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Bounded and almost automorphic solutions of a Liénard equation with a singular nonlinearity

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Abstract

We study some properties of bounded and $C^{(1)}$-almost automorphic solutions of the following Liénard equation:

$$x'' + f(x)x' + g(x) = p(t),$$

where $p : \mathbf{R} \longrightarrow \mathbf{R}$ is an almost automorphic function, f, $g : (a, b) \longrightarrow \mathbf{R}$ are continuous functions and g is strictly decreasing.

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1 Introduction

In this paper, we study some properties of bounded or $C^{(1)}$-almost automorphic solutions of the following Liénard equation:

$$x'' + f(x)x' + g(x) = p(t), \tag{1.1}$$

where $p : \mathbf{R} \longrightarrow \mathbf{R}$ is an almost automorphic function and $f, g : (a, b) \to \mathbf{R}$, $(-\infty \leq a < b \leq +\infty)$ are continuous functions. The following assumptions will be used in proving the main results:

(H1) *f and $g : (a, b) \longrightarrow \mathbf{R}$ are locally Lipschitz continuous.*

(H2) *g is strictly decreasing.*

(H3) *$f(x) \geq 0$ for all $x \in (a, b)$.*

The model of Equation (1.1) is

$$x'' + cx' + \frac{1}{x^\alpha} = p(t) \tag{1.2}$$

where $c \geq 0$, $\alpha > 0$ and $p : \mathbf{R} \longrightarrow \mathbf{R}$ is an almost automorphic function, that appears when the restoring force is a singular nonlinearity which becomes infinite at zero. Martínez-Amores and Torres in [13], then Campos and Torres in [5] describe the dynamics of Equation (1.1) in the periodic case, namely the forcing term p is periodic. Recall that the existence of periodic solutions of Equation (1.1) without friction term ($f = 0$) is proved by Lazer and Solimini in [12] and by Habets and Sanchez in [11] for some Liénard equations with singularities, more general than Equation (1.1). In [5], Campos and Torres prove that the existence of a bounded solution on $(0, +\infty)$ implies the existence of a unique periodic solution that attracts all bounded solutions on $(0, +\infty)$. Moreover, they proved that the set of initial conditions of bounded solutions on $(0, +\infty)$ is the graph of a continuous nondecreasing function. Then Cieutat extends these results to the almost periodic case in [6]. In [5], Campos and Torres use topological tools, such as free homeomorphisms (c.f. [4]), together with truncation arguments. The homeomorphisms used in [5], are the Poincaré operators of Equation (1.1), therefore these topological tools are not adapted to the almost periodic case. In [6], the method used is essentially the recurrence property of the almost periodic functions. This last property says that once a value is taken by $\phi(t)$ at some point $t \in \mathbf{R}$, it will be "almost" taken arbitrarily far in the future and in the past. Later, Ait Dads et al. [1] in the bounded case, namely the forcing term p is continuous and bounded, prove the uniqueness of the bounded solutions on $(-\infty, +\infty)$ and describe the set of initial conditions of bounded solutions on $(0, +\infty)$. Then they establish a result of existence and uniqueness of the pseudo almost periodic solution.

The notion of almost automorphic is a generalization of almost periodicity. It has been introduced in the literature by Bochner in relation to some aspect of differential geometry [2, 3] and more recently, this notion was developed by N'Guérékata (see for instance [14, 15]).

Our aim is to extend some results of [5, 6] to the almost automorphic case, namely to prove that the existence of a bounded solution on $(0, +\infty)$ implies the existence of a unique almost automorphic solution that attracts all bounded solutions on $(0, +\infty)$. Then we state and prove a result on the existence of almost automorphic solutions.

Let us first fix some notations and definitions.

