$ is integrable on $[0,t)$ for each $0 \leq t < T$, and there exists $f \in L^1([0,T],E)$, $f(t) \in F(t,x_t)$ a.e. $t \in [0,T]$, such that*

$$\begin{aligned} x(t) &= S(t)(\varphi(0) - h(0,\varphi)) + \ h(t,x_t) + \int_0^t S(t-s)f(s)ds \\ &\quad + \int_0^t AS(t-s)h(s,x_s)ds, \quad t \in [0,T], \end{aligned}$$

and

$$x(t) = \varphi(t), \quad t \in [-r, 0]\,.$$

To establish our result we consider the following conditions:

Suppose that the multimap $F : [0,T] \times C([-r,0],E) \to Kv(E)$ satisfies the following properties:

$F1)$ the multifunction $F(\cdot, u)$ has a strongly measurable selection for every $u \in C([-r,0],E)$;

$F2)$ the multimap $F : (t, \cdot) \to Kv(E)$ is upper semicontinuous for e.a. $t \in [0,T]$;

$F3)$ there exists a function $\beta \in L^1([0,T], \mathbb{R}^+)$ such that, for every $u \in C([-r,0],E)$,

$$\|F(t,u)\| \le \beta(t)(1 + \|u\|_{C([-r,0],E)}), a.e.\ t \in [0,T];$$

$F4)$ there exists a function $\kappa \in L^1([0,T], \mathbb{R}^+)$ such that for every $\Omega \subset C([-r,0],E)$,

$$\chi(F(t,\Omega)) \le \kappa(t) \sup_{s\in[-r,0]} \chi(\Omega(s)), a.e.\ t \in [0,T],$$

where, for $s \in [-r,0]$, $\Omega(s) = \{u(s); u \in \Omega\}$.

Assume also that

$H)$ there exist constants $d_1, d_2, \omega, \theta \in \mathbb{R}^+$ and $0 < \alpha < 1$, such that h is X_α-valued, and

(i) for every $u \in C([-r,0],E)$, and $t \in [0,T]$

$$\|(-A)^\alpha h(t,u)\| \le d_1 \|u\|_{C([-r,0],E)} + d_2;$$

(ii) for every bounded set $\Omega \subset C([-r,0],E)$ and $t \in [0,T]$,

$$\chi((-A)^\alpha h(t,\Omega)) \le \omega \sup_{s\in[-r,0]} \chi(\Omega(s));$$

(iii) for every $u_1, u_2 \in C([-r,0],E)$ and $t, s \in [0,T]$,

$$\|(-A)^\alpha h(t,u_1) - (-A)^\alpha h(s,u_2)\| \le \theta \|u_1 - u_2\|_{C([-r,0],E)} + \vartheta(|t-s|),$$

where $\vartheta : [0,T] \to \mathbb{R}^+$ is a continuous function, such that $\vartheta(0) = 0$.

We note that from assumptions $(F1) - (F3)$ it follows that the superposition multioperator

$$sel_F : C([-r,T],E) \to P(L^1([0,T],E))$$

defined for $x \in C([-r,T],E)$ by:

$$sel_F(x) = \{f \in L^1([0,T],E), f(t) \in F(t,x_t), a.e.\ t \in [0,T]\}$$

is correctly defined (see [8]) and is weakly closed in the following sense: if the sequences $\{x^n\}_{n=1}^{\infty} \subset C([-r,T],E), \{f_n\}_{n=1}^{\infty} \subset L^1([0,T],E)$, $f_n(t) \in F(t,x_t^n)$, a.e. $t \in [0,T], n \geq 1$ are such that $x^n \to x^0$, $f_n \underset{w}{\to} f_0$., then $f_0(t) \in F(t,x_t^0)$ a.e. $t \in [0,T]$ (see [8]). Since the family $\{S(t)\}_{t\geq 0}$ is an analytic semigroup [16], the operator function
$s \to AS(t-s)$ is continuous in the uniform operator topology on $[0,t)$ which from the estimate

$$\begin{aligned} \|(-A)S(t-s)h(s,x_s)\| &= \left\|(-A)^{1-\alpha}S(t-s)(-A)^{\alpha}h(s,x_s)\right\| \\ &\leq \frac{C_{1-\alpha}}{(t-s)^{1-\alpha}}(d_1\|x_s\|_{C([-r,0],E)}+d_2) \\ &\leq \frac{C_{1-\alpha}}{(t-s)^{1-\alpha}}(d_1\|x\|_{C([-r,T],E)}+d_2) \end{aligned}$$

and the Bochner's theorem implies that $AS(t-s)h(s,x_s)$ is integrable on $[0,t)$.

Now we shall prove our main result.

Theorem 1. *Let the assumptions $(F1)-(F4)$ and (H) be satisfied. If*

$$\left\|(-A)^{-\alpha}\right\| \max\{\omega,\theta,d_1\} < 1$$

then the mild solutions set of the problem (1.1)-(1.2) is a nonempty compact subset of the space $C([-r,T],E)$.

Proof. In the space $C([-r,T],E)$, Let define the operator
$\Gamma : C([-r,T],E) \to P(C([-r,T],E))$ in the following way:

$$\left.\begin{aligned} \Gamma(x)(t) &= \{y \in C([-r,T],E) : y(t)=\varphi(t),\ t\in[-r,0] \quad \text{and} \\ y(t) &= \Upsilon(f)(t)+h(t,x_t)+\int_0^t AS(t-s)h(s,x_s)ds;\ for\ t\ \in[0,T] \end{aligned}\right\}$$

where $f \in sel_F(x)$, and the operator $\Upsilon : L^1([0,T],E) \to C([0,T],E)$ is defined by

$$\Upsilon(f)(t) = S(t)x_0 + \int_0^t S(t-s)f(s)ds,\ t\in[0,T]$$

where $x_0 = \varphi(0) - h(0,\varphi)$.

Remark 1. *It is clear that the operator Γ is well defined, and the fixed points of Γ are mild solutions of the problem (1.1)-(1.2).*

The proof will be given in four steps.

Step 1. The multivalued operator Γ is closed.

