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LOCAL ASYMPTOTIC STABILITY FOR NONLINEAR QUADRATIC FUNCTIONAL INTEGRAL EQUATIONS

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Abstract

In the present study, using the characterizations of measures of noncompactness we prove a theorem on the existence and local asymptotic stability of solutions for a quadratic functional integral equation via a fixed point theorem of Darbo. The investigations are placed in the Banach space of real functions defined, continuous and bounded on an unbounded interval. An example is indicated to demonstrate the natural realizations of abstract result presented in the paper.

Keywords: Quadratic functional integral equation; Measure of noncompactness; Fixed point theorem; Attractive solutions, etc.,
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1 Introduction

In this paper, we are going to prove a theorem on the existence and uniform global attractivity of solutions for a quadratic functional integral equation. Our investigations will be carried out in the Banach space of real functions which are defined, continuous and bounded on the real half axis $\mathbb{R}_+$. The integral equation in question has rather general form and contains as particular cases a lot of functional equations and nonlinear integral equations of Volterra type. The main tool used in our considerations is the technique of measures of noncompactness and the fixed point theorem of Darbo [1, page 17].

The measure of noncompactness used in this paper allows us not only to obtain the existence of solutions of the mentioned functional integral equation but also to characterize the solutions in terms of uniform global asymptotic attractivity. This assertion means that all possible solutions of the functional integral equation in question are globally uniformly attractive in the sense of notion defined in the following section.

The assumptions imposed on the nonlinearities in our main existence theorem admit several natural realizations which are illustrated by an example. The results obtained in this paper generalize and extend several ones obtained earlier in a lot of papers concerning asymptotic stability of solutions of some functional integral equations (cf. [3,4,5,8]). It is worthwhile mentioning that the novelty of our approach consists mainly in the possibility of obtaining of the uniformly global attractivity of solutions for considered quadratic functional integral equations.

2 Notations, Definitions and Auxiliary facts

At the beginning of this section, we present some basic facts concerning the measures of noncompactness [1,2] in Banach spaces.

Assume that $(E, \|\cdot\|)$ is an infinite dimensional Banach space with zero element θ. Denote by $\overline{\mathcal{B}}_r(x)$ the closed ball centered at x and with radius r. Thus, $\overline{\mathcal{B}}_r(\theta)$ is the closed ball centered at origin of radius r. If X is a subset of E then the symbols $\overline{X}$, Conv$X$ stand for the closure and closed convex hull of X, respectively. Moreover, we denote by $\mathcal{P}_{bd}(E)$ the family of all nonempty and bounded subsets of E and by $\mathcal{P}_{rcp}(E)$ its subfamily consisting of all relatively compact subsets of E.

The following definition of a measure of noncompactness appears in Banas and Goebel [1].

Definition 2.1 *A mapping $\mu : \mathcal{P}_{bd}(E) \to \mathbb{R}_+ = [0, \infty)$ is said to be a measure of noncompactness in E if it satisfies the following conditions:*

1^o *The family ker$\mu = \{X \in \mathcal{P}_{bd}(E) : \mu(X) = 0\}$ is nonempty and* $\ker\mu \subset \mathcal{P}_{rcp}(E)$

2^o $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$

3^o $\mu(\overline{X}) = \mu(X)$

4^o $\mu(ConvX) = \mu(X)$

5^o $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ *for* $\lambda \in [0, 1]$

6^o *If (X_n) is a sequence of closed sets from $\mathcal{P}_{bd}(E)$ such that $X_{n+1} \subset X_n$ $(n = 1, 2, ...)$ and if $\lim_{n\to\infty} \mu(X_n) = 0$, then the intersection set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.*

The family ker μ described in 1^o is said to be *the kernel of the measure of noncompactness μ*. Observe that the intersection set X_∞ from 6^o is a member of the family ker μ. In fact, since $\mu(X_\infty) \leq \mu(X_n)$ for any n, we infer that $\mu(X_\infty) = 0$. This yields that $X_\infty \in$ ker μ. This simple observation will be essential in our further investigations.

Now we state a fixed point theorem of Darbo type which will be used in the sequel (see Banas [2, page 17]).

Theorem 2.1 *Let Ω be a nonempty, bounded, closed and convex subset of the Banach space E and let $F : \Omega \to \Omega$ be a continuous mapping. Assume that there exists a constant $k \in [0, 1)$ such that $\mu(FX) \leq k\mu(X)$ for any nonempty subset X of Ω. Then F has a fixed point in the set Ω.*

Remark 2.1 Let us denote by $Fix(F)$ the set of all fixed points of the operator F which belong to Ω. It can be shown [1] that the set $Fix\,(F)$ belongs to the family $\ker\,\mu$.