We say that a function $u \in C(\mathbf{R})$ (continuous) is *almost automorphic* if for any sequence of real numbers $(t'_n)_n$, there exists a subsequence of $(t'_n)_n$, denoted $(t_n)_n$ such that

$$v(t) = \lim_{n \mapsto +\infty} u(t + t_n) \tag{1.3}$$

is well defined for each $t \in \mathbf{R}$ and

$$\lim_{n \mapsto +\infty} v(t - t_n) = u(t) \tag{1.4}$$

for each $t \in \mathbf{R}$.

If we denote by $AA(\mathbf{R})$ the space of all almost automorphic $\mathbf{R}$-valued functions, then it turns out to be a Banach space under the sup-norm.

Because of pointwise convergence, the function $v \in L^\infty(\mathbf{R})$ (the space of essentially bounded measurable functions in $\mathbf{R}$), but not necessarily continuous. It is also clear from the definition above that almost periodic functions (in the sense of Bochner [2, 10]) are almost automorphic. If we denote $AP(\mathbf{R})$, the space of all almost periodic $\mathbf{R}$-valued functions, we have $AP(\mathbf{R}) \subset AA(\mathbf{R})$.

A function $u \in C(\mathbf{R})$ is said to be $C^{(n)}$-almost automorphic if it is almost automorphic up to its *nth* derivative. We denote the space of all such functions by $AA^{(n)}(\mathbf{R})$ (see [8]).

If the limit in (1.3) is uniform on any compact subset $K \subset \mathbf{R}$, we say that u is *compact almost automorphic*. If we denote $AA_c(\mathbf{R})$, the space of compact almost automorphic $\mathbf{R}$-valued functions and $BC(\mathbf{R})$ the space of bounded and continuous $\mathbf{R}$-valued functions, we have

$$AP(\mathbf{R}) \subset AA_c(\mathbf{R}) \subset AA(\mathbf{R}) \subset BC(\mathbf{R}). \tag{1.5}$$

Similarly $AA_c^{(n)}(\mathbf{R})$ will denote the space of all $C^{(n)}$-compact almost automorphic functions. For more details on almost automorphic functions, we refer to [14, 15].

The bounded solutions considered in this paper, are the solutions such that their range is relatively compact in the domain (a, b) of Equation (1.1). More precisely, for a bounded solution x, we impose the existence of a compact set such that

$$\forall t \in \mathbf{R}, \qquad x(t) \in K \subset (a, b).$$

In the almost periodic case, this type of conditions was assumed by Corduneanu in [7, Chapter 4] and by Yoshizawa in [18, Chapter 3]. Without these conditions, the tools of the study of almost automorphic solutions of differential equations are often unusable.

For these reasons, we say that a function $x : \mathbf{R} \longrightarrow \mathbf{R}$ is *bounded on* $\mathbf{R}$ if there exist A and $B \in \mathbf{R}$ such that

$$a < A \leq x(t) \leq B < b \quad \text{for all} \quad t \in \mathbf{R},$$

where a and b are the two constants defined in Hypothesis (H1).

We also say that a function $x : (c, +\infty) \longrightarrow \mathbf{R}$ (with $-\infty \leq c < +\infty$) is *bounded in the future* if there exist A, $B \in \mathbf{R}$ and $t_0 > c$ such that

$$a < A \leq x(t) \leq B < b \quad \text{for all} \quad t > t_0.$$

Remark that if x is a periodic solution of Equation (1.1), then x is *bounded on* $\mathbf{R}$ (in the sense of above definition), but an almost periodic solution, therefore an almost automorphic solution, is not necessarily *bounded on* $\mathbf{R}$ (of course $\sup_{t\in\mathbf{R}} \mid x(t) \mid < +\infty$), because there exists an almost periodic solution x such that $\inf_{t\in\mathbf{R}} x(t) = a$ (if $a \in \mathbf{R}$). For example, we consider $x(t) := \cos(t) - \cos(2\pi t) + 2$. Since $x(t) > 0$ for all $t \in \mathbf{R}$, then x is an almost periodic solution of Equation (1.1) where $a := 0$, $b := +\infty$, $f(x) := 0$, $g(x) := -x$ and $p(t) := ((2\pi)^2 + 1)\cos(2\pi t) - 2\cos(t) - 2$. Moreover there exists a sequence $(a_n)_n$ of integers such that $\lim_{n\to+\infty} \cos(a_n) = -1$, therefore $\lim_{n\to+\infty} x(a_n) = 0$, so x is not *bounded on* $\mathbf{R}$.

