

Three Positive Solutions for p -Laplacian Functional Dynamic Equations on Time Scales

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Abstract

In this paper, existence criteria of three positive solutions to the following p -Laplacian functional dynamic equation on time scales

$$\begin{cases} [\Phi_p(u^\Delta(t))]^\nabla + a(t)f(u(t), u(\mu(t))) = 0, & t \in (0, T), \\ u_0(t) = \varphi(t), & t \in [-r, 0], \quad u(0) - B_0(u^\Delta(\eta)) = 0, \quad u^\Delta(T) = 0, \end{cases}$$

are established by using the well-known Five Functionals Fixed Point Theorem.

Keywords: Time scale; p -Laplacian functional dynamic equation; Boundary value problem; Positive solution; fixed point theorem

MSC: 39K10, 34B15

1 Introduction

The theory of dynamic equations on time scales has been a new important mathematical branch (see, for example, [1-5]) since it was initiated by Hilger [6]. At the same time, boundary value problems (BVPs) for dynamic equation on time scales have received considerable attention [7-15]. However, to the best of our knowledge, few papers can be found in the literature for BVPs of p -Laplacian dynamic equations on time scales, especially for p -Laplacian functional dynamic equations on time scales [12, 14].

Let \mathbf{T} be a time scale, i.e., \mathbf{T} is a nonempty closed subset of R . Let $0, T$ be points in \mathbf{T} , an interval $(0, T)$ denote time scales interval, that is, $(0, T) := (0, T) \cap \mathbf{T}$. Other types of intervals are defined similarly. Some definitions concerning time scales can be found in [2-4].

In this paper, we are concerned with the existence of positive solutions for the p -Laplacian functional dynamic equation on time scale

$$\begin{cases} [\Phi_p(u^\Delta(t))]^\nabla + a(t)f(u(t), u(\mu(t))) = 0, & t \in (0, T), \\ u_0(t) = \varphi(t), & t \in [-r, 0], \quad u(0) - B_0(u^\Delta(\eta)) = 0, \quad u^\Delta(T) = 0, \end{cases} \quad (1.1)$$

where $\Phi_p(s)$ is p -Laplacian operator, i.e., $\Phi_p(s) = |s|^{p-2}s$, $p > 1$, $(\Phi_p)^{-1} = \Phi_q$, $\frac{1}{p} + \frac{1}{q} = 1$, $\eta \in (0, \rho(T))$ and

(C₁) $f : (R^+)^2 \rightarrow R^+$ is continuous ;

(C₂) $a : \mathbf{T} \rightarrow R^+$ is left dense continuous (i.e., $a \in C_{\text{ld}}(\mathbf{T}, R^+)$) and does not vanish identically on any closed subinterval of $[0, T]$, where $C_{\text{ld}}(\mathbf{T}, R^+)$ denotes the set of all left dense continuous functions from \mathbf{T} to R^+ ;

(C₃) $\varphi : [-r, 0] \rightarrow R^+$ is continuous and $r > 0$;
 (C₄) $\mu : [0, T] \rightarrow [-r, T]$ is continuous, $\mu(t) \leq 0$ for all t ;
 (C₅) $B_0 : R \rightarrow R$ is continuous and satisfies the condition that there are $A \geq B \geq 0$ such that

$$Bv \leq B_0(v) \leq Av, \text{ for all } v \geq 0.$$

We note that in [16], Li and Shen studied the problem (1.1) when $\mathbf{T} = R$, $\varphi(t) = 0, t \in [-r, 0]$ and the nonlinear term is not involved $u(\mu(t))$. They imposed conditions on f to yield at least three positive solutions to the problem (1.1), by applying the Five Functionals Fixed Point Theorem [17] (which is a generalization of the Leggett-Williams Fixed-Point Theorem [18]).

In [14], Song and Xiao considered the problem (1.1), by using a double fixed-point theorem due to Avery et al. [19] in a cone, and obtained the existence of two positive solutions.

Motivated by [14] and [16], we shall show that the problem (1.1), has at least three positive solutions by means of the Five Functionals Fixed Point Theorem.

Let γ, β, θ be nonnegative, continuous, convex functionals on P and α, ψ be nonnegative, continuous, concave functionals on P . Then, for nonnegative real numbers h, a, b, d and c , we define the convex sets

$$\begin{aligned} P(\gamma, c) &= \{x \in P : \gamma(x) < c\}, \\ P(\gamma, \alpha, a, c) &= \{x \in P : a \leq \alpha(x), \gamma(x) \leq c\}, \\ Q(\gamma, \beta, d, c) &= \{x \in P : \beta(x) \leq d, \gamma(x) \leq c\}, \\ P(\gamma, \theta, \alpha, a, b, c) &= \{x \in P : a \leq \alpha(x), \theta(x) \leq b, \gamma(x) \leq c\} \text{ and} \\ Q(\gamma, \beta, \psi, h, d, c) &= \{x \in P : h \leq \psi(x), \beta(x) \leq d, \gamma(x) \leq c\}. \end{aligned}$$

To prove our main results, we need the following Five Functionals Fixed Point Theorem [17].

