PERIODIC SOLUTIONS OF A CLASS OF INTEGRODIFFERENTIAL IMPULSIVE PERIODIC SYSTEMS WITH TIME-VARYING GENERATING OPERATORS ON BANACH SPACE

JinRong Wang[1], X. Xiang[1,2] and W. Wei[2]

1. College of Computer Science and Technology of Guizhou University
2. College of Science of Guizhou University
Guiyang, Guizhou 550025, P.R. China

ABSTRACT. This paper deals with a class of integrodifferential impulsive periodic systems with time-varying generating operators on Banach space. Using impulsive periodic evolution operator given by us, the suitable T_0-periodic PC-mild solution is introduced and *Poincaré* operator is constructed. Showing the compactness of *Poincaré* operator and using a new generalized Gronwall's inequality with impulse, mixed type integral operators and B-norm given by us, we utilize Leray-Schauder fixed point theorem to prove the existence of T_0-periodic PC-mild solutions. Our method is much different from methods of other papers. At last, an example is given for demonstration.

Keywords. Integrodifferential impulsive periodic systems, Time-varying generating operators, T_0-periodic PC-mild solution, Generalized Gronwall's inequality, Existence.

1. Introduction

It is well known that impulsive periodic motion is a very important and special phenomena not only in natural science but also in social science such as climate, food supplement, insecticide population, sustainable development. Periodic system with applications on finite dimensional spaces have been extensively studied. Particularly, impulsive periodic systems on finite dimensional spaces are considered and some important results (such as the existence and stability of periodic solution, the relationship between bounded solution and periodic solution, robustness by perturbation) are obtained (See [7], [11], [12], [32]).

Since the end of last century, many researchers pay great attention on impulsive systems on infinite dimensional spaces. Particulary, Dr. Ahmed investigated optimal control problems of system governed by impulsive system (See [3], [4], [5], [6]). Many authors including us also gave a series of results for semilinear (integrodifferential, strongly nonlinear) impulsive systems and optimal control problems (See [8], [9], [10], [13], [14], [15], [16], [26], [27], [28], [29], [30], [31]).

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E-mail: wjr9668@126.com.

Although, there are some papers on periodic solution for periodic systems on infinite dimensional spaces (See [1], [13], [24], [25]) and some results discussing integrodifferential system on finite Banach space and infinite Banach space (see [10], [14]), to our knowledge, integrodifferential impulsive periodic systems with time-varying generating operators on infinite dimensional spaces have not been extensively investigated. Recently, we discuss the impulsive periodic system and integrodifferential impulsive system on infinite dimensional spaces. For linear impulsive evolution operator is constructed and T_0-periodic PC-mild solution is introduced. Existence of periodic solutions and alternative theorem, criteria of Massera type, asymptotical stability and robustness by perturbation are established (See [17], [18], [19]). For semilinear impulsive periodic system, a suitable *Poincaré* operator is constructed and verify its compactness and continuity. By virtue of a generalized Gronwall inequality with mixed integral operator and impulse given by us, the estimate on the PC-mild solutions are derived. Some fixed point theorem such as Banach fixed point theorem, Horn's fixed point theorem and Leray-Schauder fixed point theorem are applied to obtain the existence of periodic PC-mild solutions respectively (See [20], [21], [22]). For integrodifferential impulsive system, existence of PC-mild solutions and optimal controls are presented (See [26]).

Herein, we go on studying the following integrodifferential impulsive periodic system with time-varying generating operators:

$$\begin{cases} \dot{x}(t) = A(t)x(t) + f\left(t, x, \int_0^t g(t,s,x)ds\right), & t \neq \tau_k, \\ \Delta x(t) = B_k x(t) + c_k, & t = \tau_k. \end{cases} \tag{1.1}$$

in the Banach X, where $\{A(t),\ t \in [0, T_0]\}$ is a family of closed densely defined linear unbounded operators on X and the resolvent of the unbounded operator $A(t)$ is compact. f is a T_0-periodic, with respect to $t \in [0 + \infty)$, *Carathéodory* function, g is a continuous function from $[0, \infty) \times [0, \infty) \times X$ to X and are T_0-periodic in t and s, and $B_{k+\delta} = B_k$, $c_{k+\delta} = c_k$. This paper is mainly concerned with the existence of periodic solutions for integrodifferential impulsive periodic system on infinite dimensional Banach space X.

Here, we also use Leray-Schauder fixed point theorem to obtain the existence of periodic solutions for integrodifferential impulsive periodic system with time-varying generating operators (1.1). First, by virtue of impulsive evolution operators corresponding to linear homogeneous impulsive system with time-varying generating operators, we construct a new *Poincaré* operator P for integrodifferential impulsive periodic system with time-varying generating operators (1.1), then overcome some difficulties to show the compactness of *Poincaré* operator P which is very important. By a new generalized Gronwall inequality with impulse, mixed type integral operator and B-norm given by us, the estimate of fixed point set $\{x = \lambda P x, \lambda \in [0, 1]\}$ is established. Therefore, the existence of T_0-periodic PC-mild solutions for impulsive integrodifferential periodic system with time-varying generating operators is shown.

In order to obtain the existence of periodic solutions, many authors use Horn's fixed point theorem or Banach fixed point theorem. However, the conditions for Horn's fixed point theorem are not easy to be verified sometimes and the conditions for Banach's fixed point theorem are too strong. Our methods is much different from other's and we give a new way to show the existence of periodic solutions. In addition, the new generalized Gronwall inequality with impulse, mixed type integral operators and B-norm given by us, which can be used in other problems, have played an essential role in the study of nonlinear problems on infinite dimensional spaces.