The paper is organized as follows: we announce the main results (Theorem 2.1) in Section 2 and we give its proof in Section 3. Section 4 is devoted to an example.

2 Main Result

Theorem 2.1. *Assume that hypotheses (H1)-(H3) hold, and let* $p \in AA(\mathbf{R})$. *In addition, assume that Equation (1.1) has at least one solution that is bounded in the future. Then the following statements hold true:*

i) Equation (1.1) has exactly one solution ϕ *that is bounded on* $\mathbf{R}$. *Moreover* $\phi \in AA_c^{(1)}(\mathbf{R})$.

ii) Every solution x *bounded in the future of Equation (1.1) is asymptotically almost automorphic, in the sense that:*

$$\lim_{t\to+\infty} (\mid x(t) - \phi(t) \mid + \mid x'(t) - \phi'(t) \mid) = 0. \tag{2.1}$$

The proof of Theorem 2.1 will be given in Section 3.

Remark. For the proof of Theorem 2.1, we use a result on the structure of solutions that are bounded in the future and on the uniqueness of the bounded solution on $\mathbf{R}$ when the second member p is bounded and continuous (c.f. Proposition 3.1). This last proposition is established in [1]. Firstly, for the proof of Theorem 2.1, we state the existence of a solution that is bounded in the future implies the existence of a bounded solution on the whole real line. This result is well-known when the second member p is almost periodic (for instance [9, 10]). In the almost automorphic case, this result is stated when p is compact almost automorphic. For example, Fink has established similar result [9, Lemma 2], which is valid even for the following differential system in $\mathbf{R}^n$: $x'(t) = F(t, x(t))$. We cannot use [9, Lemma 2] because we do not assume that p is compact almost automorphic, but only almost automorphic. Secondly, we prove that the unique bounded solution is compact almost periodic. Since we assume that p is only almost automorphic, we cannot use [9, Corollary 1].

Corollary 2.2. *Assume that hypotheses (H1)-(H3) hold. In addition suppose that* $p \in AA(\mathbf{R})$. *If* $\inf_{t\in\mathbf{R}} p(t)$ *and* $\sup_{t\in\mathbf{R}} p(t)$ *are in the range of* g: $g(a,b)$, *then Equation (1.1) has a unique bounded solution* x *on* $\mathbf{R}$ *which is compact almost automorphic. Moreover this solution is asymptotically almost automorphic and its derivative is also compact almost automorphic.*

Remark. In the particular case of Equation (1.2), one has the existence and uniqueness of compact almost automorphic solution, when the second member p satisfies $0 < \inf_{t\in\mathbf{R}} p(t) \leq \sup_{t\in\mathbf{R}} p(t) < +\infty$ and p is almost automorphic.

Proof of Corollary 2.2. We use Theorem 2.1. It suffices to prove the existence of a solution of Equation (1.1) that is bounded on $\mathbf{R}$. For that we adapt a result of Opial [16, Théorème 2]. In the particular case where $p(t) = p_0$ for each $t \in \mathbf{R}$, i.e. $\inf_{t\in\mathbf{R}} p(t) = \sup_{t\in\mathbf{R}} p(t)$, there exists $x_0 \in (a, b)$ such that $g(x_0) = p_0$, therefore $x(t) = x_0$ for each $t \in \mathbf{R}$, is a solution that is bounded on the $\mathbf{R}$.