The multivalued operator Γ can be written in the form $\Gamma = \sum_1^3 \Gamma_i$ where the operators $\Gamma_i, i = 1, 2, 3$ are defined as follows: the multivalued operator $\Gamma_1 : C([-r, T], E) \to P(C([-r, T], E))$ by

$$\Gamma_1(x)(t) = \left\{ \begin{array}{ll} \varphi(t) - h(0, \varphi), & t \in [-r, 0], \\ \Upsilon(f)(t), & t \in [0, T] \end{array} \right\}$$

where $f \in Sel_F(x)$, the operator $\Gamma_2 : C([-r, T], E) \to C([-r, T], E)$ by

$$\Gamma_2(x)(t) = \left\{ \begin{array}{ll} h(0, \varphi), & t \in [-r, 0], \\ h(t, x_t), & t \in [0, T] \end{array} \right.$$

and the operator $\Gamma_3 : C([-r, T], E) \to C([-r, T], E)$ by

$$\Gamma_3(x)(t) = \left\{ \begin{array}{ll} 0, & t \in [-r, 0] \\ \int_0^t AS(t-s)h(s, x_s)ds, & t \in [0, T]. \end{array} \right.$$

Let $\{x^n\}_{n=1}^\infty, \{z^n\}_{n=1}^\infty, x^n \to x^0, z^n \in \Gamma((x^n), n \geq 1$, and $z^n \to z^0$. Let $\{f_n\}_{n=1}^\infty \subset L^1([0, T], E)$ an arbitrary sequence such that, for $n \geq 1$

$$f_n(t) \in F(t, x_t^n), a.e.\ t \in [0, T],$$

and

$$z^n = \left\{ \begin{array}{ll} \varphi(t), & t \in [-r, 0], \\ \Upsilon(f_n)(t) + h(t, x_t^n) + \int_0^t AS(t-s)h(s, x_s^n)ds, & t \in [0, T]. \end{array} \right.$$

Since $\{S(t)\}_{t \geq 0}$ is a strongly continuous semigroup (see [3]), the operator Υ satisfies the properties (J_1) and (J_2) of Lemma 3, by using hypothesis $(F3)$ we have that sequence $\{f_n\}_{n=1}^\infty$ is integrably bounded. Hypothesis $(F4)$ implies that

$$\chi(\{f_n(t)\}_{n=1}^\infty) \leq \kappa(t)\chi(\{x^n(t)\}_{n=1}^\infty = 0$$

for a.e. $t \in [0, T]$, i.e., the set $\{f_n(t)\}_{n=1}^\infty$ is relatively compact for a.e. $t \in [0, T]$, thus $\{f_n\}_{n=1}^\infty$ is semicompact sequence. Consequently $\{f_n\}_{n=1}^\infty$ is weakly compact in $L^1([0, T], E)$ so we can assume without loss of generality, that $f_n \underset{w}{\to} f_0$.

By applying Lemma 3 , $\Upsilon(f_n) \to \Upsilon(f_0)$ in $C([0, T], E)$. Moreover, by using the fact that the operator sel_F is closed, we have $f_0 \in sel_F(x^0)$.

Consequently

$$z_1^n(t) \quad = \quad \left\{ \begin{array}{ll} \varphi(t) - h(0,\varphi), & t \in [-r,0], \\ \Upsilon(f_n)(t), & t \in [0,T]. \end{array} \right.$$

$$\underset{n\to\infty}{\longrightarrow} \quad z_1^0(t) = \left\{ \begin{array}{ll} \varphi(t) - h(0,\varphi), & t \in [-r,0], \\ \Upsilon(f_0)(t), & t \in [0,T]. \end{array} \right. \tag{3.1}$$

in the space $C([-r,T],E)$, with $f_0 \in sel_F(x^0)$. On the other hand, using $(H)-(iii)$, for $t \in [0,T]$ we get

$$\begin{aligned} \left\| h(t,x_t^n) - h(t,x_t^0) \right\| &\leq \left\| (-A)^{-\alpha} \right\| \left\| (-A)^{\alpha} h(t,x_t^n) - (-A)^{\alpha} h(t,x_t^0) \right\| \\ &\leq \theta \left\| (-A)^{-\alpha} \right\| \left\| x_s^n - x_s^0 \right\|_{C([-r,0],E)} \\ &\leq \theta \left\| (-A)^{-\alpha} \right\| \left\| x^n - x^0 \right\|_{C([-r,T],E)}. \end{aligned}$$

It results that

$$\left\| \Gamma_2(x^0) - \Gamma_2(x^n) \right\|_{C([-r,T],E)} \leq \theta \left\| (-A)^{-\alpha} \right\| \left\| x^n - x^0 \right\|_{C([-r,T],E)} \tag{3.2}$$

Using hypothesis $(H)-(ii)$ and the estimate in the family $\{(-A)^{1-\alpha}S(t)\}_{t>0}$, for any $t \in [0,T]$ we have

$$\begin{aligned} &\left\| \int_0^t \left[AS(t-s)h(s,x_s^n) - AS(t-s)h(s,x_s^0) \right] ds \right\| \\ \leq\ & \int_0^t \left\| AS(t-s)h(s,x_s^n) - AS(t-s)h(s,x_s^0) \right\| ds \\ \leq\ & \theta \left\| (-A)^{-\alpha} \right\| \left\| x^n - x^0 \right\|_{C([-r,T],E)} \int_0^t \left\| (-A)^{1-\alpha} S(t-s) \right\| ds \\ \leq\ & \theta \left\| (-A)^{-\alpha} \right\| \left\| x^n - x^0 \right\|_{C([-r,T],E)} \int_0^t \tfrac{C_{1-\alpha}}{(t-s)^{1-\alpha}} ds \\ \leq\ & \theta \left\| (-A)^{-\alpha} \right\| \frac{C_{1-\alpha} T^{\alpha}}{\alpha} \left\| x^n - x^0 \right\|_{C([-r,T],E)} \end{aligned}$$