This paper is organized as follows. In section 2, some results of linear impulsive periodic system with time-varying generating operators and properties of impulsive periodic evolution operator corresponding to homogeneous linear impulsive periodic system with time-varying generating operators are recalled. In section 3, the new generalized Gronwall inequality with impulse, mixed type integral operator and B-norm is established. In section 4, the T_0-periodic PC-mild solution for integrodifferential impulsive periodic system with time-varying generating operators (1.1) is introduced. We construct the suitable *Poincaré* operator P and give the relation between T_0-periodic PC-mild solution and the fixed point of P. After showing the compactness of the *Poincaré* operator P and obtaining the boundedness of the fixed point set $\{x = \lambda Px, \lambda \in [0, 1]\}$ by virtue of the generalized Gronwall inequality, we can use Leray-Schauder fixed point theorem to establish the existence of T_0-periodic PC-mild solutions for integrodifferential impulsive periodic system with time-varying generating operators. At last, an example is given to demonstrate the applicability of our result.

2. Linear Impulsive Periodic System with time-varying generating operators

In order to study the integrodifferential impulsive periodic system with time-varying generating operators, we first recall some results about linear impulsive periodic system with time-varying generating operators here. Let X be a Banach space. $\pounds(X)$ denotes the space of linear operators in X; $\pounds_b(X)$ denotes the space of bounded linear operators in X. $\pounds_b(X)$ is the Banach space with the usual supremum norm. Define $\widetilde{D}=\{\tau_1, \cdots, \tau_\delta\} \subset [0, T_0]$, where $\delta \in \mathbb{N}$ denotes the number of impulsive points between $[0, T_0]$. We introduce $PC([0, T_0]; X) \equiv \{x : [0, T_0] \to X \mid x$ is continuous at $t \in [0, T_0]\backslash\widetilde{D}$, x is continuous from left and has right hand limits at $t \in \widetilde{D}\}$ and $PC^1([0, T_0]; X) \equiv \{x \in PC([0, T_0]; X) \mid \dot{x} \in PC([0, T_0]; X)\}$. Set

$$\|x\|_{PC} = \max\left\{\sup_{t\in[0,T_0]} \|x(t+0)\|, \sup_{t\in[0,T_0]} \|x(t-0)\|\right\} \quad \text{and} \quad \|x\|_{PC^1} = \|x\|_{PC} + \|\dot{x}\|_{PC}.$$

It can be seen that endowed with the norm $\|\cdot\|_{PC}$ ($\|\cdot\|_{PC^1}$), $PC([0, T_0]; X)$ $\big(PC^1([0, T_0]; X)\big)$ is a Banach space.

Consider the following homogeneous linear impulsive periodic system with time-varying generating operators

$$\begin{cases} \dot{x}(t) = A(t)x(t), & t \neq \tau_k, \\ \Delta x(\tau_k) = B_k x(\tau_k), & t = \tau_k. \end{cases} \tag{2.1}$$

on Banach space X, where $\triangle x(\tau_k) = x(\tau_k^+) - x(\tau_k^-)$, $\{A(t),\ t \geq 0\}$ is a family of closed densely defined linear unbounded operators on X satisfying the following assumption.

Assumption [A1]: (See [2], p.158) For $t \in [0, T_0]$ one has

(P_1) The domain $D(A(t)) = D$ is independent of t and is dense in X.

(P_2) For $t \geq 0$, the resolvent $R(\lambda, A(t)) = (\lambda I - A(t))^{-1}$ exists for all λ with $Re\lambda \leq 0$, and there is a constant M independent of λ and t such that

$$\|R(\lambda, A(t))\| \leq M(1 + |\lambda|)^{-1} \qquad \text{for} \qquad Re\lambda \leq 0.$$

(P_3) There exist constants $L > 0$ and $0 < \alpha \leq 1$ such that

$$\big\|\big(A(t) - A(\theta)\big)A^{-1}(\tau)\big\| \leq L|t-\theta|^{\alpha} \quad \text{for} \quad t, \theta, \tau \in [0, T_0].$$

Lemma 2.1: (See [2], p.159) Under the assumption [A1], the Cauchy problem

$$\dot{x}(t) + A(t)x(t) = 0,\ t \in (0, T_0] \quad \text{with} \quad x(0) = \bar{x} \tag{2.2}$$

has a unique evolution system $\{U(t,\theta) \mid 0 \leq \theta \leq t \leq T_0\}$ in X satisfying the following properties:

(1) $U(t,\theta) \in \pounds_b(X)$ for $0 \leq \theta \leq t \leq T_0$.

(2) $U(t,r)U(r,\theta) = U(t,\theta)$ for $0 \leq \theta \leq r \leq t \leq T_0$.

(3) $U(\cdot,\cdot)x \in C(\Delta, X)$ for $x \in X$, $\Delta = \{(t,\theta) \in [0,T_0] \times [0,T_0] \mid 0 \leq \theta \leq t \leq T_0\}$.