Theorem 1.1. Let P be a cone in a real Banach space E . Suppose there exist positive numbers c and M , nonnegative, continuous, concave functionals α and ψ on P , and nonnegative, continuous, convex functionals γ, β and θ on P , with

$$\alpha(x) \leq \beta(x) \text{ and } \|x\| \leq M\gamma(x)$$

for all $x \in \overline{P(\gamma, c)}$. Suppose

$$F : \overline{P(\gamma, c)} \rightarrow \overline{P(\gamma, c)}$$

is completely continuous and there exist nonnegative numbers h, a, k, b , with $0 < a < b$ such that:

- (i) $\{x \in P(\gamma, \theta, \alpha, b, k, c) : \alpha(x) > b\} \neq \phi$ and $\alpha(Fx) > b$ for $x \in P(\gamma, \theta, \alpha, b, k, c)$;
- (ii) $\{x \in Q(\gamma, \beta, \psi, h, a, c) : \beta(x) < a\} \neq \phi$ and $\beta(Fx) < a$ for $x \in Q(\gamma, \beta, \psi, h, a, c)$;
- (iii) $\alpha(Fx) > b$ for $x \in P(\gamma, \alpha, b, c)$ with $\theta(Fx) > k$;
- (iv) $\beta(Fx) < a$ for $x \in Q(\gamma, \beta, a, c)$ with $\psi(Fx) < h$.

Then F has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\gamma, c)}$ such that

$$\beta(x_1) < a, b < \alpha(x_2), \text{ and } a < \beta(x_3) \text{ with } \alpha(x_3) < b.$$

2 Existence of Three Positive Solutions

We note that $u(t)$ is a solution of the BVP (1.1) if and only if

$$u(t) = \begin{cases} B_0 \left(\Phi_q \left(\int_{\eta}^T a(r) f(u(r), u(\mu(r))) \nabla r \right) \right) \\ \quad + \int_0^t \Phi_q \left(\int_s^T a(r) f(u(r), u(\mu(r))) \nabla r \right) \Delta s, & t \in [0, T], \\ \varphi(t), & t \in [-r, 0]. \end{cases}$$

Let $E = C_{\text{Id}}([0, T], R)$ be endowed with $\|u\| = \sup_{t \in [0, T]} |u(t)|$, so E is a Banach space. Define cone $P \subset E$ by

$$P = \left\{ u \in E : u \text{ is concave and nonnegative valued on } [0, T], \text{ and } u^\Delta(T) = 0 \right\}.$$

For each $u \in E$, extend $u(t)$ to $[-r, T]$ with $u(t) = \varphi(t)$ for $t \in [-r, 0]$.

Define $F: P \rightarrow E$ by

$$(Fu)(t) = B_0 \left(\Phi_q \left(\int_{\eta}^T a(r) f(u(r), u(\mu(r))) \nabla r \right) \right) \\ + \int_0^t \Phi_q \left(\int_s^T a(r) f(u(r), u(\mu(r))) \nabla r \right) \Delta s, \quad t \in [0, T].$$

We seek a point, u_1 , of F in the cone P . Define

$$u(t) = \begin{cases} u_1(t), & t \in [0, T], \\ \varphi(t), & t \in [-r, 0]. \end{cases}$$

Then $u(t)$ denotes a positive solution of the BVP (1.1).

We have the following results

Lemma 2.1. Let $u \in P$, then

- (1) $u(t) \geq \frac{t}{T} \|u\|$ for $t \in [0, T]$, and
- (2) $\tau u(\varsigma) \leq \varsigma u(\tau)$ for $0 < \tau < \varsigma < T$ and $\tau, \varsigma \in \mathbf{T}$.

Proof. (1) is Lemma 3.1 of [12]. It is easy to conclude that (2) is satisfied by the concavity of u .

Set

$$Y_1 = \{t \in [0, T] : \mu(t) < 0\}; \quad Y_2 = \{t \in [0, T] : \mu(t) \geq 0\}; \quad Y_3 = Y_1 \cap [l, T].$$

Throughout this paper, we assume $Y_3 \neq \emptyset$ and $\int_{Y_3} a(r) \nabla r > 0$.