Our further considerations will be placed in the Banach space $BC(\mathbb{R}_+, \mathbb{R})$ consisting of all real functions $x = x(t)$ defined, continuous and bounded on $\mathbb{R}_+$. This space is equipped with the standard supremum norm

$$||x|| = \sup\{|x(t)| : \ t \in \mathbb{R}_+\} \ . \tag{2.1}$$

For our purposes we will use the ball measure of noncompactness in $E = BC(\mathbb{R}_+, \mathbb{R})$ defined by

$$\beta(A) = \inf\Big\{r > 0 : A \subset \bigcup_{i=1}^{n} \mathcal{B}(x_i, r) \text{ for } x_i \in E\Big\} \tag{2.2}$$

for all bounded subsets A of $BC(\mathbb{R}_+, \mathbb{R})$, where $\mathcal{B}(x_i, r) = \{x \in X \mid \|x_i - x\| < r\}$.

The ball measure of noncompactness is also called Hausdorff measure of noncompactness since it has close connections with the Hausdorff metric in the Banach space E. We use a handy formula for ball or Hausdorff measure of noncompactness in $BC(\mathbb{R}_+, \mathbb{R})$ discussed in Banas [2]. To derive this formula, let us fix a nonempty and bounded subset X of the space $BC(\mathbb{R}_+, \mathbb{R})$ and a positive number T. For $x \in X$ and $\epsilon \geq 0$ denote by $\omega^T(x, \epsilon)$ *the modulus of continuity* of the function x on the interval $[0, T]$, i.e.

$$\omega^T(x, \epsilon) = \sup\{|x(t) - x(s)| : \ t, s \in [0, T], \ |t - s| \leq \epsilon\} \ .$$

Next, let us put

$$\omega^T(X, \epsilon) = \sup\{\omega^T(x, \epsilon) : \ x \in X\} \ ,$$

$$\omega_0^T(X) = \lim_{\epsilon \to 0} \omega^T(X, \epsilon) \ ,$$

$$\omega_0(X) = \lim_{T \to \infty} \omega_0^T(X) \ .$$

It is known that

$$\beta(A) = \frac{1}{2}\omega_0(A)$$

for any bounded subset A of $BC(\mathbb{R}_+, \mathbb{R})$ (see Banas and Goebel [1] and the reference given therein).

Now, for a fixed number $t \in \mathbb{R}_+$ let us denote

$$X(t) = \{x(t) : x \in X\}$$

and

$$\|X(t)\| = \sup\{|x(t)| \ : \ x, y \in X\} \ .$$

Finally, let us consider the function μ defined on the family $BC(\mathbb{R}_+, \mathbb{R})$ by the formula

$$\mu(X) = \omega_0(X) + \limsup_{t\to\infty} \|X(t)\| \ . \tag{2.3}$$

It can be shown as in Banas [2] that the function μ is a measure of noncompactness in the space $BC(\mathbb{R}_+, \mathbb{R})$. The kernel $\ker \mu$ of this measure consists nonempty and bounded subsets X of $BC(\mathbb{R}_+, \mathbb{R})$ such that the functions from X are locally equicontinuous on $\mathbb{R}_+$ and the thickness of the bundle formed by functions from X tends to zero at infinity. This particular characteristic of $\ker \mu$ has been utilized in establishing the local attractivity of the solutions for quadratic integral equation.

In order to introduce further concepts used in the paper let us assume that Ω is a nonempty subset of the space $BC(\mathbb{R}_+, \mathbb{R})$. Moreover, let Q be an operator defined on Ω with values in $BC(\mathbb{R}_+, \mathbb{R})$.

Consider the operator equation of the form

$$x(t) = Qx(t), \ \ t \in \mathbb{R}_+ \ . \tag{2.4}$$

Definition 2.2 We say that solutions of the equation (2.3) are locally attractive if there exists a ball $\overline{\mathcal{B}}_r(x_0)$ in the space $BC(\mathbb{R}_+, \mathbb{R})$ such that for arbitrary solutions $x = x(t)$ and $y = y(t)$ of equation (2.3) belonging to $\overline{\mathcal{B}}_r(x_0) \cap \Omega$ we have that

$$\lim_{t\to\infty} (x(t) - y(t)) = 0 \ . \tag{2.5}$$

In the case when the limit (2.4) is uniform with respect to the set $\overline{\mathcal{B}}(x_0) \cap \Omega$, i.e. when for each $\epsilon > 0$ there exists $T > 0$ such that

$$|x(t) - y(t)| \leq \epsilon \tag{2.6}$$

for all $x, y \in \overline{\mathcal{B}}(x_0) \cap \Omega$ being solutions of (2.4) and for $t \geq T$, we will say that solutions of equation (2.4) are uniformly locally attractive (or equivalently, that solutions of (2.4) are asymptotically stable).