(4) For $0 \leq \theta < t \leq T_0$, $U(t,\theta)$: $X \longrightarrow D$ and $t \longrightarrow U(t,\theta)$ is strongly differentiable in X. The derivative $\frac{\partial}{\partial t}U(t,\theta) \in \pounds_b(X)$ and it is strongly continuous on $0 \leq \theta < t \leq T_0$. Moreover,

$$\begin{aligned}
\frac{\partial}{\partial t}U(t,\theta) &= -A(t)U(t,\theta) \quad \text{for} \quad 0 \leq \theta < t \leq T_0, \\
\left\|\frac{\partial}{\partial t}U(t,\theta)\right\|_{\pounds_b(X)} &= \|A(t)U(t,\theta)\|_{\pounds_b(X)} \leq \frac{C}{t-\theta}, \\
\left\|A(t)U(t,\theta)A(\theta)^{-1}\right\|_{\pounds_b(X)} &\leq C \quad \text{for} \quad 0 \leq \theta \leq t \leq T_0.
\end{aligned}$$

(5) For every $v \in D$ and $t \in (0, T_0], U(t,\theta)v$ is differentiable with respect to θ on $0 \leq \theta \leq t \leq T_0$

$$\frac{\partial}{\partial \theta}U(t,\theta)v = U(t,\theta)A(\theta)v.$$

And, for each $\bar{x} \in X$, the Cauchy problem (2.2) has a unique classical solution $x \in C^1([0,T_0]; X)$ given by

$$x(t) = U(t,0)\bar{x}, \quad t \in [0, T_0].$$

In addition to assumption [A1], we introduce the following assumptions.

Assumption [A2]: There exits $T_0 > 0$ such that $A(t + T_0) = A(t)$ for $t \in [0, T_0]$.

Assumption [A3]: For $t \geq 0$, the resolvent $R(\lambda, A(t))$ is compact.

Then we have

Lemma 2.2: Assumptions [A1], [A2] and [A3] hold. Then evolution system $\{U(t,\theta) \mid 0 \leq \theta \leq t \leq T_0\}$ in X also satisfying the following two properties:

(6) $U(t + T_0, \theta + T_0) = U(t,\theta)$ for $0 \leq \theta \leq t \leq T_0$;

(7) $U(t,\theta)$ is compact operator for $0 \leq \theta < t \leq T_0$.

In order to introduce a impulsive evolution operator and give it's properties, we need the following assumption.

Assumption [B]: For each $k \in \mathbb{Z}_0^+$, $B_k \in \pounds_b(X)$, there exists $\delta \in \mathbb{N}$ such that $\tau_{k+\delta} = \tau_k + T_0$ and $B_{k+\delta} = B_k$.

Consider the following Cauchy problem

$$(2.3)\qquad \begin{cases} \dot{x}(t) = A(t)x(t), & t \in [0, T_0]\backslash \widetilde{D}, \\ \Delta x(\tau_k) = B_k x(\tau_k), & k = 1, 2, \cdots, \delta, \\ x(0) = \bar{x}. \end{cases}$$

For every $\bar{x} \in X$, D is an invariant subspace of B_k, using Lemma 2.1, step by step, one can verify that the Cauchy problem (2.3) has a unique classical solution $x \in PC^1([0, T_0]; X)$ represented by $x(t) = \mathscr{S}(t, 0)\bar{x}$ where $\mathscr{S}(\cdot, \cdot) : \Delta \longrightarrow \pounds(X)$ given by

$$(2.4)\qquad \mathscr{S}(t, \theta) = \begin{cases} U(t, \theta), & \tau_{k-1} \leq \theta \leq t \leq \tau_k, \\ \\ U(t, \tau_k^+)(I + B_k)U(\tau_k, \theta), & \tau_{k-1} \leq \theta < \tau_k < t \leq \tau_{k+1}, \\ \\ U(t, \tau_k^+)\left[\prod_{\theta < \tau_j < t}(I + B_j)U(\tau_j, \tau_{j-1}^+)\right](I + B_i)U(\tau_i, \theta), & \\ \qquad\qquad \tau_{i-1} \leq \theta < \tau_i \leq \cdots < \tau_k < t \leq \tau_{k+1}. \end{cases}$$

The operator $\{\mathscr{S}(t, \theta), (t, \theta) \in \Delta\}$ is called impulsive evolution operator associated with $\{B_k; \tau_k\}_{k=1}^{\infty}$.

The following lemma on the properties of the impulsive evolution operator $\{\mathscr{S}(t, \theta), (t, \theta) \in \Delta\}$ associated with $\{B_k; \tau_k\}_{k=1}^{\infty}$ are widely used in this paper.

Lemma 2.3: (See Lemma 1 of [18]) Assumptions [A1], [A2], [A3] and [B] hold. Impulsive evolution operator $\{\mathscr{S}(t, \theta), (t, \theta) \in \Delta\}$ has the following properties:

(1) For $0 \leq \theta \leq t \leq T_0$, $\mathscr{S}(t, \theta) \in \pounds_b(X)$, i.e., $\sup_{0 \leq \theta \leq t \leq T_0} \|\mathscr{S}(t, \theta)\| \leq M_{T_0}$ where $M_{T_0} > 0$.

(2) For $0 \leq \theta < r < t \leq T_0$, $r \neq \tau_k$, $\mathscr{S}(t, \theta) = \mathscr{S}(t, r)\mathscr{S}(r, \theta)$.

(3) For $0 \leq \theta \leq t \leq T_0$ and $N \in Z_0^+$, $\mathscr{S}(t + NT_0, \theta + NT_0) = \mathscr{S}(t, \theta)$.

(4) For $0 \leq t \leq T_0$ and $M \in Z_0^+$, $\mathscr{S}(MT_0 + t, 0) = \mathscr{S}(t, 0) \left[\mathscr{S}(T_0, 0)\right]^M$.

(5) $\mathscr{S}(t, \theta)$ is compact operator for $0 \leq \theta < t \leq T_0$.