Now we assume that $\inf_{t\in\mathbf{R}} p(t) < \sup_{t\in\mathbf{R}} p(t)$. By hypothesis on the range of g and by (H2), there exist A and $B \in \mathbf{R}$ such that $g(A) = \sup_{t\in\mathbf{R}} p(t)$ and $g(B) = \inf_{t\in\mathbf{R}} p(t)$ and $a < A < B < b$. Let $\tilde{f}$ and $\tilde{g}$ be extensions of $f_{/[A,B]}$ and $g_{/[A,B]}$. The extension $\tilde{f}$ is defined by $\tilde{f} : \mathbf{R} \longrightarrow \mathbf{R}$ with

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if} \quad A \leq x \leq B \\ f(A) & \text{if} \quad x < A \\ f(B) & \text{if} \quad x > B. \end{cases}$$

In a similar way we define $\tilde{g}$. Obviously $\tilde{f}$ and $\tilde{g}$ are continuous.
Now set

$$F(t,x,y) := p(t) - \tilde{f}(x)y - \tilde{g}(x),$$

$$V(y) := 2+ \mid y \mid,$$

$$T(t) := \max\left(\mid p(t) \mid, \sup_{A\le x\le B} f(x), \sup_{A\le x\le B} \mid g(x) \mid\right),$$

for each t, x and $y \in \mathbf{R}$. Then

i) $F \in C(\mathbf{R}^3, \mathbf{R})$ and $F(t, A, 0) \le 0 \le F(t, B, 0)$ for each $t \in \mathbf{R}$,

ii) V and T are nonnegative and continuous functions on $\mathbf{R}$ such that V satisfies $\int_0^{+\infty} \frac{y}{V(y)}dy = +\infty$, $V(-y) = V(y)$ and $V(y) \ge 1$ for each $y \in \mathbf{R}$,

iii) $\mid F(t,x,y) \mid\le T(t)V(y)$ for each t, $y \in \mathbf{R}$ and $x \in [A, B]$.

By using [16, Théorème 2], we can assert that the equation

$$x'' = F(t,x,x')$$

admits at least a solution x satisfying $A \le x(t) \le B$ for each $t \in \mathbf{R}$, therefore x is a solution of Equation (1.1) that is bounded on $\mathbf{R}$. This ends the proof.■

3 Proof of Theorem 2.1

The object of this section is to prove Theorem 2.1. For the reader's convenience, we recall the following results.

Proposition 3.1. (Ait Dads, Lhachimi and Cieutat [1]). *Assume that hypotheses (H1)-(H3) hold. We also suppose that* $p \in BC(\mathbf{R})$. *Then we get:*

i) Any pair of distinct solutions of Equation (1.1) x_1 *and* x_2 *bounded in the future, satisfy*

$$(x_1(t) - x_2(t))(x_1'(t) - x_2'(t)) < 0 \tag{3.1}$$

for every t *where both solutions are defined and*

$$\lim_{t\to+\infty} (\mid x_1(t) - x_2(t) \mid + \mid x_1'(t) - x_2'(t) \mid) = 0, \tag{3.2}$$

ii) Equation (1.1) has at most one bounded solution on $\mathbf{R}$.

Remark. Relation (3.1) implies that $t \longrightarrow\mid x_1(t) - x_2(t) \mid$ is strictly decreasing and any two distinct solutions bounded in the future have no common point.

Lemma 3.2. (Cieutat [6]). *Assume that* $p \in BC(\mathbf{R})$*,* f *and* $g \in C(a,b)$*. Let* $I = (t_0, +\infty)$ *with* $t_0 = -\infty$ *or* $t_0 \in \mathbf{R}$*. If* x *is a solution of Equation (1.1) which is bounded in the future (respectively bounded on* $\mathbf{R}$*), i.e.* $a < A \leq x(t) \leq B < b$ *for all* $t > t_0$ *(respectively* $t \in \mathbf{R}$*), then the derivatives* x' *and* x'' *are bounded in the future (respectively bounded on* $\mathbf{R}$*), i.e.* $\sup_{t\in I} \mid x'(t) \mid \leq c_1 < +\infty$ *and* $\sup_{t\in I} \mid x''(t) \mid \leq c_2 < +\infty$ *where*

