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Mixed semicontinuous perturbation of a second order nonconvex sweeping process

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Abstract

We prove a theorem on the existence of solutions of a second order differential inclusion governed by a class of nonconvex sweeping process with a mixed semicontinuous perturbation.

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1 Introduction

The existence of solutions for the second order differential inclusions governed by the sweeping process

$$(\mathcal{P}_F)\begin{cases}-\ddot{u}(t)\in N_{K(u(t))}(\dot{u}(t))+F(t,u(t),\dot{u}(t)), & \text{a.e. } t\in[0,T],\\ \dot{u}(t)\in K(u(t)),\\ u(0)=u_0; \quad \dot{u}(0)=v_0,\end{cases}$$

where $N_{K(u(t))}(\cdot)$ denotes the normal cone to $K(u(t))$, has been thoroughly studied (when the sets $K(x)$ are convex or nonconvex) by Castaing for the first time when $F\equiv\{0\}$ see [5], and later by many other authors see for example [2], [3], [6], [10] and [12]. Note that in this literature some existence results are established for the problem $(\mathcal{P}_F)$ with lower and upper semicontinuous perturbations.

Our aim in this paper is to prove existence results for $(\mathcal{P}_F)$ when F is a mixed semicontinuous set-valued map and $K(x)$ are nonconvex sets. For the first order sweeping process with a mixed semicontinuous perturbation we refer the reader to [9], and to [13] for the sweeping process with non regular sets and to [1] for second order differential inclusions with mixed semicontinuous perturbations.

After some preliminaries, we present our main result in the finite dimensional space H whenever the sets $K(x)$ are uniformly ρ-prox-regular ($\rho > 0$) and the set-valued mapping F is mixed semicontinuous, that is, $F(\cdot,\cdot,\cdot)$ is measurable and for every $t \in [0,T]$, at each $(x,y) \in H \times H$ where $F(t,x,y)$ is convex the set-valued map $F(t,\cdot,\cdot)$ is upper semicontinuous, and whenever $F(t,x,y)$ is not convex $F(t,\cdot,\cdot)$ is lower semicontinuous on some neighborhood of (x,y).

2 Definition and preliminary results

Let H be a real Hilbert space and let S be a nonempty closed subset of H. We denote by $d(\cdot,S)$ the usual distance function associated with S, i.e., $d(u,S) := \inf_{y\in S} \|u-y\|$. For any $x \in H$ and $r \geq 0$ the closed ball centered at x with radius r will be denoted by $\overline{\mathbf{B}}_H(x,r)$. For $x = 0$ and $r = 1$ we will put $\overline{\mathbf{B}}_H$ in place of $\overline{\mathbf{B}}_H(0,1)$. $\mathcal{L}([0,T])$ is the σ-algebra of Lebesgue-measurable sets of $[0,T]$ and $\mathcal{B}(H)$ is the σ-algebra of Borel subsets of H. By $\mathbf{L}^1_H([0,T])$ we denote the space of all Lebesgue-Bochner integrable H-valued mappings defined on $[0,T]$ and by $\mathbf{C}_H([0,T])$ the Banach space of all continuous mappings $u : [0,1] \to H$, endowed with the sup norm
We need first to recall some notation and definitions that will be used in all the paper. Let x be a point in S. We recall (see [8]) that the proximal normal cone to S at x is defined by $N^P_S(x) := \partial^P \psi_S(x)$, where ψ_S denotes the indicator function of S, i.e., $\psi_S(x) = 0$ if $x \in S$ and $+\infty$ otherwise. Note that the proximal normal cone is also given by

$$N^P_S(x) = \{\xi \in H : \exists \alpha > 0 \text{ s.t. } x \in \mathrm{Proj}_S(x + \alpha\xi)\},$$

where

$$\mathrm{Proj}_S(u) := \{y \in S : d(u,S) := \|u - y\|\}.$$

Recall now that for a given $\rho \in]0,+\infty]$ the subset S is uniformly ρ-prox-regular (see [11]) or equivalently ρ -proximally smooth (see [8]) if and only if every nonzero proximal normal to S can be realized by ρ-ball, this means that for all $\overline{x} \in S$ and all $0 \neq \xi \in N^P_S(\overline{x})$ one has