Here, we note that system (2.1) has a T_0-periodic PC-mild solution x if and only if $\mathscr{S}(T_0, 0)$ has a fixed point. The impulsive evolution operator $\{\mathscr{S}(t, \theta), (t, \theta) \in \Delta\}$ can be used to reduce the existence of T_0-periodic PC-mild solutions for linear impulsive periodic system with time-varying generating operators to the existence of fixed points for an operator equation. This implies that we can build up the new framework to study the periodic PC-mild solutions for the integrodifferential impulsive periodic system with time-varying generating operators on Banach space.

Now we introduce the PC-mild solution of Cauchy problem (2.3) and T_0-periodic PC-mild solution of the system (2.1).

Definition 2.1: For every $\bar{x} \in X$, the function $x \in PC([0, T_0]; X)$ given by $x(t) = \mathscr{S}(t, 0)\bar{x}$ is said to be the PC-mild solution of the Cauchy problem (2.3) .

Definition 2.2: A function $x \in PC([0, +\infty); X)$ is said to be a T_0-periodic PC-mild solution of system (2.1) if it is a PC-mild solution of Cauchy problem (2.3) corresponding to some $\bar{x}$ and $x(t + T_0) = x(t)$ for $t \geq 0$.

Secondly, we recall the following nonhomogeneous linear impulsive periodic system with time-varying generating operators

$$\begin{cases} \dot{x}(t) = A(t)x(t) + f(t), & t \neq \tau_k, \\ \Delta x(\tau_k) = B_k x(\tau_k) + c_k, & t = \tau_k. \end{cases} \tag{2.5}$$

where $f \in L^1([0, T_0]; X)$, $f(t + T_0) = f(t)$ for $t \geq 0$ and c_k satisfies the following assumption.

Assumption [C]: For each $k \in \mathbb{Z}_0^+$ and $c_k \in X$, there exists $\delta \in \mathbb{N}$ such that $c_{k+\delta} = c_k$.

In order to study system (2.5), we need to consider the following Cauchy problem

$$\begin{cases} \dot{x}(t) = A(t)x(t) + f(t), & t \in [0, T_0] \backslash \widetilde{D}, \\ \Delta x(\tau_k) = B_k x(\tau_k) + c_k, & k = 1, 2, \cdots, \delta, \\ x(0) = \bar{x}. \end{cases} \tag{2.6}$$

and introduce the PC-mild solution of Cauchy problem (2.6) and T_0-periodic PC-mild solution of system (2.5).

Definition 2.3: A function $x \in PC([0, T_0]; X)$, for finite interval $[0, T_0]$, is said to be a PC-mild solution of the Cauchy problem (2.5) corresponding to the initial value $\bar{x} \in X$ and input $f \in L^1([0, T_0]; X)$ if x is given by

$$x(t) = \mathscr{S}(t, 0)\bar{x} + \int_0^t \mathscr{S}(t, \theta) f(\theta) d\theta + \sum_{0 \leq \tau_k < t} \mathscr{S}\left(t, \tau_k^+\right) c_k \quad \text{for } t \in [0, T_0].$$

Definition 2.4: A function $x \in PC([0, +\infty); X)$ is said to be a T_0-periodic PC-mild solution of system (2.5) if it is a PC-mild solution of Cauchy problem (2.6) to some $\bar{x}$ and $x(t+T_0) = x(t)$ for $t \geq 0$.

3. The generalized Gronwall's inequality

In order to use Leray-Schauder theorem to show the existence of periodic solutions, we need a new generalized Gronwall's inequality with impulse, mixed type integral operator and B-norm which is much different from the classical Gronwall's inequality and can be used in other problems (such as discussion on integrodifferential equation of mixed type, see [26]). It will play an essential role in the study of nonlinear problems on infinite dimensional spaces.

We first introduce the following generalized Gronwall's inequality with impulse and B-norm.

Lemma 3.1: Let $x \in PC([0, \infty); X)$ and satisfy the following inequality

$$\|x(t)\| \leq a + b \int_0^t \|x(\theta)\|^{\lambda_1} d\theta + d \int_0^t \|x_\theta\|_B^{\lambda_3} d\theta, \tag{3.1}$$

where a, b, $d \geq 0$, $0 \leq \lambda_1, \lambda_3 \leq 1$ are constants, and $\|x_\theta\|_B = \sup_{0 \leq \xi \leq \theta} \|x(\xi)\|$. Then

$$\|x(t)\| \leq (a + 1)e^{(b+c)t}.$$

Proof. See Lemma 3.1 of [23]. □

Using Gronwall's inequality with impulse and B-norm, we can obtain the following new generalized Gronwall's Lemma.

Lemma 3.2: Let $x \in PC([0,T_0];X)$ satisfy the following inequality

$$\|x(t)\| \leq a+b\int_0^t \|x(\theta)\|^{\lambda_1} d\theta + c\int_0^{T_0} \|x(\theta)\|^{\lambda_2} d\theta + d\int_0^t \|x_\theta\|_B^{\lambda_3} d\theta + e\int_0^{T_0} \|x_\theta\|_B^{\lambda_4} d\theta \text{ for all } t \in [0,T_0],$$

where $\lambda_1, \lambda_3 \in [0,1]$, $\lambda_2, \lambda_4 \in [0,1)$, a, b, c, d, $e \geq 0$ are constants. Then exists a constant $M^* > 0$ such that

$$\|x(t)\| \leq M^*.$$

Proof. See Lemma 3.2 of [23]. □

4. Periodic solutions of Integrodifferential Impulsive Periodic System with time-varying generating operators

In this section, we consider the following integrodifferential impulsive periodic system with time-varying generating operators

$$\begin{cases} \dot{x}(t) = A(t)x(t) + f\left(t, x, \int_0^t g(t,s,x)ds\right), & t \neq \tau_k, \\ \Delta x(t) = B_k x(t) + c_k, & t = \tau_k. \end{cases} \tag{4.1}$$

and the associated Cauchy problem

$$\begin{cases} \dot{x}(t) = A(t)x(t) + f\left(t, x, \int_0^t g(t,s,x)ds\right), & t \in [0,T_0]\backslash\widetilde{D}, \\ \Delta x(\tau_k) = B_k x(\tau_k) + c_k, & k = 1,2,\cdots,\delta, \\ x(0) = \bar{x}. \end{cases} \tag{4.2}$$

By virtue of the expression of the PC-mild solution of the Cauchy problem (2.6), we can introduce the PC-mild solution of the Cauchy problem (4.2).