Definition 2.3 A line $y = m_1 t + m_2$, where m_1 and m_2 are real numbers, is called a **attractor** for the solution $x \in BC(\mathbb{R}_+, \mathbb{R})$ to the equation (2.1) if $\lim_{t\to\infty}[x(t) - (m_1 t + m_2)] = 0$. In this case the solution x to the equation (2.1) is also called to be asymptotic to the line $y = m_1 t + m_2$ and the line is an asymptote for the solution x on $\mathbb{R}_+$.

Now we introduce the following definition useful in the sequel.

Definition 2.4 The solutions of the equation (2.1) are said to be **locally asymptotically attractive** if there exists a $x_0 \in BC(\mathbb{R}_+, \mathbb{R})$ and an $r > 0$ such that for any two solutions $x = x(t)$ and $y = y(t)$ of the equation (2.1) belonging to $\overline{\mathcal{B}}_r(x_0) \cap \Omega$ the condition (2.3) is satisfied and there is a line which is a common attractor to them on $\mathbb{R}_+$. In the case when condition (2.3) is satisfied uniformly with respect to the set $\overline{\mathcal{B}}_r(x_0)\cap\Omega$, that is, if for every $\varepsilon > 0$ there exists $T > 0$ such that the inequality (2.4) is satisfied for $t \geq T$ and for all $x, y \in \overline{\mathcal{B}}_r(x_0)\cap\Omega$ being the solutions of (2.1) having a line as a common attractor, we will say that solutions of the equation (2.1) are **uniformly locally asymptotically attractive** on $\mathbb{R}_+$.

Remark 2.2 Note that two solutions x and y of the equation (2.1) existing on $\mathbb{R}_+$ are called asymptotically attractive if the condition (2.3) is satisfied and there is a line as a common attractor on $\mathbb{R}_+$. Therefore, locally asymptotically attractive solutions are asymptotically attractive, but the converse may not be true. Similarly, uniformly locally asymptotically attractive solutions are asymptotically attractive, but the converse may not be true. A asymptotically attractive solution for the operator equation (2.1) existing on $\mathbb{R}_+$ is also called **asymptotically stable** on $\mathbb{R}_+$.

Let us mention that the concept of attractivity of solutions was introduced in Hu and Yan [9] and Banas and Rzepka [3] while the concept of asymptotic attractivity is introduced in Dhage [7].

3 The Integral Equation and Stability Result

In this section we will investigate the following nonlinear quadratic functional integral equation (in short QFIE)

$$x(t) = q(t) + \Big[f(t, x(\alpha(t)))\Big] \left(\int_0^{\beta(t)} g(t, s, x(\gamma(s)))\, ds \right) \tag{3.1}$$

for all $t \in \mathbb{R}_+$, where $q : \mathbb{R}_+ \to \mathbb{R}$, $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$.

By a *solution* of the QFIE (3.1) we mean a function $x \in C(\mathbb{R}_+, \mathbb{R})$ that satisfies the equation (3.1), where $C(\mathbb{R}_+, \mathbb{R})$ is the space of continuous real-valued functions on $\mathbb{R}_+$.

Observe that the above equation includes several classes of functional, integral and functional integral equations considered in the literature (cf. [3, 5, 6, 7, 8, 9, 10] and references therein). Let us also mention that the functional integral equation considered in [3, 5] is a special case of the equation (3.1), where $\alpha(t) = \beta(t) = \gamma(t) = t$.

The equation (3.1) will be considered under the following assumptions:

(A_1) The functions $\alpha, \beta, \gamma : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous and $\lim_{t\to\infty} \alpha(t) = \infty$.

(A_2) The function $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous and there exist a bounded function $\ell : \mathbb{R}_+ \to \mathbb{R}$ with bound L such that

$$|f(t, x) - f(t, y)| \leq \ell(t)|x - y|$$

for $t \in \mathbb{R}_+$ and for $x, y \in \mathbb{R}$.

(A_3) The function $F : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $F(t) = |f(t,0)|$ is bounded on $\mathbb{R}_+$ with $F_0 = \sup_{t\geq 0} F(t)$.

(B_1) The function $q : \mathbb{R} \to \mathbb{R}$ is continuous and $\lim_{t\to\infty} q(t) = 0$.