$$c_0 := \max(\mid A \mid, \mid B \mid), \tag{3.3}$$

$$\begin{aligned} c_1 := \frac{1}{2} \sup_{t\in\mathbf{R}} \mid p(t) \mid + \frac{1}{2} \sup_{A\leq z\leq B} \mid g(z) \mid \\ +2c_0 + 4c_0 \sup_{A\leq z\leq B} \mid f(z) \mid < +\infty \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} c_2 := \sup_{t\in I} \mid p(t) \mid + \sup_{A\leq z\leq B} \mid g(z) \mid \\ +c_1 \sup_{A\leq z\leq B} \mid f(z) \mid < +\infty. \end{aligned} \tag{3.5}$$

Lemma 3.3 wil play a crucial role in the proof of Theorem 2.1. When $p \in C(\mathbf{R})$, recall that x is a (classical) solution on $\mathbf{R}$ of the differential equation (1.1), if $x \in C^2(\mathbf{R})$ (of class C^2) and $x(t)$ satisfies Equation (1.1) for each $t \in \mathbf{R}$.

Let $p \in L^\infty(\mathbf{R})$. We say that x is a *weak* solution on $\mathbf{R}$ of Equation (1.1), if $x \in C^1(\mathbf{R})$ (of class C^1) and satisfies

$$x'(t) + \int_s^t \{f(x(\sigma))x'(\sigma) + g(x(\sigma))\}\, d\sigma = x'(s) + \int_s^t p(\sigma)\, d\sigma, \tag{3.6}$$

for each s and $t \in \mathbf{R}$ such that $s \leq t$.

Obviously a classical solution is a weak solution and in the particular case where p is continuous, the notion of weak solution and classical solution are equivalent.

Lemma 3.3. *Let $e \in L^\infty(\mathbf{R})$ and $f,\ g \in C(\mathbf{R})$. We assume that u is a weak solution bounded on $\mathbf{R}$ of*

$$u'' + f(u)u' + g(u) = e(t), \tag{3.7}$$

such that $u' \in L^\infty(\mathbf{R})$ and u' is k-Lipschitzian on $\mathbf{R}$ for some constant k. If there exist a numerical sequence $(t'_n)_n$ and $e_* \in L^\infty(\mathbf{R})$ such that

$$\forall t \in \mathbf{R}, \quad \lim_{n\to+\infty} \mid e(t+t'_n) - e_*(t) \mid = 0, \tag{3.8}$$

then there exists a subsequence of $(t'_n)_n$ denoted $(t_n)_n$ such that

$$u(t+t_n) \to v(t) \qquad \text{as} \quad n \to +\infty, \tag{3.9}$$

$$u'(t+t_n) \to v'(t) \qquad \text{as} \quad n \to +\infty \tag{3.10}$$

uniformly on each compact subset of $\mathbf{R}$, where v is a weak solution bounded on $\mathbf{R}$ of

$$v'' + f(v)v' + g(v) = e_*(t), \tag{3.11}$$

such that $v' \in L^\infty(\mathbf{R})$ and v' is k-Lipschitzian on $\mathbf{R}$.

Proof. Since u is a bounded on $\mathbf{R}$, there exist A and $B \in \mathbf{R}$ such that for each $t \in \mathbf{R}$

$$a < A \le u(t) \le B < b.$$

If we denote by

$$u_n(t) := u(t+t'_n), \tag{3.12}$$

then $u_n \in C^1(\mathbf{R})$ and satisfies, for each $t \in \mathbf{R}$ and $n \in \mathbf{N}$