Definition 4.1: A function $x \in PC([0,T_0];X)$ is said to be a PC-mild solution of the Cauchy problem (4.2) corresponding to the initial value $\bar{x} \in X$ if x satisfies the following integral equation

$$x(t) = \mathscr{S}(t,0)\bar{x} + \int_0^t \mathscr{S}(t,\theta) f\left(\theta, x(\theta), \int_0^\theta g(\theta,s,x(s))ds\right) d\theta + \sum_{0\leq\tau_k<t} \mathscr{S}\left(t,\tau_k^+\right) c_k \quad \text{for} \quad t \in [0,T_0].$$

Now, we introduce the T_0-periodic PC-mild solution of system (4.1).

Definition 4.2: A function $x \in PC([0,+\infty);X)$ is said to be a T_0-periodic PC-mild solution of system (4.1) if it is a PC-mild solution of Cauchy problem (4.2) corresponding to some $\bar{x}$ and $x(t+T_0) = x(t)$ for $t \geq 0$.

We make the following assumptions.

Assumption [F]:

[F1]: $f : [0,+\infty) \times X \times X \to X$ satisfies:

(i) For each $(x,y) \in X \times X$, $t \to f(t,x,y)$ is measurable.

(ii) For each $\rho > 0$ there exists $L_f(\rho) > 0$ such that, for almost all $t \in [0,+\infty)$ and all x_1, x_2, $y_1, y_2 \in X$, $\|x_1\|$, $\|x_2\|$, $\|y_1\|$, $\|y_2\| \leq \rho$, we have

$$\|f(t,x_1,y_1) - f(t,x_2,y_2)\| \leq L_f(\rho)(\|x_1 - x_2\| + \|y_1 - y_2\|).$$

[F2]: There exists a positive constant $M_f > 0$ such that

$$\|f(t,x,y)\| \le M_f(1+\|x\|+\|y\|) \text{ for all } x,y \in X.$$

[F3]: $f(t,x,y)$ is T_0-periodic in t, i.e., $f(t+T_0,x,y) = f(t,x,y), t \ge 0$.

Assumption [G]:

[G1]: Let $D = \{(t,s) \in [0+\infty) \times [0+\infty); 0 \le s \le t\}$. The function $g : D \times X \to X$ is continuous for each $\rho > 0$ there exists $L_g(\rho) > 0$ such that, for each $(t,s) \in D$ and each x, $y \in X$ with $\|x\|$, $\|y\| \le \rho$, we have

$$\|g(t,s,x) - g(t,s,y)\| \le L_g(\rho)\|x-y\|.$$

[G2]: There exists a positive constant $M_g > 0$ such that

$$\|g(t,s,x)\| \le M_g(1+\|x\|) \text{ for all } x \in X.$$

[G3]: $g(t,s,x)$ are T_0-periodic in t and s, i.e., $g(t+T_0,s+T_0,x) = g(t,s,x), t \ge s \ge 0$ and

$$\int_0^{T_0} g(t,s,x)ds = 0, t \ge s \ge 0.$$

Lemma 4.1: Under assumptions [G1] and [G2], one has the following properties:

(1) $\int_0^{\cdot} g(\cdot,s,x(s))\,ds : PC([0,T_0];X) \to PC([0,T_0];X)$.

(2) For all $x_1, x_2 \in PC([0,T_0];X)$ and $\|x_1\|_{PC([0,T_0];X)}, \|x_2\|_{PC([0,T_0];X)} \le \rho$,

$$\left\| \int_0^t g(t,s,x_1(s))\,ds - \int_0^t g(t,s,x_2(s))\,ds \right\| \le L_g(\rho) T_0 \|(x_1)_t - (x_2)_t\|_B .$$

(3) For $x \in PC([0,T_0];X)$,

$$\left\| \int_0^t g(t,s,x(s))\,ds \right\| \le M_g T_0 (1+\|x_t\|_B).$$

Proof. See Lemma 4.3 of [23]. □

Now we present the existence of PC-mild solution for system (4.2).

Theorem 4.1: Assumptions [A1], [F1], [F2], [G1] and [G2] hold. Then system (4.2) has a unique PC-mild solution given by the following integral equation

$$x(t,\bar{x}) = \mathscr{S}(t,0)\bar{x} + \int_0^t \mathscr{S}(t,\theta) f\left(\theta, x(\theta), \int_0^\theta g(\theta,s,x(s))ds\right) d\theta + \sum_{0 \le \tau_k < t} \mathscr{S}\left(t,\tau_k^+\right) c_k.$$

Proof. In order to make the process clear we divide it into three steps.