$$\left\langle \frac{\xi}{\|\xi\|}, x - \overline{x} \right\rangle \leq \frac{1}{2\rho} \|x - \overline{x}\|^2,$$

for all $x \in S$. We make the convention $\frac{1}{\rho} = 0$ for $\rho = +\infty$. Recall that for $\rho = +\infty$ the uniform ρ-prox-regularity of S is equivalent to the convexity of S. The following proposition summarizes some important consequences of the uniform prox-regularity needed in the sequel. For the proof of these results we refer the reader to [11].

Proposition 2.1 *Let S be a nonempty closed subset in H and $x \in S$. The following assertions hold:*

(1) $\partial^P d(x,S) = N_S^P(x) \cap \overline{\mathbf{B}}_H$*;*
(2) let $\rho \in]0,+\infty]$*. If* S *is uniformly* ρ*-prox-regular, then*
(2.1) for all $x \in H$ *with* $d(x,S) < \rho$*; one has* $Proj_S(x) \neq \emptyset$*;*
(2.2) the proximal subdifferential of $d(\cdot,S)$ *coincides with its Clarke subdifferential at all points* $x \in H$ *satisfying* $d(x,S) < \rho$*. So, in such a case, the subdiferential* $\partial d(x,S) := \partial^P d(x,S) = \partial^C d(x,S)$ *is a closed convex set in* H*;*
(2.3) for all $x_i \in S$ *and all* $v_i \in N_S^P(x_i)$ *with* $\|v_i\| \leq \rho$ $(i = 1,2)$ *one has*

$$\langle v_1 - v_2, x_1 - x_2 \rangle \geq - \|x_1 - x_2\|^2 .$$

As a consequence of **(2.3)** we get that for uniformly ρ-prox-regular sets, the proximal normal cone to S coincides with all the normal cones contained in the Clarke normal cone at all points $x \in S$, i.e., $N_S^P(x) = N_S^C(x)$. In such a case, we put $N_S(x) := N_S^P(x) = N_S^C(x)$. Here and above $\partial^C d(x,S)$ and $N_S^C(x)$ denote respectively the Clarke subdifferential of $d(\cdot,S)$ and the Clarke normal cone to S (see [8]).

Now, we recall some preliminaries concerning set-valued mappings. Let $T > 0$. Let $C : [0,T] \rightrightarrows H$ and $K : H \rightrightarrows H$ be two set-valued mappings. We say that C is absolutely continuous provided that there exists an absolutely continuous nonnegative function $a : [0,T] \to \mathbb{R}_+$ with $a(0) = 0$ such that

$$|d(x, C(t)) - d(y, C(s))| \leq \|x - y\| + |a(t) - a(s)|$$

for all $x, y \in H$ and all $s, t \in [0,T]$.
We will say that K is Hausdorff-continuous (resp. Lipschitz with ratio $\lambda > 0$) if for any $x \in H$ one has

$$\lim_{x' \to x} \mathcal{H}(K(x), K(x')) = 0$$

(resp. if for any $x, x' \in H$ one has

$$\mathcal{H}(K(x), K(x')) \leq \lambda \|x - x'\|.)$$

We close this section with the following theorem in [4], which is an important closedness property of the subdifferential of the distance function associated with a set-valued mapping.