$$a < A \le u_n(t) \le B < b. \tag{3.13}$$

Moreover, since $u' \in L^\infty(\mathbf{R})$, then for each $t \in \mathbf{R}$

$$\mid u'_n(t) \mid \le c := \sup_{t\in\mathbf{R}} \mid u'(t) \mid < +\infty, \tag{3.14}$$

and thus we obtain

$$\mid u_n(t) - u_n(s) \mid \le c \mid t-s \mid \tag{3.15}$$

for each $s,\ t \in \mathbf{R}$ and $n \in \mathbf{N}$. From (3.13) and (3.15), we deduce that for each $t \in \mathbf{R}$, $\{u_n(t); n \in \mathbf{N}\}$ is a bounded subset of $\mathbf{R}$ and the sequence $(u_n)_n$ is equicontinuous. By help of Arzela Ascoli's theorem [17, p. 312], we can assert that $\{u_n; n \in \mathbf{N}\}$ is a relatively compact subset of $C(\mathbf{R})$ endowed with the topology of compact convergence. From the sequence $(t'_n)_n$, we can extract a subsequence $(t_n)_n$ such that there exists $v \in C(\mathbf{R})$ and (3.9) holds. Moreover since u' is k-Lipschitzian on $\mathbf{R}$, then one has

$$| u'(t+t_n) - u'(s+t_n) | \leq k \mid t-s \mid \tag{3.16}$$

for each s, $t \in \mathbf{R}$ and $n \in \mathbf{N}$. Using (3.14), (3.16) and applying Arzela Ascoli's theorem, we deduce that there exist $w \in C(\mathbf{R})$ and a subsequence of $(t_n)_n$ (which we denote by the same) such that

$$u'(t+t_n) \to w(t) \qquad \text{as} \quad n \to +\infty$$

uniformly on each compact subset of $\mathbf{R}$. With (3.9), we deduce that $w = v'$, consequently (3.10) holds. By assumptions, $u \in C^1(\mathbf{R})$, $u' \in L^\infty(\mathbf{R})$ and u' is k-Lipschitzian, then the convergence (3.9) and (3.10) and relations (3.13), (3.14) and (3.16) imply that $v \in C^1(\mathbf{R})$, v is bounded on $\mathbf{R}$, $v' \in L^\infty(\mathbf{R})$ and v' is k-Lipschitzian.

It remains to prove that v is a weak solution of Equation (3.11). Since u is a weak solution of Equation (3.7), then for each $s \leq t$, we have

$$u'(t) + \int_s^t \{f(u(\sigma))u'(\sigma) + g(u(\sigma))\}\, d\sigma = u'(s) + \int_s^t e(\sigma)\, d\sigma,$$

therefore

$$u'(t+t_n) + \int_s^t \{f(u(\sigma+t_n))u'(\sigma+t_n) + g(u(\sigma+t_n))\}\, d\sigma$$

$$= u'(s+t_n) + \int_s^t e(\sigma+t_n)\, d\sigma. \tag{3.17}$$

Moreover, we have $| e(\sigma+t_n) | \leq \sup_{t\in\mathbf{R}} | e(t) | < +\infty$ for each $\sigma \in [s,t]$ and by Lebesgue's dominated convergence theorem, we obtain

$$\lim_{n\to+\infty} \int_s^t e(\sigma+t_n)\, d\sigma = \int_s^t e_*(\sigma)\, d\sigma. \tag{3.18}$$

By (3.9), (3.10), (3.17) and (3.18), we deduce that

$$v'(t) + \int_s^t \{f(v(\sigma)v'(\sigma) + g(v(\sigma))\}\, d\sigma = v'(s) + \int_s^t e_*(\sigma)\, d\sigma,$$

therefore v is a weak solution of Equation (3.11).■

Proof of Theorem 2.1. i) let $(t_n)_n$ a sequence of real numbers such that

$$\lim_{n\to+\infty} t_n = +\infty. \tag{3.19}$$

Since p is almost automorphic, then there exists a subsequence of $(t_n)_n$ (which denote by the same)) such that for each $t \in \mathbf{R}$