Step 1, we consider the following general integro-differential equation without impulse

$$\begin{cases} \dot{x}(t) = A(t)x(t) + f\left(t, x, \int_0^t g(t,s,x)ds\right), \ t \in [s,\tau], \\ x(s) = \bar{x} \in X. \end{cases} \tag{4.3}$$

In order to obtain the local existence of mild solution for system (4.3), we only need to set up the framework for use of the contraction mapping theorem. Consider the ball given by

$$\mathfrak{B} = \{x \in C([s,t_1];X) \mid \|x(t)-\bar{x}\| \le 1, s \le t \le t_1\}$$

where t_1 would be chosen, and $\|x(t)\| \leq 1 + \|\bar{x}\| = \bar{\rho}$, $s \leq t \leq t_1$. $\mathfrak{B} \subseteq C([s,t_1],X)$ is a closed convex set. Define a map $\mathbf{Q}$ on $\mathfrak{B}$ given by

$$(\mathbf{Q}x)(t) = U(t,0)\bar{x} + \int_s^t U(t,\theta) f\left(\theta, x(\theta), \int_0^\theta g(\theta,s,x(s))ds\right) d\theta.$$

Under the assumptions [A1], [F1], [F2], [G1], [G2] and Lemma 3.1, one can verify that map $\mathbf{Q}$ is a contraction map on $\mathfrak{B}$ with chosen $t_1 > 0$. This means that system (4.3) has a unique mild solution $x \in C([s,t_1];X)$ given by

$$x(t) = U(t,0)\bar{x} + \int_s^t U(t,\theta) f\left(\theta, x(\theta), \int_0^\theta g(\theta,s,x(s))ds\right) d\theta \text{ on } [s,t_1].$$

Again, using the Lemma 3.1 and (3) of Lemma 4.1, we can obtain the a prior estimate of the mild solutions for system (4.3) and present the global existence of mild solutions.

Step 2, for $t \in (\tau_k, \tau_{k+1}]$, consider Cauchy problem

$$\begin{cases} \dot{x}(t) = A(t)x(t) + f\left(t, x, \int_0^t g(t,s,x)ds\right), \ t \in (\tau_k, \tau_{k+1}], \\ x(\tau_k) = x_k. \end{cases} \tag{4.4}$$

where $x_k \equiv (I + B_k)x(\tau_k) + c_k \in X$.

By Step 1, Cauchy problem (4.4) also has a unique PC-mild solution

$$x(t) = U(t,\tau_k)x_k + \int_{\tau_k}^t U(t,\theta) f\left(\theta, x(\theta), \int_0^\theta g(\theta,s,x(s))ds\right) d\theta.$$

Step 3, combining the all of solutions on $(\tau_k, \tau_{k+1}]$ $(k = 1, \cdots, \delta)$, one can obtain the PC-mild solution of the Cauchy problem (4.2) given by

$$x(t,\bar{x}) = \mathscr{S}(t,0)\bar{x} + \int_0^t \mathscr{S}(t,\theta) f\left(\theta, x(\theta), \int_0^\theta g(\theta,s,x(s))ds\right) d\theta + \sum_{0 \leq \tau_k < t} \mathscr{S}\left(t, \tau_k^+\right) c_k.$$

This completes the proof. □

To establish the periodic solutions for the system (4.1), we define a *Poincaré* operator from X to X as following

$$\begin{aligned} P(\bar{x}) &= x(T_0,\bar{x}) \\ &= \mathscr{S}(T_0,0)\bar{x} + \int_0^{T_0} \mathscr{S}(T_0,\theta) f\left(\theta, x(\theta,\bar{x}), \int_0^\theta g(\theta,s,x(s,\bar{x}))ds\right) d\theta + \sum_{0 \leq \tau_k < T_0} \mathscr{S}\left(T_0, \tau_k^+\right) c_k \end{aligned} \tag{4.5}$$

where $x(\cdot,\bar{x})$ denote the PC-mild solution of the Cauchy problem (4.2) corresponding to the initial value $x(0) = \bar{x}$, then, examine whether P has a fixed point.

We first note that a fixed point of P gives rise to a periodic solution.

Lemma 4.2: System (4.1) has a T_0-periodic PC-mild solution if and only if P has a fixed point.

Proof. Suppose $x(\cdot) = x(\cdot + T_0)$, then $x(0) = x(T_0) = P(x(0))$. This implies that $x(0)$ is a fixed point of P. On the other hand, if $Px_0 = x_0$, $x_0 \in X$, then for the PC-mild solution $x(\cdot, x_0)$ of the Cauchy problem (4.2) corresponding to the initial value $x(0) = x_0$, we can define $y(\cdot) = x(\cdot + T_0, x_0)$, then $y(0) = x(T_0, x_0) = Px_0 = x_0$. Now, for $t > 0$, we can use the (2), (3) and (4) of Lemma 2.3 and assumptions [A2], [B], [C], [F3], [G3] to arrive at