Then

$$\begin{aligned} &\left\| \Gamma_3(x^0) - \Gamma_3(x^n) \right\|_{C([-r,T],E)} \\ &\qquad \leq \theta \left\| (-A)^{-\alpha} \right\| \frac{C_{1-\alpha} T^{\alpha}}{\alpha} \left\| x^n - x^0 \right\|_{C([-r,T],E)}. \end{aligned} \tag{3.3}$$

From the inequalities (3.1)-(3.3) follows immediately that $z^n \to z^0$ in the space $C([-r,T],E)$, with

$$z^0(t) = \left\{ \begin{array}{ll} \varphi(t), & t \in [-r,0], \\ \Upsilon(f_0)(t) + h(t,x_t^0) + \int_0^t AS(t-s)h(s,x_s^0)ds, & t \in [0,T]. \end{array} \right\}$$

and $f_0 \in sel_F(x^0)$. Thus $z^0 \in \Gamma(x^0)$ and hence Γ is closed. Now in the space $C([-r,T],E)$ we consider the measure of noncompactness Θ defined in the following way: for every bounded subset $\Omega \subset C([-r,T],E)$

$$\Theta(\Omega) = (\chi(\Omega([-r,0]), \Psi(\Omega), mod_c\Omega)$$

where

$$\Psi(\Omega) = \sup_{t\in[0,T]} \left(e^{-Lt} \sup_{s\in[0,t]} \chi(\Omega(s)) \right)$$

and $mod_c\Omega$ is the module of equicontinuity of the set $\Omega \subset C([-r,T],E)$ given by:

$$mod_c\Omega = \limsup_{\delta\to 0} \max_{x\in\Omega |t_1-t_2|\le\delta} \|x(t_1) - x(t_2)\|$$

and $L > 0$ is chosen so that

$$\begin{aligned}
M \sup_{t\in[0,T]} \int_0^t e^{-L(t-s)}\kappa(s)ds &\le q_1 < 1 \\
\omega \sup_{t\in[0,T]} \int_0^t \frac{C_{1-\alpha}}{(t-s)^{1-\alpha}} e^{-L(t-s)}ds &\le q_2 < 1 \\
d_1 \sup_{t\in[0,T]} \int_0^t \frac{e^{-L(t-s)}}{(t-s)^{1-\alpha}} c_{1-\alpha}ds &\le q_3 < 1 \\
M \sup_{t\in[0,T]} \int_0^t e^{-L(t-s)}\beta(s)ds &\le q_4 < 1
\end{aligned}$$

where M is the constant from the estimation in the family $\{S(t)\}_{t\ge 0}$, the constants d_1, d_2 from $(H)-(i)$, the constant ω from $(H)-(ii)$, the function β from the hypothesis $(F3)$ and the function κ from the hypothesis $(F4)$. From the Arzelá-Ascoli theorem, the measure Θ give a nonsingular and regular measure of noncompactness in $C([-r,T],E)$.

Remark 2. *If $\xi \in L^1([0,T],E)$, it is clear that*

$$\sup_{t\in[0,T]} \int_0^t e^{-L(t-s)}\xi(s)ds \underset{L\to+\infty}{\to} 0.$$

Step 2. The miltioperator Γ is Θ condensing on every bounded subset of $C([-r,T],E)$.

Let $\Omega \subset C([-r,T],E)$ be a bounded subset such that

$$\Theta(\Gamma(\Omega)) \ge \Theta(\Omega), \tag{3.4}$$

where the inequality is taking in the sense of the order in $\mathbb{R}^3$ induced by the positive cone $\mathbb{R}^3_+$. We will show that (3.4) implies that Ω is relatively compact in $C([-r,T],E)$. From the inequality (3.4) follows immediately that

$$\chi(\Omega([-r,0]))=0. \tag{3.5}$$

Indeed, we have

$$\chi(\Gamma\Omega)[-r,0])=\chi\{\varphi([-r,0])\}=0\geq\chi(\Omega[-r,0])\geq 0.$$

Remark that from (3.5) it follows that

$$\sup_{\alpha\in[-r,0]}\chi(\Omega(\alpha))=0.$$

Therefore, for $s\in[0,T]$

$$\begin{aligned}\sup_{\alpha\in[s-r,s]}\chi(\Omega(\alpha)) &\leq \sup_{\alpha\in[-r,0]}\chi(\Omega(\alpha))+\sup_{\alpha\in[0,s]}\chi(\Omega(\alpha))\\ &\leq \sup_{\alpha\in[0,s]}\chi(\Omega(\alpha)).\end{aligned} \tag{3.6}$$

We give now an upper estimate for $\chi(\{f(s),f\in sel_F(\Omega)\}$, for $s\in[0,t]$, $t\leq T$. By using (3.6) and the assumption $(F4)$ we have

$$\begin{aligned}\chi(\{f(s),f\in sel_F(\Omega)\} &\leq \chi(F(s,\Omega_s))\\ &\leq e^{Ls}\kappa(s)e^{-Ls}\sup_{\alpha\in[-r,0]}\chi(\Omega_s(\alpha))\\ &\leq e^{Ls}\kappa(s)e^{-Ls}\sup_{\alpha\in[s-r,s]}\chi(\Omega(\alpha))\\ &\leq e^{Ls}\kappa(s)e^{-Ls}\sup_{\alpha\in[0,s]}\chi(\Omega(\alpha))\\ &\leq e^{Ls}\kappa(s)\sup_{s\in[0,t]}e^{-Ls}\sup_{\alpha\in[0,s]}\chi(\Omega(\alpha)).\\ &\leq e^{Ls}\kappa(s)\sup_{s\in[0,T]}e^{-Ls}\sup_{\alpha\in[0,s]}\chi(\Omega(\alpha))\\ &\leq e^{Ls}\kappa(s)\Psi(\Omega)\end{aligned}$$

Using Lemma 2 with $D=M$, we get

$$\chi(\{\Upsilon(f)(s),f\in sel_F(\Omega)\}\leq M\Psi(\Omega)\int_0^s e^{L\lambda}\kappa(\lambda)d\lambda.$$