$$\lim_{n\to+\infty} p(t+t_n) = p_*(t), \tag{3.20}$$

$$\lim_{n\to+\infty} p_*(t-t_n) = p(t). \tag{3.21}$$

Let x be a solution that is bounded in the future; therefore there exist A, B and $t_0 \in \mathbf{R}$ such that

$$a < A \leq x(t) \leq B < b \qquad \text{for all } t > t_0 \tag{3.22}$$

and for each s and $t \in \mathbf{R}$ such that $t_0 < s \leq t$

$$x'(t) + \int_s^t \{f(x(\sigma)x'(\sigma) + g(x(\sigma))\}\, d\sigma = x'(s) + \int_s^t p(\sigma)\, d\sigma. \tag{3.23}$$

By Lemma 3.2, there exists c_1 and $c_2 > 0$ such that

$$\sup_{t>t_0} \mid x'(t) \mid \leq c_1 < +\infty, \tag{3.24}$$

$$\sup_{t>t_0} \mid x''(t) \mid \leq c_2 < +\infty \tag{3.25}$$

and by using the mean value theorem, we obtain

$$\mid x'(t) - x'(s) \mid \leq c_2 \mid t-s \mid \tag{3.26}$$

for each s and $t \in \mathbf{R}$ such that $s,\ t > t_0$. Given any interval $(\tau, +\infty)$, for $n \in \mathbf{N}$ sufficiently large $(\tau + t_n \geq t_0)$, $t \to x(.+t_n)$ is defined on $(\tau, +\infty)$. Moreover (3.22), (3.24) and (3.25) imply

$$a < A \leq x(t+t_n) \leq B < b \quad \text{for all } t \in (\tau, +\infty), \tag{3.27}$$

$$\mid x'(t+t_n) \mid \leq c_1 \qquad \text{for all } t \in (\tau, +\infty), \tag{3.28}$$

$$\mid x''(t+t_n) \mid \leq c_2 \qquad \text{for all } t \in (\tau, +\infty). \tag{3.29}$$

Taking τ as a sequence going to $-\infty$ and applying Arzela Ascoli's theorem and using a diagonal argument, we can assert that there exist $x_* \in C^1(\mathbf{R})$ and a subsequence of $(t_n)_n$ such that

$$x(t+t_n) \to x_*(t) \qquad \text{as} \quad n \to +\infty, \tag{3.30}$$

$$x'(t+t_n) \to x'_*(t) \qquad \text{as} \quad n \to +\infty \tag{3.31}$$

uniformly on each compact subset of $\mathbf{R}$. Since x satisfies (3.23), then for each $s \le t$ and for $n \in \mathbf{N}$ sufficiently large, we have

$$x'(t+t_n) + \int_s^t \{f(x(\sigma+t_n)x'(\sigma+t_n) + g(x(\sigma+t_n))\}\, d\sigma$$

$$= x'(s+t_n) + \int_s^t p(\sigma+t_n)\, d\sigma. \tag{3.32}$$

Now applying the Lebesgue's dominated convergence theorem, we obtain that (3.20) implies

$$\lim_{n\to+\infty} \int_s^t p(\sigma+t_n)\, d\sigma = \int_s^t p_*(\sigma)\, d\sigma, \tag{3.33}$$

thus with (3.30)-(3.33), we deduce that x_* is a weak solution on $\mathbf{R}$ of

$$x''_* + f(x_*)x'_* + g(x_*) = p_*(t). \tag{3.34}$$

From (3.26)-(3.28), (3.30) and (3.31), we deduce that x_* is bounded on $\mathbf{R}$ and $x'_* \in L^\infty(\mathbf{R})$ and x'_* is Lipschitzian. Applying Lemma 3.3, $u = x_*$, $e = p_*$ and the sequence $(-t_n)_n$ (c.f. (3.21)), we obtain the existence of a weak solution ϕ of Equation (1.1) that is bounded on $\mathbf{R}$. Since p is a continuous function, then ϕ is a classical solution on $\mathbf{R}$ of Equation (1.1). The uniqueness of the bounded solution of Equation (1.1) follows from Proposition 3.1.