Theorem 2.1 *Let* $\rho \in]0,+\infty]$*,* Ω *be an open subset in* H*, and* $K : \Omega \rightrightarrows H$ *be a Hausdorff-continuous set-valued mapping. Assume that* $K(z)$ *is uniformly* ρ*-prox-regular for all* $z \in \Omega$*. Then for a given* $0 < \delta < \rho$*, the following holds:*
"for any $\overline{z} \in \Omega$*,* $\overline{x} \in K(\overline{z}) + (\rho - \delta)\overline{\mathbf{B}}_H$*,* $x_n \to \overline{x}$*,* $z_n \to \overline{z}$ *with* $z_n \in \Omega$ *(*x_n *not necessarily in* $K(z_n)$*) and* $\xi_n \in \partial d(x_n, K(z_n))$ *with* $\xi_n \to^w \overline{\xi}$ *one has* $\overline{\xi} \in \partial d(\overline{x}, K(\overline{z}))$*."*
Here $\to^w$ *means the weak convergence in* H*.*

Remark 2.1 *As a direct consequence of this theorem, we have for every* $\rho \in]0,+\infty]$*, for a given* $0 < \delta < \rho$*, and for every set-valued mapping* $K : \Omega \rightrightarrows H$ *with uniformly* ρ*-prox regular values, the set-valued mapping* $(z,x) \mapsto \partial d(x, K(z)))$ *is upper semicontinuous from* $\{(z,x) \in \Omega \times H : \ x \in K(z) + (\rho - \delta)\}$ *to* H *endowed with the weak topology, which is equivalent to the upper semicontinuity of the function* $(z,x) \mapsto \sigma(\partial d(x, K(z))), p)$ *on* $\{(z,x) \in \Omega \times H : \ x \in K(z) + (\rho - \delta)\}$*, for any* $p \in H$*. Here* $\sigma(S,p)$ *denotes the support function to* S *defined by* $\sigma(S,p) = \sup_{s \in S} \langle s, p \rangle$*.*

3 Existence results under mixed semicontinuous perturbation.

Our existence result is stated in a finite dimensional space H under the following assumptions.
(H_1) For each $x \in H$, $K(x)$ is a nonempty closed subset in H and uniformly ρ-prox-regular for some fixed $\rho \in]0, +\infty]$;

(H_2) K is Lipschitz with ratio $\lambda > 0$;

(H_3) $l = \sup_{x\in H} |K(x)| < +\infty$.

The proof of our main theorem uses existence results for the first order sweeping process, the selection theorem proved in Tolstonogov [14] and the Kakutani fixed point theorem for set-valued mappings. We begin by recalling them.

Proposition 3.1 (Proposition 1.1 in [7]) *Let H be a finite dimensional space, $T > 0$ and let $C : I := [0, T] \rightrightarrows H$ be a nonempty closed valued set-valued mapping satisfying the following assumptions.*
(A_1) For each $t \in I$, $C(t)$ is ρ-prox-regular for some fixed $\rho \in]0, +\infty]$;
(A_2) $C(t)$ varies in an absolutely continuous way, that is, there exists a nonnegative absolutely continuous function $v : I \to \mathbb{R}$ such that

$$|d(x, C(t)) - d(y, C(s))| \le \|x - y\| + |v(t) - v(s)|$$

for all $x, y \in H$ and $s, t \in I$.
Then for any mapping $h \in \mathbf{L}^1_H([0, T])$, the differential inclusion

$$\begin{cases} -\dot{u}(t) \in N_{C(t)}(u(t)) + h(t), & a.e.\ t \in [0, T], \\ u(0) = u_0 \in C(0) \end{cases}$$

admits one and only one absolutely continuous solution $u(\cdot)$ and

$$\|\dot{u}(t) + h(t)\| \le |\dot{v}(t)| + \|h(t)\|.$$

Further, let m be a nonnegative Lebesgue-integrable function defined on $[0, T]$ and let

$$\mathcal{K} = \{h \in \mathbf{L}^1_H([0, T]) : \ \ \|h(t)\| \le m(t)\ a.e\}.$$

Then the solutions set $\{u_h : \ \ h \in \mathcal{K}\}$, where u_h is the unique absolutely continuous solution of the above inclusion, is compact in $\mathbf{C}_H([0, T])$, and the mapping $h \mapsto u_h$ is continuous on $\mathcal{K}$ when $\mathcal{K}$ is endowed with the weak topology $w(\mathbf{L}^1_H([0, T]), \mathbf{L}^\infty_H([0, T]))$.