(B_2) The function $g : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous and there exist continuous functions $a, b : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|g(t,s,x)| \leq a(t)b(s)$$

for $t, s \in \mathbb{R}_+$. Moreover, we assume that

$$\lim_{t\to\infty} a(t) \int_0^{\beta(t)} b(s)ds = 0 \ .$$

Now we formulate the main result of this paper.

Theorem 3.1 *Assume that the hypotheses (A_1) through (A_3) and (B_1) through (B_2) hold. Furthermore, if $LK_2 < 1$, where $K_2 = \sup_{t\geq 0} a(t) \int_0^{\beta(t)} b(s)ds$, then the quadratic functional integral equation (3.1) has at least one solution in the space $BC(\mathbb{R}_+, \mathbb{R})$. Moreover, solutions of the equation (3.1) are uniformly locally asymptotically stable on $\mathbb{R}_+$.*

Proof. Set $E = BC(\mathbb{R}_+, \mathbb{R})$. Consider the operator Q defined on the Banach space E by the formula

$$Qx(t) = q(t) + \Big[f(t, x(\alpha(t)))\Big] \left(\int_0^{\beta(t)} g(t,s,x(\gamma(s)))\, ds \right) \tag{3.2}$$

for all $t \in \mathbb{R}_+$. Observe that in view of our assumptions, for any function $x \in E$, Qx is a real-valued continuous function on $\mathbb{R}_+$. Since the function $v : \mathbb{R}_+ \to \mathbb{R}$ defined by

$$v(t) = \lim_{t\to\infty} a(t) \int_0^{\beta(t)} b(s)\, ds \tag{3.3}$$

is continuous and in view of hypothesis (B_2), the number $K_2 = \sup_{t\geq 0} v(t)$ exists. Define a closed ball $\overline{\mathcal{B}}_r(\theta)$ in E centered at the origin θ of radius equal to $r = \dfrac{K_1 + K_2}{1 - LK_2}$, $LK_2 < 1$. Let $x \in \overline{\mathcal{B}}_r(\theta)$ be arbitrarily fixed. Then, by hypotheses (A_1)-(A_2) and (B_1)-(B_2) we obtain:

$$|Qx(t)| \ \leq \ |q(t)| + |f(t, x(\alpha(t)))| \Big(\int_0^{\beta(t)} |g(t,s,x(s))\, ds \Big)$$

$$
\begin{aligned}
&\le |q(t)| + \Big[|f(t, x(\alpha(t))) - f(t, 0)| + |f(t, 0)|\Big]\Big(a(t)\int_0^{\beta(t)} b(s)\,ds\Big) \\
&\le |q(t)| + \Big[\ell(t)|x(\alpha(t))| + F(t)\Big]\, v(t) \\
&\le |q(t)| + \Big[L|x(\alpha(t))| + F_0\Big]\, K_2 \\
&\le K_1 + LK_2\|x\| + F_0K_2 \\
&= \frac{K_1 + F_0K_2}{1 - LK_2}
\end{aligned}
$$

for all $t \in \mathbb{R}_+$. Taking the supremum over t, we obtain the estimate :

$$
||Qx|| \le \frac{K_1 + F_0K_2}{1 - LK_2} \tag{3.4}
$$

for all $x \in \overline{\mathcal{B}}_r(\theta)$. As a result, Q defines a mapping $Q : \overline{\mathcal{B}}_r(\theta) \to \overline{\mathcal{B}}_r(\theta)$.

Now we show that the operator Q is continuous on the ball $\overline{\mathcal{B}}_r(\theta)$. To do this, let us fix arbitrarily $\epsilon > 0$ and take $x, y \in \overline{\mathcal{B}}_r(\theta)$ such that $||x - y|| \le \epsilon$. Then by hypotheses (A_1)-(A_2) and (B_1)-(B_2) we get:

