Exact solutions to some nonlinear PDEs, travelling profiles method

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ABSTRACT
We suggest finding exact solutions of equation:

$$\frac{\partial u}{\partial t} = (\frac{\partial^m}{\partial x^m} u)^p, \quad t \geq 0, \quad x \in \mathbb{R}, \; m, \; p \in \mathbb{N}, \; p > 1,$$

by a new method that we call the travelling profiles method. This method allows us to find several forms of exact solutions including the classical forms such as travelling-wave and self-similar solutions.

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1 Introduction

Consider the following equation :

$$\frac{\partial u}{\partial t} = A_x u, \tag{1.1}$$

where $A_x u$ is a nonlinear differential operator.
For seeking exact solutions to nonlinear PDEs (1.1), there are three approaches in general:
1- Travelling-wave solutions (see for example [4, 10, 17]):
The principle of this method is to seek a solution in the form :

$$u = u(z), \qquad z = x + \lambda t \tag{1.2}$$

where the function u is solution of following differential equation :

$$A_z u - \lambda u_z' = 0.$$

2- Self-similar solutions (see for examples [4, 8, 10, 14, 16]):

This method is largely used, its principle is to seek a solution in the form :

$$u = t^{\beta}u(\xi), \qquad \xi = xt^{-\gamma}$$

where β and γ are some constants, the function u is determined by the differential equation :

$$A_{\xi}u - \beta u + \gamma\xi u'_{\xi} = 0.$$

There exists also a general form of self similar solutions in the form

$$u(x,t) = \varphi(t)u(\frac{x}{\psi(t)}), \tag{1.3}$$

where $\varphi(t)$ and $\psi(t)$ are chosen for reason of convenience in the specific problem.
3- Separation of variables (see [5, 6, 11, 12]):
For this method, there are several forms of solutions including the following forms

$$u(x,t) = F(\varphi_1(x)\psi_1(t) + \psi_2(t)), \qquad u(x,t) = F(\varphi_1(x)\psi_1(t) + \varphi_2(x)).$$

The profile F and the functions $\varphi_1(x)$, $\varphi_2(x)$, $\psi_1(t)$, $\psi_2(t)$ are to be determined.
In this paper we propose a new approach to find exact solutions to some nonlinear PDEs in the form:

$$\frac{\partial u}{\partial t} = (\frac{\partial^m}{\partial x^m}u)^p, \quad t \geq 0, \quad x \in \mathbb{R}, \ m, \ p \in \mathbb{N}, \ p > 1\,. \tag{1.4}$$

This equation engenders many vell known problems such as the porous medium equation (PME) for $m = 2$ (see [1, 8, 9]).
The new approach which we will present is called the travelling profiles method (TPM).

2 The travelling profiles method (TPM):

The principle of this method is to seek the solution of the problem (1.4) under the form

$$u\,(x,t) = c\,(t)\,\psi\left[\frac{x - b\,(t)}{a\,(t)}\right], \tag{2.1}$$

where ψ is in L^2, that one will call the based-profile. The parameters $a\,(t)\,, b\,(t)\,, c\,(t)$ are real valued functions of t.
If we put $\xi = \frac{x-b(t)}{a(t)}$ then $u\,(x,t) = c\,(t)\,\psi\,(\xi)$ and

$$\frac{\partial u}{\partial t} = \dot{c}\,(t)\,\psi - \frac{\dot{a}\,(t)}{a(t)}c(t)\xi\psi'_{\xi} - \frac{\dot{b}\,(t)}{a(t)}c(t)\psi'_{\xi}, \qquad \frac{\partial u}{\partial x} = \frac{c\,(t)}{a(t)}\psi'_{\xi}. \tag{2.2}$$

If we replace (2.2) in (1.4) we obtain

$$\dot{c}\,(t)\,\psi - \frac{\dot{a}\,(t)}{a(t)}c(t)\xi\psi'_{\xi} - \frac{\dot{b}\,(t)}{a(t)}c(t)\psi'_{\xi} = \frac{c^p\,(t)}{a^{mp}(t)}(\frac{d^m}{d\xi^m}\psi)^p.$$