Therefore,

$$\sup_{s\in[0,t]}\chi(\{\Upsilon(f)(s),f\in sel_F(\Omega)\}\leq M\Psi(\Omega)\int_0^t e^{L\lambda}\kappa(\lambda)d\lambda.$$

By multiplying both sides with e^{-Lt} and bearing in mind the definition of q_1, we get

$$\begin{aligned}\sup_{t\in[0,T]} e^{-Lt}\sup_{s\in[0,t]}\chi(\{\Upsilon(f)(s), f &\in sel_F(\Omega)\})\\ &= \Psi(\{\Upsilon(f),\ f\in sel_F(\Omega)\})\\ &\leq \Psi(\Omega)M\sup_{t\in[0,T]}\int_0^t e^{-L(t-\lambda)}\kappa(\lambda)d\lambda\\ &\leq q_1\Psi(\Omega) \qquad (3.7)\end{aligned}$$

Since the measure χ is monotone, by using $(H_1)-(iii)$ and Lemma 1, we obtain for $s\in[0,t]$, $t\leq T$

$$\begin{aligned}\chi(h(s,\Omega_s)) &\leq e^{Lt}e^{-Lt}\chi\left((A)^{-\alpha}(-A)^{\alpha}h(s,\Omega_s)\right)\\ &\leq e^{Lt}\omega\left\|(-A)^{-\alpha}\right\| e^{-Lt}\sup_{\alpha\in[0,s]}\chi(\Omega(\alpha))\\ &\leq e^{Lt}\omega\left\|(-A)^{-\alpha}\right\| e^{-Lt}\sup_{\alpha\in[0,t]}\chi(\Omega(\alpha))\\ &\leq e^{Lt}\omega\left\|(-A)^{-\alpha}\right\|\sup_{t\in[0,T]} e^{-Lt}\sup_{\alpha\in[0,t]}\chi(\Omega(\alpha))\\ &\leq e^{Lt}\omega\left\|(-A)^{-\alpha}\right\|\Psi(\Omega).\end{aligned}$$

Then,

$$\sup_{s\in[0,t]}\chi(h(s,\Omega_s))\leq .e^{Lt}\omega\left\|(-A)^{-\alpha}\right\|\Psi(\Omega)$$

By multiplying both sides with e^{-Lt}, we have

$$\sup_{t\in[0,T]} e^{-Lt}\sup_{s\in[0,t]}\chi(h(s,\Omega_s))\leq .\omega\left\|(-A)^{-\alpha}\right\|\Psi(\Omega). \qquad (3.8)$$

The multifunction $G: s\rightarrow AS(t-s)h(s,\Omega_s)$, $s\in[0,t)$ is integrable and integrably bounded. Indeed for any $x\in\Omega$ we have:

$$\begin{aligned}\|(-A)S(t-s)h(s,x_s)\| &= \left\|(-A)^{1-\alpha}S(t-s)(-A)^{\alpha})h(s,x_s)\right\|\\ &\leq \frac{C_{1-\alpha}}{(t-s)^{1-\alpha}}(d_1\|x_s\|_{C([-r,0],E)}+d_2)\\ &\leq \frac{C_{1-\alpha}}{(t-s)^{1-\alpha}}(d_1\|x\|_{C([-r,T],E)}+d_2)\\ &\leq \frac{C_{1-\alpha}}{(t-s)^{1-\alpha}}(d_1\sup_{x\in\Omega}\|x\|_{C([-r,T],E)}+d_2)\end{aligned}$$

Using the assumption $(H)-(ii)$ and Lemma 1, we get

$$\begin{aligned}\chi\left(AS(t-s)h(s,x_s)\right) &= \chi\big((-A)^{1-\alpha}(-A)^{\alpha}S(t-s)h(s,x_s)\big)\\ &\le \big\|(-A)^{1-\alpha}S(t-s)\big\|\,\chi\left((-A)^{\alpha}h(s,x_s)\right)\\ &\le \frac{C_{1-\alpha}}{(t-s)^{1-\alpha}}\omega\sup_{\lambda\in[0,s]}\chi(\Omega(\lambda))\\ &\le \frac{\omega C_{1-\alpha}}{(t-s)^{1-\alpha}}e^{Ls}\sup_{s\in[0,T]}e^{-Ls}\sup_{\lambda\in[0,s]}\chi(\Omega(\lambda))\\ &\le \frac{\omega C_{1-\alpha}}{(t-s)^{1-\alpha}}e^{Ls}\Psi(\Omega)\end{aligned}$$

By lemma 2, we get for every $s\in[0,t]$

$$\begin{aligned}\chi\left(\int_0^s(-A)S(t-\lambda)h(\lambda,x_\lambda)d\lambda\right) &\le \Psi(\Omega)\int_0^s\frac{\omega C_{1-\alpha}}{(t-\lambda)^{1-\alpha}}e^{L\lambda}d\lambda\\ &\le \Psi(\Omega)\int_0^t\frac{\omega C_{1-\alpha}}{(t-\lambda)^{1-\alpha}}e^{L\lambda}d\lambda.\end{aligned}$$

Thus,

$$\sup_{s\in[0,t]}\chi\left(\int_0^s(-A)S(t-\lambda)h(\lambda,x_\lambda)d\lambda\right)\le\Psi(\Omega)\int_0^t\frac{\omega C_{1-\alpha}}{(t-\lambda)^{1-\alpha}}e^{L\lambda}d\lambda$$

By multiplying both sides with e^{-Lt} and bearing in mind the definition

of q_2, we get

$$\begin{aligned}&\sup_{t\in[0,T]}e^{-Lt}\sup_{s\in[0,t]}\chi\left(\int_0^s(-A)S(t-\lambda)h(\lambda,x_\lambda)d\lambda\right)\\ \le\ &\Psi(\Omega)\sup_{t\in[0,T]}\int_0^t\frac{\omega C_{1-\alpha}}{(t-\lambda)^{1-\alpha}}e^{-L(t-\lambda)}d\lambda\\ \le\ &q_2\Psi(\Omega).\end{aligned}\tag{3.9}$$