$$
\begin{aligned}
&|Qx(t) - Qy(t)| \\
&\quad\le \Big|f(t, x(\alpha(t)))\int_0^{\beta(t)} g(t, s, x(\gamma(s)))\,ds - f(t, y(\alpha(t)))\int_0^{\beta(t)} g(t, s, y(\gamma(s)))\,ds\Big| \\
&\quad\le \Big|f(t, x(\alpha(t)))\int_0^{\beta(t)} g(t, s, x(\gamma(s)))\,ds - f(t, y(\alpha(t)))\int_0^{\beta(t)} g(t, s, x(\gamma(s)))\,ds\Big| \\
&\qquad + \Big|f(t, y(\alpha(t)))\int_0^{\beta(t)} g(t, s, x(\gamma(s)))\,ds - f(t, y(\alpha(t)))\int_0^{\beta(t)} g(t, s, y(\gamma(s)))\,ds\Big| \\
&\quad\le |f(t, x(\alpha(t))) - f(t, y(\alpha(t)))|\Big|\int_0^{\beta(t)} g(t, s, x(\gamma(s)))\,ds\Big| \\
&\qquad + \Big[|f(t, y(\alpha(t))) - f(t, 0)| + |f(t, 0)|\Big] \\
&\qquad\quad \times \Big(\int_0^{\beta(t)} [|g(t, s, x(\gamma(s)))| + |g(t, s, x(\gamma(s)))|]\,ds\Big) \\
&\quad\le \ell(t)\Big[|x(\alpha(t)) - y(\alpha(t))|\Big]\Big(a(t)\int_0^{\beta(t)} b(s)\,ds\Big) \\
&\qquad + 2\Big[\ell(t)|y(\alpha(t))| + F_0\Big]\Big(a(t)\int_0^{\beta(t)} b(s)\,ds\Big) \\
&\quad\le L\|x - y\|v(t) + 2L\|y\|v(t)
\end{aligned}
$$

$$\leq LK_2\,\epsilon + 2Lrv(t).$$

Hence, in virtue of hypothesis (B_2), we infer that there exists a $T > 0$ such that $v(t) \leq \frac{\epsilon}{2Lr}$ for $t \geq T$. Thus, for $t \geq T$ from the estimate (3.3) we derive that

$$|Qx(t) - Qy(t)| \leq (LK_2 + 1)\epsilon\ . \tag{3.5}$$

Further, let us assume that $t \in [0, T]$. Then, evaluating similarly as above we get:

$$\begin{aligned}
|Qx(t) - Qy(t)| &\leq |f(t, x(\alpha(t))) - f(t, y(\alpha(t)))| \Big| \int_0^{\beta(t)} g(t, s, x(\gamma(s)))\, ds \Big| \\
&\quad + \big[|f(t, y(\alpha(t))) - f(t, 0)|\big] \Big(\int_0^{\beta(t)} [|g(t, s, x(\gamma(s))) - g(t, s, y(\gamma(s)))|]\, ds \Big) \\
&\quad + |f(t, 0)| \Big(\int_0^{\beta(t)} [|g(t, s, x(\gamma(s))) - g(t, s, y(\gamma(s)))|]\, ds \Big) \\
&\leq L\, K_2\, |x(\alpha(t)) - y(\alpha(t))| \\
&\quad + (Lr + F_0) \Big(\int_0^{\beta_T} [|g(t, s, x(\gamma(s))) - g(t, s, y(\gamma(s)))|]\, ds \Big) \\
&\leq LK_2\,\epsilon + \beta_T\, \omega_r^T(g, \epsilon)\ ,
\end{aligned} \tag{3.6}$$

where we have denoted

$$\beta_T = \sup\{\beta(t) :\ t \in [0, T]\}\ ,$$

and

$$\begin{aligned}
\omega_r^T(g, \epsilon) = \sup\{&|g(t, s, x) - g(t, s, y)| : \\
&t, s \in [0, T], s \in [0, \beta_T],\ x, y \in [-r, r],\ |x - y| \leq \epsilon\}\ .
\end{aligned} \tag{3.7}$$

Obviously, in view of continuity of β, we have that $\beta_T < \infty$. Moreover, from the uniform continuity of the function $g(t, s, x)$ on the set $[0, T] \times [0, \beta_T] \times [-r, r]$ we derive that $\omega_r^T(g, \epsilon) \to 0$ as $\epsilon \to 0$. Now, linking the inequalities (3.5), (3.6) and the above established facts we conclude that the operator Q maps continuously the ball $\overline{\mathcal{B}}_r(\theta)$ into itself.

Further on, let us take a nonempty subset X of the ball $\mathcal{B}_r(\theta)$. Next, fix arbitrarily $T > 0$ and $\epsilon > 0$. Let us choose $x \in X$ and $t_1, t_2 \in [0, T]$ with $|t_2 - t_1| \leq \epsilon$. Without loss of generality we may assume that $t_1 < t_2$. Then, taking into account our assumptions, we get:

$$\begin{aligned}
|Qx(t_1) - Qx(t_2)| &\leq |q(t_1) - q(t_2)| \\
&\quad + \Big| f(t_1, x(\alpha(t_1))) \int_0^{\beta(t_1)} g(t_1, s, x(\gamma(s)))\, ds
\end{aligned}$$