For the proof of our theorem we will also need the following theorem which is a direct consequence of Theorem 2.1 in [14].

Theorem 3.1 *Let H be a finite dimensional space and let $M : [0, T] \times H \times H \rightrightarrows H$ be a closed valued set-valued mapping satisfying the following hypotheses.*
(i) *M is $\mathcal{L}([0, T]) \otimes \mathcal{B}(H) \otimes \mathcal{B}(H)$-measurable;*

(ii) *for every* $t \in [0,1]$, *at each* $(x,y) \in H \times H$ *such that* $M(t,x,y)$ *is convex,* $M(t,\cdot,\cdot)$ *is upper semicontinuous, and whenever* $M(t,x,y)$ *is not convex,* $M(t,\cdot,\cdot)$ *is lower semicontinuous on some neighborhood of* (x,y)*;*
(iii) *there exists a Caratheodory function* $\zeta : [0,1] \times H \times H \to \mathbb{R}_+$ *which is integrably bounded and such that* $M(t,x,y) \bigcap \overline{\mathbf{B}}_H(0,\zeta(t,x,y)) \neq \emptyset$ *for all* $(t,x,y) \in [0,1] \times H \times H$*.*
Then for any $\varepsilon > 0$ *and any compact set* $K \subset \mathbf{C}_H([0,T])$ *there is a nonempty closed convex valued multifunction* $\Phi : K \rightrightarrows \mathbf{L}^1_H([0,T])$ *which has a strongly-weakly sequentially closed graph such that for any* $u \in K$ *and* $\phi \in \Phi(u)$ *one has*

$$\phi(t) \in M(t,u(t),\dot{u}(t));$$

$$\|\phi(t)\| \leq \zeta(t,u(t),\dot{u}(t)) + \varepsilon,$$

for almost every $t \in [0,T]$*.*

Now we are able to prove our main result.

Theorem 3.2 *Let* H *be a finite dimensional space,* $K : H \rightrightarrows H$ *be a set-valued mapping satisfying assumptions* (H_1), (H_2) *and* (H_3)*. Let* $T > 0$ *and let* $F : [0,T] \times H \times H \rightrightarrows H$ *be a set-valued mapping satisfying hypotheses* ***(i)*** *and* ***(ii)*** *of Theorem 3.1 and the following one*
(iv) *there exist nonnegative Lebesgue-integrable functions* m, p *and* q *defined on* $[0,T]$ *such that*

$$F(t,x,y) \subset (m(t) + p(t)\|x\| + q(t)\|y\|)\overline{\mathbf{B}}_H$$

for all $(t,x,y) \in [0,T] \times H \times H$*.*
Then for all $u_0 \in H$ *and* $v_0 \in K(u_0)$*, there exist two Lipschitz mappings* $u, v : [0,T] \to H$ *such that*

$$\begin{cases} u(t) = u_0 + \int_0^t v(s)ds, \quad \forall t \in [0,T]; \\ -\dot{v}(t) \in N_{K(u(t))}(v(t)) + F(t,u(t),v(t)), \quad a.e \text{ on } [0,T]; \\ v(t) \in K(u(t)), \quad \forall t \in [0,T]; \\ u(0) = u_0; \quad v(0) = v_0 \end{cases}$$

with $\|\dot{v}(t)\| \leq \lambda l + 2(m(t) + p(t)(\|u_0\| + lT) + q(t)l)$ *a.e.* $t \in [0,T]$*.*
In other words, there is a Lipschitz solution $u : [0,T] \to H$ *to the Cauchy problem* $(\mathcal{P}_F)$*.*