From the inequalities (3.7)-(3.9), remark 2 and the fact that $\omega \left\| (-A)^{\alpha} \right\| < 1$, we get

$$\begin{aligned}
\Psi(\Gamma(\Omega)) &= \sup_{t\in[0,T]} e^{-Lt} \sup_{s\in[0,t]} \chi \left\{ \Upsilon(f)(s) + h(s,\Omega_s) + \int_0^s AS(t-s)h(\lambda,\Omega_\lambda)d\lambda \right\} \\
&\leq \Psi(\Omega\Big[\sup_{t\in[0,T]} M \int_0^t e^{-L(t-s)}\kappa(s)ds + \left\| (-A)^{\alpha} \right\| \omega \\
&\quad + \sup_{t\in[0,T]} \int_0^t \frac{\omega C_{1-\alpha}}{(t-s)^{1-\alpha}} e^{-L(t-S)} ds \Big] \\
&\leq \Psi(\Omega) \left[q_1 + q_2 + \omega \left\| (-A)^{\alpha} \right\| \right] \\
&< \Psi(\Omega).
\end{aligned}$$

Using the inequality (3.4), the last inequality implies that

$$\Psi(\Omega) = 0. \tag{3.10}$$

We shall give now an upper estimate for $mod_c\Gamma(\Omega)$. We have shown that
$\chi \left\{ \Upsilon(f)(t),\ f \in sel_F(x), x \in \Omega \right\} = 0$, for any $t \in [0,T]$. From the conditions $(F3)$ and $(F4)$ follows that the sequence $\{f \in sel_F(x), x \in \Omega\}$ is semicompact in $L^1([0,T],E)$, and hence the set

$$\{y; y(t) = \Upsilon f(t), t \in [0,T], f \in sel_F(x), x \in \Omega\}$$

is relatively compact in $C([0,T],E)$ (see [3]). Therefore, the set

$$\begin{aligned}
\Gamma_1(\Omega) &= \{ y(t) = \varphi(t) - h(0,\varphi), t \in [-r,0]; \\
&\qquad y(t) = \Upsilon(f)(t), t \in [0,T], f \in sel_F(x), x \in \Omega \}
\end{aligned}$$

is relatively compact in $C([-r,T],E)$. Consequently

$$mod_c\Gamma_1(\Omega) = 0. \tag{3.11}$$

Let $\delta > 0$, and $t, t' \in [0, T]$, such that For $0 \leq t' - t < \delta$, and let $x \in \Omega$, we have

$$\begin{aligned}
\|\Gamma_2(x)(t) - \Gamma_2(x)(t')\| &= \|h(t, x_t) - h(t', x_{t'})\| \leq \\
&\leq \left\|(-A)^{-\alpha}(-A)^{\alpha}h(t, x_t) - (-A)^{-\alpha}(-A)^{\alpha}h(t', x_{t'})\right\| \\
&\leq \left\|(-A)^{-\alpha}\right\| \left[\theta \left\|x_t - x_{t'}\right\|_{C([-r,0],E)} + \sup_{t'-t<\delta} \vartheta(t' - t)\right] \\
&\leq \left\|(-A)^{-\alpha}\right\| \left[\theta \sup_{\substack{\alpha\in[-r,0] \\ t'-t<\delta}} \|x(t+\alpha) - x(t'+\alpha)\| + \sup_{t'-t<\delta} \vartheta(t' - t)\right] \\
&\leq \left\|(-A)^{-\alpha}\right\| \left[\theta \sup_{\substack{s,s'\in[t-r,t'] \\ |s-s'|<\delta}} \|x(s) - x(s')\| + \sup_{t'-t<\delta} \vartheta(t' - t)\right] \\
&\leq \left\|(-A)^{-\alpha}\right\| \left[\theta \sup_{\substack{s,s'\in[-r,T] \\ |s-s'|<\delta}} \|x(s) - x(s')\| + \sup_{t'-t<\delta} \vartheta(t' - t)\right]
\end{aligned}$$

Since

$$\lim_{\delta\to 0} \sup_{t'-t<\delta} \vartheta(t' - t) = \vartheta(0) = 0$$

It results that

$$mod_c \Gamma_2(\Omega) \leq \theta \left\|(-A)^{-\alpha}\right\| mod_c \Omega \tag{3.12}$$

Now we will show that the set

$$\Gamma_3(\Omega) = \left\{ y; y(t) = \left\{ \begin{array}{ll} 0, & t \in [-r, 0], \\ \int_0^t A e^{A(t-s)} h(s, x_s) ds, & t \in [0, T], \end{array} \right\} \right.$$

where $x \in \Omega$, is equicontinuous on $C([-r,T],E)$. Let $0 \leq t \leq t' \leq T$, and $x \in \Omega$. We have

$$\begin{aligned}
&\left\| \int_0^{t'} \left[AS(t'-s)h(s,x_s) - \int_0^t AS(t-s)h(s,x_s) \right] ds \right\| \\
\leq &\left\| (S(t'-t) - I) \int_0^t AS(t-s)h(s,x_s)ds \right\| + \left\| \int_t^{t'} AS(t'-s)h(s,x_s)ds \right\| \\
\leq &\left\| (S(t'-t) - I) \int_0^t AS(t-s)h(s,x_s)ds \right\| \\
&+ C_{1-\alpha}(d_1 \sup_{x \in \Omega} \|x\|_{C([-r,T],E)} + d_2) \frac{(t'-t)^\alpha}{\alpha}
\end{aligned}$$

Since $\chi\left(\int_0^t AS(t-s)h(s,\Omega_s)ds\right) = 0$, i.e., the set $\left\{\int_0^t AS(t-s)h(s,\Omega_s)ds\right\}$ is relatively compact for every $t \in [0,T]$, the first term on the right hand side converge to zero when $t' \to t$ uniformly on $x \in \Omega$. As consequence we get

$$mod_c \Gamma_3(\Omega) = 0. \tag{3.13}$$

Since

$$mod_c \Gamma(\Omega) \leq \sum_{i=1}^{3} mod_c \Gamma_i(\Omega)$$

From the inequalities (3.11)-(3.13), we obtain

$$mod_c \Gamma(\Omega) \leq \theta \left\| (-A)^{-\alpha} \right\| mod_c \Omega.$$

Since $\theta \left\| (-A)^{-\alpha} \right\| < 1$, from the inequality (3.4) follows

$$mod_c(\Omega) = 0. \tag{3.14}$$

Finally from the inequalities (3.5), (3.10) and (3.14) we get

$$\Theta(\Omega) = (0,0,0).$$

This shows that the subset Ω is relatively compact, concluding the proof of Step 2.