$$
\begin{aligned}
&\qquad - f(t_2, x(\alpha(t_2))) \int_0^{\beta(t_2)} g(t_2, s, x(\gamma(s)))\, ds\Big| \\
&\le |q(t_1) - q(t_2)| \\
&\quad + \Big| f(t_1, x(\alpha(t_1))) \int_0^{\beta(t_1)} g(t_1, s, x(\gamma(s)))\, ds \\
&\qquad - f(t_2, x(\alpha(t_2))) \int_0^{\beta(t_1)} g(t_1, s, x(\gamma(s)))\, ds\Big| \\
&\quad + \Big| f(t_2, x(\alpha(t_2))) \int_0^{\beta(t_1)} g(t_1, s, x(\gamma(s)))\, ds \\
&\qquad - f(t_2, x(\alpha(t_2))) \int_0^{\beta(t_2)} g(t_2, s, x(\gamma(s)))\, ds\Big| \\
&\le |q(t_1) - q(t_2)| \\
&\quad + \Big| f(t_1, x(\alpha(t_1))) - f(t_2, x(\alpha(t_2)))\Big| \int_0^{\beta(t_1)} |g(t_1, s, x(\gamma(s)))|\, ds \\
&\quad + \Big| f(t_2, x(\alpha(t_2)))\Big| \Big| \int_0^{\beta(t_1)} g(t_1, s, x(\gamma(s)))\, ds - \int_0^{\beta(t_2)} g(t_2, s, x(\gamma(s)))\, ds\Big| \\
&\le |q(t_1) - q(t_2)| + \Big| f(t_1, x(\alpha(t_1))) - f(t_2, x(\alpha(t_1)))\Big|\, v(t_1) \\
&\qquad + \Big| f(t_2, x(\alpha(t_1))) - f(t_2, x(\alpha(t_2)))\Big|\, v(t_1) \\
&\quad + (Lr + F_0)\Big| \int_0^{\beta(t_1)} g(t_1, s, x(\gamma(s)))\, ds - \int_0^{\beta(t_2)} g(t_2, s, x(\gamma(s)))\, ds\Big| \\
&\le |q(t_1) - q(t_2)| \\
&\quad + K_2 \Big| f(t_1, x(\alpha(t_1))) - f(t_2, x(\alpha(t_1)))\Big| + L\, K_2\, |x(\alpha(t_1)) - x(\alpha(t_2))| \\
&\qquad + (Lr + F_0)\Big| \int_0^{\beta(t_1)} g(t_1, s, x(\gamma(s)))\, ds - \int_0^{\beta(t_2)} g(t_2, s, x(\gamma(s)))\, ds\Big| \qquad (3.8)
\end{aligned}
$$

Again,

$$
\begin{aligned}
&\Big| \int_0^{\beta(t_1)} g(t_1, s, x(\gamma(s)))\, ds - \int_0^{\beta(t_2)} g(t_2, s, x(\gamma(s)))\, ds\Big| \\
&\qquad \le \Big| \int_0^{\beta(t_1)} g(t_1, s, x(\gamma(s)))\, ds - \int_0^{\beta(t_1)} g(t_2, s, x(\gamma(s)))\, ds\Big| \\
&\qquad\quad + \Big| \int_0^{\beta(t_1)} g(t_2, s, x(\gamma(s)))\, ds - \int_0^{\beta(t_2)} g(t_2, s, x(\gamma(s)))\, ds\Big| \\
&\qquad \le \int_0^{\beta(t_1)} |g(t_1, s, x(\gamma(s))) - g(t_2, s, x(\gamma(s)))|\, ds \\
&\qquad\quad + \Big| \int_{\beta(t_2)}^{\beta(t_1)} |g(t_2, s, x(\gamma(s)))|\, ds\Big|
\end{aligned}
$$

$$\leq \int_0^{\beta_T} |g(t_1, s, x(\gamma(s))) - g(t_2, s, x(\gamma(s)))|\, ds + |v(t_1) - v(t_2)| \tag{3.9}$$