Proof. Step 1. Put $I := [0,T]$, $M(t) = m(t) + p(t)(\|u_0\| + lT) + q(t)l$, and let us consider the sets

$$\mathcal{X} = \{u \in \mathbf{C}_H(I) : \ u(t) = u_0 + \int_0^t \dot{u}(s)ds, \, \forall t \in I \, and \, \|\dot{u}(t)\| \leq l, \ a.e. on \, I\},$$

$$\mathcal{U} = \{v \in \mathbf{C}_H(I) : \ v(t) = v_0 + \int_0^t \dot{v}(s)ds, \, \forall t \in I \, and \, \|\dot{v}(t)\| \leq \lambda l + 2M(t), \ a.e. on \, I\},$$

$$\mathcal{K} = \{h \in \mathbf{L}^1_H(I) : \ \|h(t)\| \leq M(t), \ a.e. on \, I\}.$$

It is clear that $\mathcal{K}$ is a convex $w(\mathbf{L}^1_H(I), \mathbf{L}^\infty_H(I))$-compact subset of $\mathbf{L}^1_H(I)$, and by Ascoli-Arzelà theorem $\mathcal{X}$ and $\mathcal{U}$ are convex compact sets in $\mathbf{C}_H(I)$. Observe now, that for all $f \in \mathcal{X}$ the set valued mapping $K \circ f$ is Lipschitz with ratio λl. Indeed, for all $t, t' \in I$

$$\begin{aligned}
\mathcal{H}((K \circ f)(t), (K \circ f)(t')) &= \mathcal{H}(K(f(t)), K(f(t'))) \\
&\leq \lambda \|f(t) - f(t')\| \\
&= \lambda \|x_0 + \int_0^t \dot{f}(s)ds - x_0 - \int_0^{t'} \dot{f}(s)ds\| \\
&\leq \lambda \int_{t'}^t |\dot{f}(s)|ds \leq \lambda l |t - t'|.
\end{aligned}$$

By Proposition 3.1, for all $(f, h) \in \mathcal{X} \times \mathcal{K}$, there exists a unique solution $u_{f,h}$ to the problem

$$(P) \begin{cases} -\dot{u}_{f,h}(t) \in N_{K(f(t))}(u_{f,h}(t)) + h(t), \;\; a.e. \; on \; I; \\ u_{f,h}(t) \in K(f(t)), \forall t \in I; \\ u_{f,h}(0) = v_0, \end{cases}$$

and for almost all $t \in I$, $\|\dot{u}_{f,h}(t)\| \leq \lambda l + 2M(t)$, i.e., $u_{f,h} \in \mathcal{U}$.

Let us consider the mapping $\mathcal{A} : \mathcal{X} \times \mathcal{K} \to \mathcal{U}$ defined by $\mathcal{A}(f, h) = u_{f,h}$, where $u_{f,h}$ is the unique solution of (P). We wish to show that $\mathcal{A}$ is continuous. Let $(f_n, h_n)_n$ be a sequence in $\mathcal{X} \times \mathcal{K}$ such that $(f_n)_n$ converges uniformly to $f \in \mathcal{X}$ and $(h_n)_n$ converges $w(\mathbf{L}^1_H(I), \mathbf{L}^\infty_H(I))$ to $h \in \mathcal{K}$, and since $(u_{f_n,h_n})_n \subset \mathcal{U}$ we may suppose that it converges uniformly to some mapping $v \in \mathcal{U}$. For each $n \in \mathbb{N}$ we have

$$(P_n) \begin{cases} -\dot{u}_{f_n,h_n}(t) \in N_{K(f_n(t))}(u_{f_n,h_n}(t)) + h_n(t), \;\; a.e. \; on \; I; \\ u_{f_n,h_n}(t) \in K(f_n(t)), \forall t \in I; \\ u_{f_n,h_n}(0) = v_0. \end{cases}$$

Since $u_{f_n,h_n}(t) \in K(f_n(t))$ for all $t \in I$, it follows from the Lipschitz property of K

$$d(u_{f_n,h_n}(t), K(f(t))) \leq \lambda \|f_n(t) - f(t)\| \to 0$$

and hence, one obtains $v(t) \in K(f(t))$, because the set $K(f(t))$ is closed.
According to (P_n) one has

$$\dot{u}_{f_n,h_n}(t) + h_n(t) \in -N_{K(f_n(t))}(u_{f_n,h_n}(t)),$$

and

$$\|\dot{u}_{f_n,h_n}(t) + h_n(t)\| \leq \lambda l + M(t) := c(t), \text{ i.e., } \dot{u}_{f_n,h_n}(t) + h_n(t) \in c(t)\overline{\mathbf{B}}_H.$$