Now in the space $C([-r,T],E)$ we introduce the equivalent norm, given by

$$\|x\|_* = \sup_{t \in [-r,0]} \|x(t)\| + \sup_{t \in [0,T]} e^{-Lt} \sup_{s \in [0,t]} \|x(s)\|$$

Consider the ball

$$B_r(0) = \{x \in C([-r,T],E); \|x\|_* \leq r\}$$

where r is a constant chosen so that

$$r \geq \frac{\|\varphi\|_{C([-r,0],E)} + \|h(0,\varphi)\| + M\left(\|x_0\| + \|\beta\|_{L^1}\right) + d_2 C_{1-\alpha}\frac{T^\alpha}{\alpha}}{1 - d_1\left\|(-A)^{-\alpha}\right\|}$$

where $x_0 = \varphi(0) - h(0,\varphi)$. Since $d_1\left\|(-A)^{-\alpha}\right\| < 1$, the last inequality implies

$$d_1\left\|(-A)^{-\alpha}\right\| r + \|\varphi\|_{C([-r,0],E)} + \|h(0,\varphi)\| + M\left(\|x_0\| + \|\beta\|_{L^1}\right) + d_2 C_{1-\alpha}\frac{T^\alpha}{\alpha} \leq r.$$

Step 3. The multioperator Γ maps the ball $B_r(0)$ into itself.

Let $x \in B_r(0)$ and $y \in \Gamma(x)$,

$$\begin{aligned} y(t) = S(t)x_0 + \quad h(t,x_t) + \int_0^t S(t-s)f(s)ds \\ + \int_0^t AS(t-s)h(s,x_s)ds, \quad t \in [0,T] \\ y(t) = \varphi(t), t \in [-r,0] \end{aligned}$$

where $f \in sel_F(x)$. Remark first that

$$y = y_1 + y_2 + y_3$$

where

$$y_1(t) = \left\{\begin{array}{ll} \varphi(t) - h(0,\varphi), & t \in [-r,0] \\ y(t) = \Upsilon f(t), & t \in [0,T] \end{array}\right\},$$

$$y_2(t) = \left\{\begin{array}{ll} h(0,\varphi), & t \in [-r,0] \\ h(t,x_t) & t \in [0,T] \end{array}\right. ,$$

and

$$y_3(t) = \left\{\begin{array}{ll} 0, & t \in [-r,0], \\ \int_0^t AS(t-s)h(s,x_s)ds, & t \in [0,T]. \end{array}\right. .$$

Therefore,

$$\|y\|_\star \leq \|y_1\|_\star + \|y_2\|_\star + \|y_3\|_\star$$

Let give an upper estimate for each $\|y_i\|_\star$; $i = 1,2,3$. For $s \in [-r,0]$, we have

(3.15)

$$\sup_{s\in[-r,0]} \|y_1(s)\| = \sup_{s\in[-r,0]} \|\varphi(t) - h(0,\varphi)\| \leq \|\varphi\|_{C([-r,0],E)} + \|h(0,\varphi)\|.$$

For $s \in [0,t]$, $t \le T$, using the hypothesis (F_3), we have

$$\begin{aligned}
\|y_1(s)\| &\le \|S(s)x_0\| + \int_0^s \|S(s-\tau)\| \, \|f(\tau)\| \, d\tau \\
&\le M\|x_0\| + M\|\beta\|_{L^1} + M \int_0^s \beta(\tau) \|x_\tau\|_2 \, d\tau \\
&\le M\|x_0\| + M\|\beta\|_{L^1} + M \int_0^s \beta(\tau) \sup_{\mu \in [\tau - r, \tau]} \|x(\mu)\| \, d\tau \\
&\le M\|x_0\| + M\|\beta\|_{L^1} \\
&+ M \int_0^s e^{L\tau} \beta(\tau) \left[e^{-L\tau} \sup_{\mu \in [-r,0]} \|x(\mu)\| + e^{-L\tau} \sup_{\mu \in [0,\tau]} \|x(\mu)\| \right] d\tau \\
&\le M\|x_0\| + M\|\beta\|_{L^1} \\
&+ M \int_0^s e^{L\tau} \beta(\tau) \left[\sup_{\mu \in [-r,0]} \|x(\mu)\| + \sup_{\tau \in [0,T]} e^{-L\tau} \sup_{\mu \in [0,\tau]} \|x(\mu)\| \right] d\tau \\
&\le M\left(\|x_0\| + \|\beta\|_{L^1}\right) + M\|x\|_\star \int_0^s e^{L\tau} \beta(\tau) d\tau
\end{aligned}$$

Thus,

$$\sup_{s \in [0,t]} \|y_1(s)\| \le M\left(\|x_0\| + \|\beta\|_{L^1}\right) + M\|x\|_\star \int_0^t e^{L\tau} \beta(\tau) d\tau$$

By multiplying both sides with e^{-Lt} and bearing in mind the definition of q_4, we obtain

$$\begin{aligned}
\sup_{t \in [0,T]} e^{-Lt} \sup_{s \in [0,t]} \|y_1(s)\| &\le M\left(\|x_0\| + \|\beta\|_{L^1}\right) + \|x\|_\star M \sup_{t \in [0,T]} \int_0^t e^{-L(t-\tau)} \beta(\tau) d\tau \\
&\le M\left(\|x_0\| + \|\beta\|_{L^1}\right) + q_4 \|x\|_\star . \qquad (3.16)
\end{aligned}$$