Let $l \in \mathbf{T}$ be fixed such that $0 < \eta < l < T$, and define the nonnegative, continuous, concave functionals α , ψ and the nonnegative, continuous, convex functionals β , θ , γ on the cone P respectively as

$$\begin{aligned} \gamma(u) &= \theta(u) = \max_{t \in [0, \eta]} u(t) = u(\eta), \quad \alpha(u) = \min_{t \in [l, T]} u(t) = u(l), \\ \beta(u) &= \max_{t \in [0, l]} u(t) = u(l), \quad \psi(u) = \min_{t \in [\eta, T]} u(t) = u(\eta). \end{aligned}$$

We observe that $\alpha(u) = \beta(u)$ for each $u \in P$.

In addition, by Lemma 2.1, we have $\gamma(u) = u(\eta) \geq \frac{\eta}{T} \|u\|$. Hence $\|u\| \leq \frac{T}{\eta} \gamma(u)$ for all $u \in P$. For convenience, we define

$$\begin{aligned}\mu &= (A + \eta)\Phi_q \left(\int_0^T a(r)\nabla r \right), \delta = (B + l)\Phi_q \left(\int_{Y_3} a(r)\nabla r \right), \\ \lambda &= (A + l)\Phi_q \left(\int_0^T a(r)\nabla r \right).\end{aligned}$$

We now state growth conditions on f so that the BVP (1.1) has at least three positive solutions.

Theorem 2.1. Let $0 < a < \frac{l}{T}b < \frac{\eta l}{T^2}c$, $\mu b < \delta c$, and suppose that f satisfies the following conditions:

- (H₁) $f(u, \varphi(s)) < \Phi_p \left(\frac{c}{\mu} \right)$, if $0 \leq u \leq \frac{T}{\eta}c$, uniformly in $s \in [-r, 0]$, and $f(u_1, u_2) < \Phi_p \left(\frac{c}{\mu} \right)$, if $0 \leq u_i \leq \frac{T}{\eta}c, i = 1, 2$;
- (H₂) $f(u, \varphi(s)) > \Phi_p \left(\frac{b}{\delta} \right)$, if $b \leq u \leq \left(\frac{T}{\eta} \right)^2 b$, uniformly in $s \in [-r, 0]$;
- (H₃) $f(u, \varphi(s)) < \Phi_p \left(\frac{a}{\lambda} \right)$, if $0 \leq u \leq \frac{T}{l}a$, uniformly in $s \in [-r, 0]$, and $f(u_1, u_2) < \Phi_p \left(\frac{a}{\lambda} \right)$, if $0 \leq u_i \leq \frac{T}{l}a, i = 1, 2$.

Then the BVP (1.1) has at least three positive solutions of the form

$$u(t) = \begin{cases} u_i(t), & t \in [0, T], \quad i = 1, 2, 3, \\ \varphi(t), & t \in [-r, 0], \end{cases}$$

where $\max_{t \in [0, l]} u_1(t) < a$, $\min_{t \in [l, T]} u_2(t) > b$, and $a < \max_{t \in [0, l]} u_3(t)$ with $\min_{t \in [l, T]} u_3(t) < b$.

Proof. By [12], it is known that $F : P \rightarrow P$ is completely continuous.

Let $u \in P(\gamma, c)$, then $\gamma(u) = \max_{t \in [0, \eta]} u(t) = u(\eta) \leq c$, consequently, $0 \leq u(t) \leq c$ for $t \in [0, \eta]$. Since $u(\eta) \geq \frac{\eta}{T}u(T)$, so $\|u\| = u(T) \leq \frac{T}{\eta}u(\eta) \leq \frac{T}{\eta}c$, this implies

$$0 \leq u(t) \leq \frac{T}{\eta}c, \text{ for } t \in [0, T].$$

From (H₁), we have

$$\begin{aligned}\gamma(Fu) &= (Fu)(\eta) \\ &= B_0 \left(\Phi_q \left(\int_{\eta}^T a(r)f(u(r), u(\mu(r)))\nabla r \right) \right) + \int_0^{\eta} \Phi_q \left(\int_s^T a(r)f(u(r), u(\mu(r)))\nabla r \right) \Delta s \\ &\leq A\Phi_q \left(\int_0^T a(r)f(u(r), u(\mu(r)))\nabla r \right) + \eta\Phi_q \left(\int_0^T a(r)f(u(r), u(\mu(r)))\nabla r \right) \\ &= (A + \eta)\Phi_q \left[\int_{Y_1} a(r)f(u(r), \varphi(\mu(r)))\nabla r + \int_{Y_2} a(r)f(u(r), u(\mu(r)))\nabla r \right] \\ &< (A + \eta)\Phi_q \left(\int_0^T a(r)\nabla r \right) \frac{c}{\mu} \\ &= c.\end{aligned}$$