$$\begin{aligned}
y(t) &= x(t+T_0, x_0) \\
&= \mathscr{S}(t+T_0, T_0)\mathscr{S}(T_0, 0)x_0 \\
&\quad + \int_0^{T_0} \mathscr{S}(t+T_0, T_0)\mathscr{S}(T_0, \theta) f\left(\theta, x(\theta, x_0), \int_0^\theta g(\theta, s, x(s, x_0))ds\right) d\theta \\
&\quad + \sum_{0 \le \tau_k < T_0} \mathscr{S}(t+T_0, T_0)\mathscr{S}\left(T_0, \tau_k^+\right) c_k \\
&\quad + \int_{T_0}^{t+T_0} \mathscr{S}(t+T_0, \theta) f\left(\theta, x(\theta, x_0), \int_0^\theta g(\theta, s, x(s, x_0))ds\right) d\theta \\
&\quad + \sum_{T_0 \le \tau_{k+\delta} < t+T_0} \mathscr{S}\left(t+T_0, \tau_{k+\delta}^+\right) c_{k+\delta} \\
&= \mathscr{S}(t, 0)\Bigg\{\mathscr{S}(T_0, 0)x_0 + \int_0^{T_0} \mathscr{S}(T_0, \theta) f\left(\theta, x(\theta, x_0), \int_0^\theta g(\theta, s, x(s, x_0))ds\right) d\theta \\
&\quad + \sum_{0 \le \tau_k < T_0} \mathscr{S}\left(T_0, \tau_k^+\right) c_{k+\delta}\Bigg\} \\
&\quad + \int_0^t \mathscr{S}(t+T_0, \theta+T_0) f\left(\theta+T_0, x(\theta+T_0, x_0), \int_0^{\theta+T_0} g(\theta+T_0, s, x(s, x_0))ds\right) d\theta \\
&\quad + \sum_{T_0 \le \tau_{k+\delta} < t+T_0} \mathscr{S}\left(t+T_0, \tau_{k+\delta}^+\right) c_{k+\delta} \\
&= \mathscr{S}(t, 0)x(T_0) \\
&\quad + \int_0^t \mathscr{S}(t+T_0, \theta+T_0) f\left(\theta+T_0, x(\theta+T_0, x_0), \int_{T_0}^{\theta+T_0} g(\theta+T_0, s, x(s, x_0))ds\right) d\theta \\
&\quad + \sum_{T_0 \le \tau_{k+\delta} < t+T_0} \mathscr{S}\left(t+T_0, \tau_{k+\delta}^+\right) c_{k+\delta} \\
&= \mathscr{S}(t, 0)x(T_0) + \int_0^t \mathscr{S}(t, \theta) f\left(\theta, x(\theta+T_0, x_0), \int_0^\theta g(\theta+T_0, s+T_0, x(s+T_0, x_0))ds\right) d\theta \\
&\quad + \sum_{T_0 \le \tau_{k+\delta} < t+T_0} \mathscr{S}\left(t+T_0, \tau_{k+\delta}^+\right) c_{k+\delta} \\
&= \mathscr{S}(t, 0)x(T_0) + \int_0^t \mathscr{S}(t, \theta) f\left(\theta, y(\theta, y(0)), \int_0^\theta g(\theta, s, y(s, y(0)))ds\right) d\theta + \sum_{0 \le \tau_k < t} \mathscr{S}\left(t, \tau_k^+\right) c_k. \\
&= \mathscr{S}(t, 0)y(0) + \int_0^t \mathscr{S}(t, \theta) f\left(\theta, y(\theta, y(0), \int_0^\theta g(\theta, s, y(s, y(0)))ds\right) d\theta + \sum_{0 \le \tau_k < t} \mathscr{S}\left(t, \tau_k^+\right) c_k.
\end{aligned}$$

This implies that $y(\cdot, y(0))$ is a PC-mild solution of Cauchy problem (4.2) with initial value $y(0) = x_0$. Thus the uniqueness implies that $x(\cdot, x_0) = y(\cdot, y(0)) = x(\cdot + T_0, x_0)$, so that $x(\cdot, x_0)$ is a T_0-periodic. □

Next, we show that P defined by (4.5) is a continuous and compact operator.

Lemma 4.3: The operator P is a continuous and compact operator.

Proof. (1) Show that P is a continuous operator on X.

Let $\bar{x}, \bar{y} \in \Xi \subset X$, where Ξ is a bounded subset of X. Suppose $x(\cdot, \bar{x})$ and $x(\cdot, \bar{y})$ are the PC-mild solutions of Cauchy problem (4.2) corresponding to the initial value $\bar{x}$ and $\bar{y} \in X$ respectively given by

$$\begin{aligned} x(t,\bar{x}) &= \mathscr{S}(t,0)\bar{x} + \int_0^t \mathscr{S}(t,\theta) f\Big(\theta, x(\theta,\bar{x}), \int_0^\theta g(\theta,s,x(s,\bar{x}))ds\Big)d\theta + \sum_{0\le\tau_k<t} \mathscr{S}\left(T_0,\tau_k^+\right)c_k;\\ x(t,\bar{y}) &= \mathscr{S}(t,0)\bar{y} + \int_0^t \mathscr{S}(t,\theta) f\Big(\theta, x(\theta,\bar{y}), \int_0^\theta g(\theta,s,x(s,\bar{y}))ds\Big)d\theta + \sum_{0\le\tau_k<t} \mathscr{S}\left(T_0,\tau_k^+\right)c_k. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} \|x(t,\bar{x})\| &\le M_{T_0}\|\bar{x}\| + (1+M_gT_0)M_{T_0}M_fT_0 + M_{T_0}\sum_{0\le\tau_k<T_0}\|c_k\| + M_{T_0}M_f\int_0^t \|x(\theta,\bar{x})\|d\theta\\ &\quad + M_{T_0}M_fM_gT_0\int_0^t\|x(s,\bar{x})\|ds\\ &\le a_0 + M_{T_0}M_f\int_0^t\|x(\theta,\bar{x})\|d\theta + M_{T_0}M_fM_gT_0\int_0^t\|x_{s,\bar{x}}\|_B ds, \end{aligned}$$

and

$$\begin{aligned} \|x(t,\bar{y})\| &\le M_{T_0}\|\bar{y}\| + (1+M_gT_0)M_{T_0}M_fT_0 + M_{T_0}\sum_{0\le\tau_k<T_0}\|c_k\| + M_{T_0}M_f\int_0^t \|x(\theta,\bar{y})\|d\theta\\ &\quad + M_{T_0}M_fM_gT_0\int_0^t\|x(s,\bar{y})\|ds\\ &\le b_0 + M_{T_0}M_f\int_0^t\|x(\theta,\bar{y})\|d\theta + M_{T_0}M_fM_gT_0\int_0^t\|x_{s,\bar{y}}\|_B ds \end{aligned}$$

where $\|x_{s,\bar{x}}\|_B = \sup_{0\le\xi\le s}\|x(\xi,\bar{x})\|$ and $\|x_{s,\bar{y}}\|_B = \sup_{0\le\xi\le s}\|x(\xi,\bar{y})\|$.