From inequalities (3.15) and (3.16), we get

$$\|y_1\|_\star \le \|\varphi\|_{C([-r,0],E)} + \|h(0,\varphi)\| + M\left(\|x_0\| + \|\beta\|_{L^1}\right) + q_4 \|x\|_\star \qquad (3.17)$$

Let now give an upper estimate fore $\|y_2\|_\star$. For $s \in [-r,0]$, we have

$$\sup_{s \in [-r,0]} \|y_2(s)\|_\star = \sup_{s \in [-r,0]} \|h(0,\varphi)\| = \|h(0,\varphi)\| \qquad (3.18)$$

For $s \in [0,t]$, $t \leq T$, using the hypothesis $(H)-(i)$, we have

$$\begin{aligned}
\|y_2(s)\| &\leq \|h(t,x_t)\| \\
&\leq d_2 \left\|(-A)^{-\alpha}\right\| + d_1 \left\|(-A)^{-\alpha}\right\| \|x_s\|_{C([-r,0],E)} \\
&\leq d_2 \left\|(-A)^{-\alpha}\right\| + d_1 \left\|(-A)^{-\alpha}\right\| \left[\sup_{\mu\in[-r,0]} \|x(\mu)\| + \sup_{\mu\in[0,s]} \|x(\mu)\|\right] \\
&\leq d_2 \left\|(-A)^{-\alpha}\right\| + d_1 \left\|(-A)^{-\alpha}\right\| e^{Lt} \left[\sup_{\mu\in[-r,0]} \|x(\mu)\| + e^{-Lt} \sup_{\mu\in[0,t]} \|x(\mu)\|\right] \\
&\leq d_2 \left\|(-A)^{-\alpha}\right\| + d_1 \left\|(-A)^{-\alpha}\right\| e^{Lt} \left[\sup_{\mu\in[-r,0]} \|x(\mu)\| + \sup_{t\in[0,T]} e^{-Lt} \sup_{\mu\in[0,t]} \|x(\mu)\|\right] \\
&\leq d_2 \left\|(-A)^{-\alpha}\right\| + d_1 \left\|(-A)^{-\alpha}\right\| e^{Lt} \|x\|_\star
\end{aligned}$$

It follows that

$$\sup_{s\in[0,t]} \|y_2(s)\| \leq d_2 \left\|(-A)^{-\alpha}\right\| + e^{Lt} d_1 \left\|(-A)^{-\alpha}\right\| \|x\|_\star$$

By multiplying both sides with e^{-Lt}, we get

$$\sup_{t\in[0,T]} e^{-Lt} \sup_{s\in[0,t]} \|y_2(s)\| \leq d_2 \left\|(-A)^{-\alpha}\right\| + d_1 \left\|(-A)^{-\alpha}\right\| \|x\|_\star \tag{3.19}$$

From the inequalities (3.18) and (3.19), it follows that

$$\begin{aligned}
\|y_2\|_\star &\leq \|h(0,\varphi)\| + d_2 \left\|(-A)^{-\alpha}\right\| \\
&\quad + d_1 \left\|(-A)^{-\alpha}\right\| \|x\|_\star
\end{aligned} \tag{3.20}$$

It remains to give an upper estimate for $\|y_3\|_\star$. For $s \in [-r,0]$, we have

$$\sup_{s\in[-r,0]} \|y_3(s)\| = 0. \tag{3.21}$$

For $s \in [0,t]$, $t \leq T$, by using $(H)-(i)$, we have

$$\begin{aligned}
\|y_3(s)\| &= \left\| \int_0^s AS(s-\tau)h(\tau, x_\tau)d\tau \right\| \\
&\leq d_1 \int_0^s \frac{C_{1-\alpha}}{(s-\tau)^{1-\alpha}} \sup_{\mu\in[\tau-r,\tau]} \|x(\mu)\| \, d\tau + d_2 \int_0^s \frac{C_{1-\alpha}}{(s-\tau)^{1-\alpha}} d\tau \\
&\leq d_1 \int_0^s \frac{C_{1-\alpha}}{(s-\tau)^{1-\alpha}} \left[\sup_{\mu\in[-r,0]} \|x(\mu)\| + \sup_{\mu\in[0,\tau]} \|x(\mu)\| \right] d\tau + d_2 \int_0^T \frac{C_{1-\alpha}}{(T-\tau)^{1-\alpha}} d\tau \\
&\leq d_1 \int_0^t \frac{C_{1-\alpha}}{(t-s)^{1-\alpha}} e^{L\tau} d\tau \left[\sup_{\mu\in[-r,0]} \|x(\mu)\| + \sup_{\tau\in[0,T]} e^{-L\tau} \sup_{\mu\in[0,\tau]} \|x(\mu)\| \right] + d_2 C_{1-\alpha} \frac{T^\alpha}{\alpha} \\
&\leq d_1 \int_0^s \frac{C_{1-\alpha}}{(t-s)^{1-\alpha}} e^{L\tau} d\tau \, \|x\|_\star + d_2 C_{1-\alpha} \frac{T^\alpha}{\alpha}
\end{aligned}$$

Therefore

$$\sup_{s\in[0,t]} \|y_3(s)\| \leq \|x\|_\star \, d_1 \int_0^t \frac{C_{1-\alpha}}{(t-\tau)^{1-\alpha}} e^{L\tau} d\tau + d_2 C_{1-\alpha} \frac{T^\alpha}{\alpha}.$$

By multiplying both sides with e^{-Lt} and bearing in mind the definition of q_3, we get

$$\begin{aligned}
&\sup_{t\in[0,T]} e^{-Lt} \sup_{s\in[0,t]} \|y_3(s)\| \\
&\leq \|x\|_\star \left(d_1 \sup_{t\in[0,T]} \int_0^t \frac{C_{1-\alpha}}{(t-\tau)^{1-\alpha}} e^{-L(t-\tau)} d\tau \right) + d_2 C_{1-\alpha} \frac{T^\alpha}{\alpha} \\
&\leq q_3 \|x\|_\star + d_2 C_{1-\alpha} \frac{T^\alpha}{\alpha} \qquad (3.22)
\end{aligned}$$