Now combining (3.8) and (3.9) we obtain,

$$\begin{aligned} |Qx(t_2) - Qx(t_1)| &\leq |q(t_1) - q(t_2)| + K_2 \left| f(t_1, x(\alpha(t_1))) - f(t_2, x(\alpha(t_1))) \right| \\ &\quad + L\, K_2 |x(\alpha(t_1)) - x(\alpha(t_2))|\, v(t_1) \\ &\quad + (Lr + F_0) \int_0^{\beta_T} |g(t_1, s, x(\gamma(s))) - g(t_2, s, x(\gamma(s)))|\, ds \\ &\quad + (Lr + F_0) |v(t_1) - v(t_2)| \\ &\leq \omega^T(q, \epsilon) + LK_2\, \omega^T(x, \omega^T(\alpha, \epsilon)) + K_2\, \omega_r^T(f, \epsilon) \\ &\quad + (Lr + F_0) \int_0^{\beta_T} \omega_r^T(g, \epsilon)\, ds + (Lr + F_0)\, \omega^T(v, \epsilon) \end{aligned} \tag{3.10}$$

where we have denoted

$$\omega^T(q, \epsilon) = \sup\{|q(t_2) - q(t_1)| : \ t_1, t_2 \in [0, T], \ |t_2 - t_1| \leq \epsilon, \} \ ,$$

$$\omega^T(\alpha, \epsilon) = \sup\{|\alpha(t_2) - \alpha(t_1)| : \ t_1, t_2 \in [0, T], \ |t_2 - t_1| \leq \epsilon, \} \ ,$$

$$\omega^T(v, \epsilon) = \sup\{|v(t_2) - v(t_1)| : \ t_1, t_2 \in [0, T], \ |t_2 - t_1| \leq \epsilon, \} \ ,$$

$$\omega_r^T(f, \epsilon) = \sup\{|f(t_2, x) - f(t_1, x)| : \ t_1, t_2 \in [0, T], \ |t_2 - t_1| \leq \epsilon, \ x \in [-r, r]\} \ ,$$

$$\omega_r^T(f, \epsilon) = \sup\{|f(t_2, x) - f(t_1, x)| : \ t_1, t_2 \in [0, T], \ |t_2 - t_1| \leq \epsilon, \ x \in [-r, r]\} \ ,$$

$$\omega_r^T(g, \epsilon) = \sup\{|g(t_2, s, x) - g(t_1, s, x)| : \ t_1, t_2 \in [0, T], \ |t_2 - t_1| \leq \epsilon, \ s \in [0, \beta_T], \ x \in [-r, r]\} \ .$$

From the above estimate we derive the following one

$$\begin{aligned} \omega^T(Q(X), \epsilon) &\leq \omega^T(q, \epsilon) + LK_2\, \omega^T(X, \omega^T(\alpha, \epsilon)) + K_2\, \omega_r^T(f, \epsilon) \\ &\quad + (Lr + F_0) \int_0^{\beta_T} \omega_r^T(g, \epsilon)\, ds + (Lr + F_0) \omega^T(v, \epsilon) \end{aligned} \tag{3.11}$$

Observe that $\omega_r^T(f, \epsilon) \to 0$ and $\omega_r^T(g, \epsilon) \to 0$ as $\epsilon \to 0$, which is a simple consequence of the uniform continuity of the functions f and g on the sets $[0, T] \times [-r, r]$ and $[0, T] \times [0, \beta_T] \times [-r, r]$, respectively. Moreover, from the uniform continuity of q, α, v on $[0, T]$, it follows that $\omega^T(q, \epsilon) \to 0$, $\omega^T(\alpha, \epsilon) \to 0$, $\omega^T(v, \epsilon) \to 0$ as $\epsilon \to 0$. Thus, linking the established facts with the estimate (3.10) we get

$$\omega_0^T(Q(X)) \leq L\, K_2 \omega_0^T(X) \ .$$

Consequently, we obtain

$$\omega_0(Q(X)) \leq L\,K_2\,\omega_0(X)\ . \tag{3.12}$$

Similarly, for any $x \in X$, one has

$$\begin{aligned}
|Qx(t)| &\leq |q(t)| + \big[f(t, x(\alpha(t)))\big] \left(\int_0^{\beta(t)} |g(t, s, x(\gamma(s)))|\, ds \right) \\
&\leq |q(t)| + \big[|f(t, x(\alpha(t))) - f(t,0)| + |f(t,0)|\big] \left(a(t) \int_0^{\beta(t)} b(s)\, ds \right) \\
&\leq |q(t)| + \big[\ell(t)|x(\alpha(t))| + F_0\big] v(t) \\
&\leq |q(t)| + LK_2|x(\alpha(t))| + F_0 v(t)
\end{aligned}$$

for all $t \in \mathbb{R}_+$. Therefore, from the above inequality, it follows that

$$\|QX(t)\| \leq |q(t)| + LK_2\|X(\alpha(t))\| + F_0 v(t)$$

for all $t \in \mathbb{R}_+$. Taking the supremum over t,

$$\begin{aligned}
\limsup_{t\to\infty} \|QX(t)\| &\leq \limsup_{t\to\infty} |q(t)| + LK_2 \limsup_{t\to\infty} \|X(\alpha(t))\| + F_0 \limsup_{t\to\infty} v(t) \\
&\leq LK_2 \limsup_{t\to\infty} \|X(\alpha(t))\| \\
&\leq LK_2 \limsup_{t\to\infty} \|X(t)\|.
\end{aligned}$$