To check that ϕ and its derivative ϕ' are compact almost automorphic, we have to prove that if $(t_n)_n$ is any sequence of real numbers, then one can pick up a subsequence of $(t_n)_n$ such that

$$\phi(t+t_n) \to \phi_*(t) \qquad \text{as} \quad n \to +\infty, \tag{3.35}$$

$$\phi'(t+t_n) \to \phi'_*(t) \qquad \text{as} \quad n \to +\infty \tag{3.36}$$

uniformly on each compact subset of $\mathbf{R}$ and

$$\forall t \in \mathbf{R}, \quad \lim_{n\mapsto+\infty} \phi_*(t-t_n) = \phi(t), \tag{3.37}$$

$$\forall t \in \mathbf{R}, \quad \lim_{n \mapsto +\infty} \phi'_*(t - t_n) = \phi'(t). \tag{3.38}$$

In fact by assumption, we can choose a subsequence of $(t_n)_n$ such that (3.20) and (3.21) hold. By applying Lemma 3.3 with $u = \phi$, $e = p$ and the sequence $(t_n)_n$ we obtain (3.35) and (3.36) where ϕ_* is a weak solution on $\mathbf{R}$ of Equation (3.34), which satisfies all hypotheses of Lemma 3.3. Applying again Lemma 3.3 to $u = \phi_*$, $e = p_*$ and the sequence $(-t_n)_n$, we obtain that

$$\forall t \in \mathbf{R}, \quad \lim_{n \mapsto +\infty} \phi_*(t - t_n) = \psi(t), \tag{3.39}$$

$$\forall t \in \mathbf{R}, \quad \lim_{n \mapsto +\infty} \phi'_*(t - t_n) = \psi'(t) \tag{3.40}$$

(for a subsequence) where ψ is a weak solution on $\mathbf{R}$ of Equation (1.1). Since p is continuous, then ψ is a classical solution on $\mathbf{R}$ of Equation (1.1). By uniqueness of the solution of Equation (1.1) that is bounded on $\mathbf{R}$, we deduce that $\psi = \phi$, therefore (3.35)-(3.38) are fulfilled, thus ϕ and ϕ' are compact almost automorphic.

ii) It is straightforward from Proposition 3.1.$\blacksquare$

4 Example

For illustration, we propose the following Liénard equation:

$$x''(t) + x^2(t)x'(t) + \frac{1}{x^\alpha(t)} = 1 + \varepsilon + \sin \frac{1}{2 + \cos t + \cos \sqrt{2}t}, \tag{4.1}$$

where α and $\varepsilon > 0$. Equation (4.1) presents a singular nonlinearity $g : (0, +\infty) \longrightarrow \mathbf{R}$ with $g(x) = \dfrac{1}{x^\alpha}$, which becomes infinite at zero. Its second member p defined by

$$p(t) = 1 + \varepsilon + \sin \frac{1}{2 + \cos t + \cos \sqrt{2}t}$$

is almost automorphic, but not almost periodic. (Example due to Levitan; see also [14]). Since $g(0, +\infty) = (0, +\infty)$ and $0 < \inf_{t \in \mathbf{R}} p(t) = \varepsilon < \sup_{t \in \mathbf{R}} p(t) < +\infty$, by Corollary 2.2, we deduce that Equation (4.1) admits a unique bounded solution x on $\mathbf{R}$:

$$0 < \inf_{t \in \mathbf{R}} x(t) = \varepsilon \leq \sup_{t \in \mathbf{R}} x(t) < +\infty.$$

Moreover $x \in AA^1_c(\mathbf{R})$ and x is asymptotically almost automorphic (in the sense of Theorem 2.1).

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