Thus to have an exact solution in form (2.1), one must determine $a(t), b(t), c(t)$ and the profile ψ. The coefficients $c(t), a(t), b(t)$ are in principle determined by the solution of minimizition problem:

$$\min_{\dot{c},\dot{a},\dot{b}} \int_{-\infty}^{+\infty} \left| \frac{\partial u}{\partial t} - (\frac{\partial^m}{\partial x^m} u)^p \right|^2 dx,$$

therefore, we obtain three orthogonality equations which are read as:

$$\begin{cases} \langle \frac{\partial u}{\partial t} - (\frac{\partial^m}{\partial x^m} u)^p, \psi \rangle = 0 \\ \langle \frac{\partial u}{\partial t} - (\frac{\partial^m}{\partial x^m} u)^p, \xi \psi'_\xi \rangle = 0 \\ \langle \frac{\partial u}{\partial t} - (\frac{\partial^m}{\partial x^m} u)^p, \psi'_\xi \rangle = 0 \end{cases} \tag{2.3}$$

where $\langle ., . \rangle$ is the inner product in L^2 space.
The PDE (1.4) is then transformed into a set of three coupled ODE's :

$$\begin{cases} \frac{\dot{c}}{c} \langle \psi, \psi \rangle - \frac{\dot{a}}{a} \langle \xi \psi'_\xi, \psi \rangle - \frac{\dot{b}}{a} \langle \psi'_\xi, \psi \rangle = \frac{c^{p-1}}{a^{mp}} \langle (\frac{d^m}{d\xi^m} \psi)^p, \psi \rangle \\ \frac{\dot{c}}{c} \langle \xi \psi'_\xi, \psi \rangle - \frac{\dot{a}}{a} \langle \xi \psi'_\xi, \xi \psi'_\xi \rangle - \frac{\dot{b}}{a} \langle \xi \psi'_\xi, \psi'_\xi \rangle = \frac{c^{p-1}}{a^{mp}} \langle (\frac{d^m}{d\xi^m} \psi)^p, \xi \psi'_\xi \rangle \\ \frac{\dot{c}}{c} \langle \psi, \psi'_\xi \rangle - \frac{\dot{a}}{a} \langle \xi \psi'_\xi, \psi'_\xi \rangle - \frac{\dot{b}}{a} \langle \psi'_\xi, \psi'_\xi \rangle = \frac{c^{p-1}}{a^{mp}} \langle (\frac{d^m}{d\xi^m} \psi)^p, \psi'_\xi \rangle. \end{cases} \tag{2.4}$$

2.1 A priori estimates of solutions :

Let:

$$V_t = \{\psi,\ \xi\psi'_\xi,\ \psi'_\xi\}$$

the subspace of L^2 generated by associated functions to ψ at the moment t.
From relations (2.3), it is deduced that $\frac{\partial u}{\partial t} - (\frac{\partial^m}{\partial x^m} u)^p$ is orthogonal to subspace V_t.
In particular we have $\frac{\partial u}{\partial t} \epsilon V_t$, then $\langle \frac{\partial u}{\partial t} - (\frac{\partial^m}{\partial x^m} u)^p, \frac{\partial u}{\partial t} \rangle = 0$, thus if also $(\frac{\partial^m}{\partial x^m} u)^p$ belongs to V_t then the method provides us a weakly exact solution, which is written under the form

$$u(x,t) = c(t)\, \psi \left[\frac{x - b(t)}{a(t)} \right]. \tag{2.5}$$

2.2 Exact solutions:

Theorem :
The function $u(x,t) = c(t)\, \psi \left[\frac{x-b(t)}{a(t)} \right]$ is an exact solution of problem (1.4), if the based profile ψ is a solution of following differential equation

$$(\frac{d^m}{d\xi^m} \psi)^p = \alpha \psi + \beta \xi \psi'_\xi + \gamma \psi'_\xi, \qquad \text{where } \alpha, \beta, \gamma \in \mathbb{R}, \text{ with } \alpha, \beta, \gamma \neq 0,$$

in this case, the coefficients $c(t), a(t), b(t)$ are given by :

$$\begin{aligned} a(t) &= \left[A(-K_0^{p-1} \beta t + K_1) \right]^{\frac{1}{A}}, \\ b(t) &= \tfrac{\gamma}{\beta} \left[A(-K_0^{p-1} \beta t + K_1) \right]^{\frac{1}{A}} + K'_0\ , \\ c(t) &= K_0 \left[A(-K_0^{p-1} \beta t + K_1) \right]^{\frac{-\alpha}{\beta A}}, \end{aligned} \tag{2.6}$$

with K_0, K_0', K_1 constants and $A = mp + \frac{\alpha}{\beta}(p-1)$.

Proof

According to the estimation principle of this method, if $(\frac{\partial^m}{\partial x^m}u)^p = \frac{c^p}{a^{mp}}(\frac{d^m}{d\xi^m}\psi)^p$ belongs to the subspace V_t, then the function $u(x,t) = c(t)\,\psi\left[\frac{x-b(t)}{a(t)}\right]$ is an exact solution to equation (1.4). In this case the term $(\frac{d^m}{d\xi^m}\psi)^p$ can be expressed as a linear combination of functions $\psi, \xi\psi_\xi'$, and ψ_ξ'. In other words we have $(\frac{d^m}{d\xi^m}\psi)^p = \alpha\psi + \beta\xi\psi_\xi' + \gamma\psi_\xi'$, for $\alpha, \beta, \gamma \in \mathbb{R}$.