Therefore we get by Proposition 2.1 (1)

$$\dot{u}_{f_n,h_n}(t) + h_n(t) \in -c(t)\partial d(u_{f_n,h_n}(t), K(f_n(t))), \; a.e. \, on \, I.$$

Now, as $(\dot{u}_{f_n,h_n} + h_n)_n$ converges weakly to $\dot{v} + h \in \mathbf{L}^1_H(I)$, Mazur's lemma ensures that for a.e. $t \in I$

$$\dot{v}(t) + h(t) \in \bigcap_n \overline{co}\{\dot{u}_{f_k,h_k}(t) + h_k(t) : \ k \geq n\}.$$

Fix such t in I and any μ in H, then the last relation gives

$$\begin{aligned} \langle \dot{v}(t) + h(t), \mu\rangle &\leq \limsup_n \sigma(-c(t)\partial d(u_{f_n,h_n}(t), K(f_n(t))), \mu) \\ &\leq \sigma(-c(t)\partial d(v(t), K(f(t))), \mu), \end{aligned}$$

where the second inequality follows from Remark 2.1 and Theorem 2.1. As the set $\partial d(v(t), K(f(t)))$ is closed and convex (see Proposition 2.1), we obtain

$$\dot{v}(t) + h(t) \in -c(t)\partial d(v(t), K(f(t))) \subset -N_{K(f(t))}(v(t)),$$

because $v(t) \in K(f(t))$. This can be rephrased as

$$\begin{cases} -\dot{v}(t) \in N_{K(f(t))}(v(t)) + h(t), \ \ \textit{a.e. on } I; \\ v(t) \in K(f(t)), \forall t \in I; \\ v(0) = v_0. \end{cases}$$

In other words, v is of the form $u_{f,h}$ with

$$\begin{cases} -\dot{u}_{f,h}(t) \in N_{K(f(t))}(u_{f,h}(t)) + h(t), \ \ \textit{a.e. on } I; \\ u_{f,h}(t) \in K(f(t)), \forall t \in I; \\ u_{f,h}(0) = v_0. \end{cases}$$

We conclude that $\mathcal{A}$ is continuous. Hence, the mapping $P : \mathcal{X} \times \mathcal{K} \to \mathbf{C}_H(I)$ defined by $P(f, h)$, where for all $t \in I$

$$P(f, h)(t) = u_0 + \int_0^t \mathcal{A}(f, h)(s)ds = u_0 + \int_0^t u_{f,h}(s)ds$$

is also continuous when $\mathcal{X}$ is endowed with the topology of uniform convergence and $\mathcal{K}$ is endowed with the weak topology. Observe that for all $t \in I$, $u_{f,h}(t) \in K(f(t))$ and then by (H_3), we have $\|u_{f,h}(t)\| \leq l$, we conclude that $P(f, h) \in \mathcal{X}$.

Step 2. By Theorem 3.1, there is a nonempty closed convex valued set-valued mapping $\Phi : \mathcal{X} \rightrightarrows \mathbf{L}^1_H(I)$ such that for any $u \in \mathcal{X}$ and $\phi \in \Phi(u)$

$$\phi(t) \in F(t, u(t), \dot{u}(t)) \ \ \textit{and} \ \ \|\phi(t)\| \leq m(t) + p(t)\|u(t)\| + q(t)\|\dot{u}(t)\|$$

for almost all $t \in I$. Since $u \in \mathcal{X}$, we have $\|\dot{u}(t)\| \leq l$ and