From the inequalities (3.21) and (3.22), it follows that

$$\|y_3\|_\star \leq q_3 \|x\|_\star + d_2 C_{1-\alpha} \frac{T^\alpha}{\alpha} \qquad (3.23)$$

Finally from (3.17), (3.20) , (3.23) and Remark 2, we get

$$\begin{aligned}
\|y\|_\star &\leq \|y_1\|_\star + \|y_2\|_\star + \|y_3\|_\star \\
&\leq \|\varphi\|_{C([-r,0],E)} + \|h(0,\varphi)\| + M\left(\|x_0\| + \|\beta\|_{L^1}\right) + d_2 C_{1-\alpha} \frac{T^\alpha}{\alpha} \\
&\quad + \left[d_1 \left\|(-A)^{-\alpha}\right\| + q_4 + q_3 \right] \|x\|_\star
\end{aligned}$$

$$\leq \|\varphi\|_{C([-r,0],E)} + \|h(0,\varphi)\| + M\left(\|x_0\| + \|\beta\|_{L^1}\right) + d_2 C_{1-\alpha}\frac{T^\alpha}{\alpha}$$
$$+\left[d_1 \left\|(-A)^{-\alpha}\right\| + q_4 + q_3\right] r$$
$$\leq r$$

According to Lemma 4, the problem (1.1)-(1.2) has at least one mild solution.

Step 4. The solutions set is compact.

The solution set is a priori bounded. In fact, if x is a mild solution of the problem (1.1)-(1.2), and the function $\upsilon(.) : [0,T] \to \mathbb{R}^+$ is such that $\upsilon(t) = \sup\limits_{\mu\in[0,t]} \|x(\mu)\|$ then as above for $t \in [0,T]$ we have

$$\begin{aligned}
\upsilon(t) &= \sup_{\mu\in[0,t]} \|x(\mu)\| \\
&\leq M\left(\|x_0\| + \|\beta\|_{L^1}\right) + M\int_0^t \beta(\tau)\left[\sup_{\mu\in[-r,0]}\|x(\mu)\| + \sup_{\mu\in[0,\tau]}\|x(\mu)\|\right]d\tau \\
&\quad + d_2\left\|(-A)^{-\alpha}\right\| + d_1\left\|(-A)^{-\alpha}\right\|\left[\sup_{\mu\in[-r,0]}\|x(\mu)\| + \sup_{\mu\in[0,t]}\|x(\mu)\|\right] \\
&\quad + d_1\int_0^t \frac{C_{1-\alpha}}{(t-s)^{1-\alpha}}d\tau\left[\sup_{\mu\in[-r,0]}\|x(\mu)\| + \sup_{\mu\in[0,\tau]}\|x(\mu)\|\right] + d_2\int_0^t \frac{C_{1-\alpha}}{(t-s)^{1-\alpha}}d\tau \\
&\leq M\left(\|x_0\| + \|\beta\|_{L^1}\right) + d_2\left\|(-A)^{-\alpha}\right\| \\
&+ \left[M\|\beta\|_{L^1} + d_1\left\|(-A)^{-\alpha}\right\| + d_2 C_{1-\alpha}\frac{T^\alpha}{\alpha}\right]\|\varphi\|_{C([-r,0],E)} \\
&\quad + d_1\left\|(-A)^{-\alpha}\right\|\sup_{s\in[0,t]}\|x(s)\| + \int_0^t\left[M\beta(\tau) + d_1\frac{C_{1-\alpha}}{(t-s)^{1-\alpha}}\right]\sup_{\mu\in[0,\tau]}\|x(\mu)\|\,d\tau \\
&\leq M\left(\|x_0\| + \|\beta\|_{L^1}\right) + d_2\left\|(-A)^{-\alpha}\right\| \\
&\quad + \left[M\|\beta\|_{L^1} + d_1\left\|(-A)^{-\alpha}\right\| + d_2 C_{1-\alpha}\frac{T^\alpha}{\alpha}\right]\|\varphi\|_{C([-r,0],E)} \\
&\quad + d_1\left\|(-A)^{-\alpha}\right\|\upsilon(t) + \int_0^t\left[M\beta(\tau) + d_1\frac{C_{1-\alpha}}{(t-s)^{1-\alpha}}\right]\upsilon(\tau)d\tau
\end{aligned}$$

it results that

$$\upsilon(t) \leq \frac{1}{d_1\left\|(-A)^{-\alpha}\right\|}\left[\xi + \int_0^t\left[M\beta(\tau) + d_1\frac{C_{1-\alpha}}{(t-s)^{1-\alpha}}\right]\upsilon(\tau)d\tau\right]$$

where

$$\xi = M\left(\|x_0\| + \|\beta\|_{L^1}\right) + d_2\left\|(-A)^{-\alpha}\right\|$$
$$+\left[M\|\beta\|_{L^1} + d_1\left\|(-A)^{-\alpha}\right\| + d_2 C_{1-\alpha}\frac{T^\alpha}{\alpha}\right]\|\varphi\|_{C([-r,0],E)}$$

Applying Gromwall-Bellmann type inequality, we get

$$\|\upsilon(t)\| \leq \frac{\xi}{1 - d_1\|(-A)^{-\alpha}\|}e^\gamma$$

where,

$$\gamma = \frac{1}{1 - d_1\|(-A)^{-\alpha}\|}\left[M\|B\|_{L^1} + T^\alpha\frac{d_1 C_{1-\alpha}}{\alpha}\right].$$

therefore,

$$\upsilon(T) = \sup_{\mu\in[0,T]}\|x(\mu)\| \leq \frac{\xi}{1 - d_1\|(-A)^{-\alpha}\|}e^\gamma$$

Consequently

$$\begin{aligned}\|x\|_{C([-r,T],E)} &\leq \sup_{\mu\in[-r,0]}\|x(\mu)\| + \sup_{\mu\in[0,T]}\|x(\mu)\| \\ &\leq \|\varphi\|_{C([-r,0],E)} + \frac{\xi}{1 - d_1\|(-A)^{-\alpha}\|}e^\gamma\end{aligned}$$

To complete the proof it remains to apply Lemma 5.

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