The coefficients $c(t), a(t), b(t)$ are obtained as follow:

When one replaces $(\frac{d^m}{d\xi^m}\psi)^p$ by the combination $\alpha\psi + \beta\xi\psi_\xi' + \gamma\psi_\xi'$ in system (2.4), we obtain:

$$MX = \frac{c^{p-1}}{a^{mp}}MF \tag{2.7}$$

with

$$M = \begin{pmatrix} \langle\psi,\psi\rangle & \langle\xi\psi_\xi',\psi\rangle & \langle\psi_\xi',\psi\rangle \\ \langle\psi,\xi\psi_\xi'\rangle & \langle\xi\psi_\xi',\xi\psi_\xi'\rangle & \langle\psi_\xi',\xi\psi_\xi'\rangle \\ \langle\psi,\psi_\xi'\rangle & \langle\xi\psi_\xi',\psi_\xi'\rangle & \langle\psi_\xi',\psi_\xi'\rangle \end{pmatrix}, \quad X = \begin{pmatrix} \frac{\dot{c}}{c} \\ -\frac{\dot{a}}{a} \\ -\frac{\dot{b}}{a} \end{pmatrix} \text{ and } F = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

where $\langle .,.\rangle$ is the inner product in L^2.

The matrix M in the system (2.7) is symmetric and invertible, therefore (2.7) can be written under the form:

$$\begin{aligned} \dot{c} &= \tfrac{c^p}{a^{mp}}\alpha \\ \dot{a} &= -\tfrac{c^{p-1}}{a^{mp-1}}\beta \\ \dot{b} &= -\tfrac{c^{p-1}}{a^{mp-1}}\gamma. \end{aligned} \tag{2.8}$$

From (2.8) we have

$$\begin{cases} c(t) = K_0\ a(t)^{\frac{-\alpha}{\beta}}, \\ b(t) = \frac{\gamma}{\beta}\ a(t) + K_0' \end{cases}, \text{ with } K_0,\ K_0' \text{ constants.} \tag{2.9}$$

If we replace (2.9) in (2.8), then we deduct (2.6).

2.3 Example:

Let us consider the equation

$$\frac{\partial u}{\partial t} = (u_x)^2, \tag{2.10}$$

If we seek an exact solution like $u(x,t) = c(t)\psi\left(\frac{x-b(t)}{a(t)}\right)$, the based-profile ψ must verify the following ODE:

$$(\psi_\xi')^2 = \alpha\psi + \beta\xi\psi_\xi' + \gamma\psi_\xi', \quad \text{with } \xi = \frac{x-b(t)}{a(t)}. \tag{2.11}$$

It is clear that the solution of this equation depends on constants α, β, and γ,
for example, if $\alpha = -\beta$ and for any γ, we have a solution of equation (2.11) given by

$$\psi(x) = (\alpha x - \gamma)k + \alpha k^2, \qquad \text{with} \;\; k \text{ constant.} \tag{2.12}$$

Then we obtain an exact solution to equation (2.10) under the form:

$$u(x,t) = c(t)\left[(\alpha \frac{x - b(t)}{a(t)} - \gamma)k + \alpha k^2\right] \tag{2.13}$$

where $c(t), a(t)$, and $b(t)$ are given (from 2.6) by:

$$\begin{array}{l} a(t) = -\alpha K_0 t + K_1 \\ c(t) = K_0(-\alpha K_0 t + K_1) \\ b(t) = \frac{\gamma}{\alpha}\left[-\alpha K_0 t + K_1\right] + K_0' \end{array},$$

with K_0, K_0', K_1 constants.

3 Some particular forms:

In our approach we can find the particular forms of well-known solutions such as travelling-wave and self-similar solutions.

3.1 Travelling-wave solutions :

If we seek a solution to equation (1.4), like

$$u(x,t) = \psi(x - b(t)), \tag{3.1}$$

we obtain a class of travelling-wave solutions, where the based profile ψ is solution of following ODE:

$$(\frac{d^m}{dz^m}\psi)^p = \gamma \psi_z', \qquad \gamma \neq 0 \tag{3.2}$$

with $z = x - b(t)$.
The parameter $b(t)$ is determined in our approach by the equation

$$\dot{b}(t) = -\frac{\langle (\frac{d^m}{dz^m}\psi)^p, \psi_z' \rangle}{\langle \psi_z', \psi_z' \rangle}, \tag{3.3}$$

from (3.2) we obtain

$$\dot{b}(t) = -\gamma \Rightarrow b(t) = -\gamma t + b_0, \qquad \text{where } b_0 = b(0). \tag{3.4}$$