$$\|u(t)\| = \|u_0 + \int_0^t \dot{u}(s)ds\| \leq \|u_0\| + lT,$$

hence

$$\|\phi(t)\| \leq m(t) + p(t)(\|u_0\| + lT) + q(t)l = M(t). \tag{3.1}$$

The relation (3.1) shows that Φ has $w(\mathbf{L}^1_H(I), \mathbf{L}^\infty_H(I))$-compact values in $\mathbf{L}^1_H(I)$. Now, let us consider the set-valued mapping $\Psi : \mathcal{X} \rightrightarrows \mathcal{X}$ defined by

$$\Psi(f) = \{P(f,h) : \ \ h \in \Phi(f)\}.$$

It is clear that Ψ has nonempty convex values since Φ has nonempty convex values. Furthermore, for all $f \in \mathcal{X}$, $\Psi(f)$ is compact in $\mathcal{X}$. Indeed, Let $(v_n)_n$ be a sequence in $\Psi(f)$, then, for each n, there is $h_n \in \Phi(f)$ such that $v_n = P(f, h_n)$. Since $(h_n)_n \subset \Phi(f)$, by extracting a subsequence (that we do not relabel) we may suppose that $(h_n)_n$ $w(\mathbf{L}^1_H(I), \mathbf{L}^\infty_H(I))$-converges to some mapping $h \in \Phi(f)$, and by the continuity of P we get $v_n = P(f, h_n) \to v = P(f,h) \in \mathcal{X}$. This shows the compactness of $\Psi(f)$.
We will prove that Ψ is upper semicontinuous, or equivalently the graph of Ψ
$\mathrm{gph}(\Psi) = \{(x,y) \in \mathcal{X} \times \mathcal{X} : \ \ y \in \Psi(x)\}$ is closed. Let $(x_n, y_n)_n$ be a sequence in $\mathrm{gph}(\Psi)$ converging to $(x,y) \in \mathcal{X} \times \mathcal{X}$. For all $n \in \mathbb{N}$, $y_n \in \Psi(x_n)$, so, there is $h_n \in \Phi(x_n)$ such that $y_n = P(x_n, h_n)$. since $(h_n)_n \subset \mathcal{K}$, by extracting a subsequence (that we do not relabel) we may suppose that $(h_n)_n$ $w(\mathbf{L}^1_H(I), \mathbf{L}^\infty_H(I))$-converges to some mapping $h \in \mathcal{K}$. As the sequence $(x_n)_n$ converges uniformly to $x \in \mathcal{X}$ and since $\mathrm{gph}(\Phi)$ is strongly-weakly sequentially closed we conclude that $h \in \Phi(x)$. On the other hand, by the continuity of the mapping P we get

$$y = \lim_{n \to +\infty} y_n = \lim_{n \to +\infty} P(x_n, h_n) = P(x,h).$$

Hence $(x,y) \in \mathrm{gph}(\Psi)$. This says that Ψ is upper semicontinuous. An application of Kakutani Theorem gives a fixed point of Ψ, that is, there is $f \in \mathcal{X}$ such that $f \in \Psi(f)$, which means that there is $h \in \Phi(f)$ such that $f = P(f,h)$. Consequently

$$f(t) = u_0 + \int_0^t u_{f,h}(s)ds \ \ and \ \ h(t) \in F(t, f(t), u_{f,h}(t))$$

with

$$\begin{cases} -\dot{u}_{f,h}(t) \in N_{K(f(t))}(u_{f,h}(t)) + h(t), \ \ a.e. \ on \ I; \\ u_{f,h}(t) \in K(f(t)), \forall t \in I; \\ u_{f,h}(0) = v_0, \end{cases}$$

or, by putting $u = f$

$$\begin{cases} -\ddot{u}(t) \in N_{K(u(t))}(\dot{u}(t)) + F(t, u(t), \dot{u}(t)), \ \ a.e. \ on \ I; \\ \dot{u}(t) \in K(u(t)), \forall t \in I; \\ u(0) = u_0; \ \ \dot{u}(0) = v_0, \end{cases}$$

with, for almost all $t \in I$, $\|\ddot{u}(t)\| \leq \lambda l + 2M(t)$. This finish the proof of our theorem. ■

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