Therefore

$$Fu \in \overline{P(\gamma, c)}.$$

We now turn to property (i) of Theorem 1.1. Choosing $u \equiv \frac{T}{\eta}b$, $k = \frac{T}{\eta}b$, it follows that

$$\alpha(u) = u(l) = \frac{T}{\eta}b > b, \quad \theta(u) = u(\eta) = \frac{T}{\eta}b = k, \quad \gamma(u) = u(\eta) = \frac{T}{\eta}b < c,$$

which shows that $\{u \in P(\gamma, \theta, \alpha, b, k, c) : \alpha(u) > b\} \neq \emptyset$, and for $u \in P(\gamma, \theta, \alpha, b, \frac{T}{\eta}b, c)$, we have

$$b \leq u(t) \leq \left(\frac{T}{\eta}\right)^2 b, \quad \text{for } t \in [l, T].$$

From (H₂), we have

$$\begin{aligned} \alpha(Fu) &= (Fu)(l) \\ &= B_0 \left(\Phi_q \left(\int_{\eta}^T a(r) f(u(r), u(\mu(r))) \nabla r \right) \right) + \int_0^l \Phi_q \left(\int_s^T a(r) f(u(r), u(\mu(r))) \nabla r \right) \Delta s \\ &\geq B \Phi_q \left(\int_l^T a(r) f(u(r), u(\mu(r))) \nabla r \right) + l \Phi_q \left(\int_l^T a(r) f(u(r), u(\mu(r))) \nabla r \right) \\ &\geq (B + l) \Phi_q \left(\int_{Y_3} a(r) f(u(r), \varphi(\mu(r))) \nabla r \right) \\ &> (B + l) \Phi_q \left(\int_{Y_3} a(r) \nabla r \right) \frac{b}{\delta} \\ &= b. \end{aligned}$$

We conclude that (i) of Theorem 1.1 is satisfied.

We next address (ii) of Theorem 1.1. If we take $u \equiv \frac{\eta}{T}a$, $h = \frac{\eta}{T}a$, then

$$\gamma(u) = u(\eta) = \frac{\eta}{T}a < c, \quad \psi(u) = u(\eta) = \frac{\eta}{T}a = h, \quad \beta(u) = u(l) = \frac{\eta}{T}a < a.$$

From this we know that $\{u \in Q(\gamma, \beta, \psi, h, a, c) : \beta(u) < a\} \neq \emptyset$. If $u \in Q(\gamma, \beta, \psi, \frac{\eta}{T}a, a, c)$, then

$$0 \leq u(t) \leq \frac{T}{l}a, \quad \text{for } t \in [0, T].$$

From (H₃), we have

$$\begin{aligned} \beta(Fu) &= (Fu)(l) \\ &= B_0 \left(\Phi_q \left(\int_{\eta}^T a(r) f(u(r), u(\mu(r))) \nabla r \right) \right) + \int_0^l \Phi_q \left(\int_s^T a(r) f(u(r), u(\mu(r))) \nabla r \right) \Delta s \\ &\leq A \Phi_q \left(\int_0^T a(r) f(u(r), u(\mu(r))) \nabla r \right) + l \Phi_q \left(\int_0^T a(r) f(u(r), u(\mu(r))) \nabla r \right) \\ &= (A + l) \Phi_q \left[\int_{Y_1} a(r) f(u(r), \varphi(\mu(r))) \nabla r + \int_{Y_2} a(r) f(u(r), u(\mu(r))) \nabla r \right] \\ &< (A + l) \Phi_q \left(\int_0^T a(r) \nabla r \right) \frac{a}{\lambda} \\ &= a. \end{aligned}$$

Now we show that (iii) of Theorem 1.1 is satisfied. If $u \in P(\gamma, \alpha, b, c)$ and $\theta(Fu) = Fu(\eta) > \frac{T}{\eta}b$, then

$$\alpha(Fu) \geq (Fu)(l) = \frac{l}{T}Fu(l) \geq \frac{l}{T}Fu(\eta) > \frac{l}{\eta}b > b.$$

Finally, if $u \in Q(\alpha, \beta, a, c)$ and $\psi(Fu) = Fu(\eta) < \frac{T}{\eta}a$, then from (2) of the Lemma 2.1 we have

$$\beta(Fu) = Fu(l) \leq \frac{T}{l}Fu(l) \leq \frac{T}{\eta}Fu(\eta) < a.$$

which shows that condition (iv) of Theorem 1.1 is fulfilled.

Thus, all the conditions of Theorem 1.1 are satisfied. Hence, F has at least three fixed points u_1, u_2, u_3 satisfying

$$\beta(u_1) < a, \quad b < \alpha(u_2), \quad \text{and} \quad a < \beta(u_3) \quad \text{with} \quad \alpha(u_3) < b.$$

Let

$$u(t) = \begin{cases} u_i(t), & t \in [0, T], \quad i = 1, 2, 3, \\ \varphi(t), & t \in [-r, 0], \end{cases}$$

which are three positive solutions of the BVP (1.1).

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