Hence,

$$\begin{aligned}
\mu(QX) &= \omega_0(QX) + \limsup_{t\to\infty} \|QX(t)\| \\
&\leq LK_2\omega_0(X) + LK_2 \limsup_{t\to\infty} \|X(t)\| \\
&\leq LK_2\big[\omega_0(X) + \limsup_{t\to\infty} \|X(t)\|\big] \\
&\leq LK_2\,\mu(X)
\end{aligned}$$

where, $LK_2 < 1$.This shows that Q is a set-contraction on $\overline{\mathcal{B}}_r(0)$ with the contraction constant $k = LK_2$.

Now, we apply Theorem 2.1 to the operator equation $Qx = x$ and deduce that the operator Q has a fixed point x in the ball $\overline{\mathcal{B}}_r(\theta)$. Obviously x is a solution of the functional integral equation (3.1). Moreover, taking into account that the image of the space X under the operator Q is contained in the ball $\overline{\mathcal{B}}_r(\theta)$ we infer that the set $Fix(Q)$ of all fixed points of Q is contained in $\overline{\mathcal{B}}_r(\theta)$. Obviously, the set $Fix(Q)$ contains all solutions of the equation (3.1). On the other hand, from Remark 2.3 we conclude that the set $Fix(Q)$ belongs to the family ker μ. Now, taking into account the description of sets belonging to ker μ (given in Section 2) we deduce that all solutions of the equation (3.1) are uniformly locally asymptotically stable on $\mathbb{R}_+$ and the common attractor is the line $x(t) = 0$. This completes the proof. □

4 An Example

Consider the nonlinear QFIE of the form

$$x(t) = \frac{1}{t+1} + \left[\frac{1}{2}\sin x(2t)\right]\left(\int_0^{t/2} \frac{e^{-t}x(s^2)}{1+|x(s^2)|}\,ds\right) \tag{4.1}$$

for all $t \in \mathbb{R}_+$. Comparing QFIE (4.1) with (3.1), we obtain

$$\alpha(t) = 2t, \;\; \beta(t) = t/2, \;\; \gamma(t) = t^2 \text{ and } q(t) = \frac{1}{t+1}$$

for all$t \in \mathbb{R}_+$; and

$$f(t,x) = \frac{1}{2}\sin x \quad \text{and} \quad g(t,s,x) = \frac{e^{-t}x}{1+|x|}$$

for all $t, s \in \mathbb{R}_+$ and $x \in \mathbb{R}$. We shall show that all the above functions satisfy the conditions of Theorem 3.1.

Clearly, the functions α, β and γ are continuous and map the half real line $\mathbb{R}_+$ into itself with $\lim\limits_{t\to\infty} \alpha(t) = \lim\limits_{t\to\infty} 2t = \infty$. Next, q is continuous and

$$\lim_{t\to\infty} q(t) = 0 \quad \text{and} \quad K_1 = \sup_{t\geq 0}|q(t)| = \sup_{t\geq 0}\frac{t}{t+1} = 1.$$

Further on, the function f is continuous on $\mathbb{R}_+ \times \mathbb{R}$ and satisfies (A_2) with $L = 1/2$. T see this, let $x, y \in \mathbb{R}$. Then

$$|f(t,x) - f(t,y)| \leq \frac{1}{2}|\sin x - \sin y| \leq \frac{1}{2}|x-y|$$

for all $t \in \mathbb{R}_+$.

Finally, the function g is continuous on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$ and

$$g(t,s,x)| = \left|\frac{xe^{-t}}{1+|x|}\right| \leq e^{-t} = a(t)b(s)$$

for all $t, s \in \mathbb{R}_+$ and $x \in \mathbb{R}$. Moreover,

$$\lim_{t\to\infty} a(t)\int_0^{\beta(t)} b(s)\,ds = \lim_{t\to\infty} e^{-t}\int_0^{t/2} ds = 0 \;\; \text{and} \;\; K_2 = \sup_{t\geq 0} e^{-t}\int_0^{t/2} ds \leq 1.$$

As $LK_2 = \frac{1}{2} < 1$, we apply Theorem 3.1 to yield that the QFIE (3.1) has a solution and all solutions are uniformly locally asymptotically stable on $\mathbb{R}_+$.

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