Then we have here a travelling-wave solution in the form:

$$u(x,t) = \psi(x + \gamma t - b_0)$$

3.2 Self-similar solutions:

Now if we seek a solution to equation (1.4), like

$$u(x,t) = c(t)\psi\left(\frac{x}{a(t)}\right), \tag{3.5}$$

we obtain a class of self-similar solutions, where the based profile ψ is solution of following ODE:

$$(\frac{d^m}{dz^m}\psi)^p = \alpha\psi + \beta z\psi'_z, \quad \alpha \neq 0, \beta \neq 0,$$

with $z = \frac{x}{a(t)}$.
The parameters $a(t)$ and $c(t)$ are given by (2.6) as:

$$\begin{aligned} a(t) &= \left[A(-K_0^{p-1}\beta t + K_1)\right]^{\frac{1}{A}}, \\ c(t) &= K_0\left[A(-K_0^{p-1}\beta t + K_1)\right]^{\frac{-\alpha}{\beta A}} \end{aligned}, \tag{3.6}$$

with K_0, K_1 constants and $A = mp + \frac{\alpha}{\beta}(p-1)$.
Then we obtain here a general form of self-similar solutions, but in our approach, the functions $\varphi(t)$ and $\psi(t)$ are explicitly determined.

4 Conclusion:

We have presented a new approach to determine exact solutions to some type of nonlinear PDEs. The approach that we presented, is called the travelling profiles method (TPM). The principle of this method is based on a decomposition of the differential operator in a subspace of L^2 generated by associated functions to based profile ψ.

References:

[1] D. G. Aronson, *The porous medium equation*, Non linear Diffusion problems - lecture notes mathematics, vol 1224, Springer, Berlin, 1986.

[2] Barenblatt, G. I., *Dimensional Analysis*, Gordon and Breach Publ, New York, 1989.

[3] C. Basdevant, M. Holshneider, V. Perrier, *Méthode des ondelettes mobiles*, C.R.Acad.Sci.Paris, Série, P647-652, 1990.

[4] Ibragimov, N. H.(Editor), CRC Handbook of Lie group Analysis of Differential Equations, Vol 1 *Symmetries, Exact solutions and Conservation Laws*, CRS Press , Boca Raton, 1994.

[5] V.A. Galaktionov, Posashkov, *New exact solutions of parabolic equation with quadratic non-linearities*, USSR. Compt. Math. Match. Phys, Vol 29, N° 2 pp112-119, 1989.

[6] V.A. Galaktionov, V. A. Posashkov, S. A. Svirshchevskii, S. R., *Generalized separation of variables for differential equations with polynomial right-hand sides,* Dif. Uravneniya, 31(2), 253, 1995.

[7] V.A. Galaktionov, J.L. Vázques, *Regional blow up in a semilinear heat equation with convergence to a Hamilton-Jacobi equation,* SIAM J.Math. Anal.24, N°5, 1254-1276, 1993.

[8] J. Hulshof, J. L. Vazquez, *Self similar solutions of the second kind for the modified porous medium equation*, Euro J. App. Math, 5, 391-403, 1994.

[9] Ki-Ahm Lee, J. L. Vazquez, *Geometrical properties of solutions of porous medium equation for large times*, Indiana Universiy Mathematics Journal, Vol. 52, N° 4, 2003.

[10] Polyanin. A.D, Zaitsev. V. F, Handbook of Nonlinear Partial Equation, Chapman&Hall/CRC, Boca Raton, 2004.

[11] Polyanin. A.D, Zaitsev. V. F, Handbook of Exact Solutions for Ordinary Differential Equations, CRC Press, Boca Raton, New York, 1995.

[12] Polyanin, A.D, Alexei I. Zhurov, Andrei V. Vyazmin2, *Generalized Separation of Variables and Mass Transfer Equations,* J. Non-Equilib. Thermodyn, Vol. 25 pp 251-267, 2000.

[13] Michael Winkler , *Blow up of solutions to a degenerate parabolic equation not in divergence form*, J. Differential equations 192, 445-474, MR 2004d 35122, 2003.

[14] Olver. P. J., *Application of Lie Groups to Differential Equations*, Springer-Verlag, NewYork, 1986.

[15] Ovsyannikov, L. V., *Group Analysis of Differential Equations*, Academic Press, New York, 1982.

[16] Rudykh. G. A, Semenov. E. I, *On new exact solutions of one-dimensional nonlinear diffusion equation with a source*, Zhurn. Vychisl. Matem. i Matem. Fiziki, Vol. 38, N° 6, pp 971-977,1998.

[17] W.I. Newman, *Some exact solutions to a nonlinear diffusion problem in population genetics and combustion*, J. Theor. Biol, 85, 325-334, 1980.

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