By Lemma 3.1, one can verify that there exist constants M_1^* and $M_2^* > 0$ such that

$$\|x(t,\bar{x})\| \le M_1^* \quad \text{and} \quad \|x(t,\bar{y})\| \le M_2^*.$$

Let $\rho = \max\{M_1^*, M_2^*\} > 0$, then $\|x(\cdot,\bar{x})\|, \|x(\cdot,\bar{y})\| \le \rho$ which imply that they are locally bounded.

By assumptions [F1], [F2], [G1], [G2] and (2) of Lemma 4.1, we obtain

$$\begin{aligned} &\|x(t,\bar{x}) - x(t,\bar{y})\|\\ \le\ & \|\mathscr{S}(t,0)\|\|\bar{x}-\bar{y}\|\\ &+ \int_0^t \|\mathscr{S}(t,\theta)\| \left\| f\Big(\theta,x(\theta,\bar{x}),\int_0^\theta g(\theta,s,x(s,\bar{x}))ds\Big) - f\Big(\theta,x(\theta,\bar{y}),\int_0^\theta g(\theta,s,x(s,\bar{y}))ds\Big)\right\| d\theta\\ \le\ & M_{T_0}\|\bar{x}-\bar{y}\| + M_{T_0}L_f(\rho)\int_0^t\|x(\theta,\bar{x})-x(\theta,\bar{y})\|d\theta + M_{T_0}L_f(\rho)L_g(\rho)T_0\int_0^t\|x_{s,\bar{x}}-x_{s,\bar{y}}\|_B ds. \end{aligned}$$

By Lemma 3.1 again, one can verify that there exists constant $M_3^* > 0$ such that

$$\|x(t,\bar{x}) - x(t,\bar{y})\| \le M_3^* M_{T_0}\|\bar{x}-\bar{y}\| \equiv L\|\bar{x}-\bar{y}\|, \text{ for all } t\in[0,T_0],$$

which implies that

$$\|P(\bar{x}) - P(\bar{y})\| = \big\|x(T_0,\bar{x}) - x(T_0,\bar{y})\big\| \le L\|\bar{x}-\bar{y}\|.$$

Hence, P is a continuous operator on X.

(2) Verify that P takes a bounded set into a precompact set in X.

Let Γ is a bounded subset of X. Define $K = P\Gamma = \{P(\bar{x}) \in X \mid \bar{x} \in \Gamma\}$.

For $0 < \varepsilon \le T_0$, define

$$K_\varepsilon = P_\varepsilon \Gamma = \mathscr{S}(T_0, T_0-\varepsilon)\big\{x(T_0-\varepsilon, \bar{x}) \mid \bar{x} \in \Gamma\big\}.$$

Next, we show that K_ε is precompact in X. In fact, for $\bar{x} \in \Gamma$ fixed, we have

$$\begin{aligned}
& \|x(T_0-\varepsilon, \bar{x})\| \\
\le\ & \|\mathscr{S}(T_0-\varepsilon, 0)\bar{x}\| + \int_0^{T_0-\varepsilon} \left\| \mathscr{S}(T_0-\varepsilon, \theta) f\left(\theta, x(\theta, \bar{x}), \int_0^\theta g(\theta, s, x(s, \bar{x}))ds\right) \right\| d\theta \\
& + \sum_{0 \le \tau_k < T_0-\varepsilon} \left\| \mathscr{S}\left(T_0-\varepsilon, \tau_k^+\right) c_k \right\| \\
\le\ & M_{T_0}\|\bar{x}\| + M_{T_0} M_f T_0 (1 + M_g T_0) + M_{T_0} M_f \int_0^{T_0} \|x(\theta, \bar{x})\| d\theta \\
& + M_{T_0} \sum_{0 \le \tau_k < T_0} \|c_k\| + M_{T_0} M_f M_g T_0 \int_0^{T_0} \|x_{s,\bar{x}}\|_B ds \\
\le\ & M_{T_0}\|\bar{x}\| + M_{T_0} M_f T_0 (1 + M_g T_0) + (1 + M_g T_0) M_{T_0} M_f T_0 \rho + M_{T_0} \sum_{0 \le \tau_k < T_0} \|c_k\|.
\end{aligned}$$

This implies that the set $\big\{x(T_0-\varepsilon, \bar{x}) \mid \bar{x} \in \Gamma\big\}$ is totally bounded.

By virtue of (5) of Lemma 2.3, $\mathscr{S}(T_0, T_0-\varepsilon)$ is a compact operator. Thus, K_ε is precompact in X.

On the other hand, for arbitrary $\bar{x} \in \Gamma$,

$$P_\varepsilon(\bar{x}) = \mathscr{S}(T_0, 0)\bar{x} + \int_0^{T_0-\varepsilon} \mathscr{S}(T_0, \theta) f\left(\theta, x(\theta, \bar{x}), \int_0^\theta g(\theta, s, x(s, \bar{x}))ds\right) d\theta + \sum_{0 \le \tau_k < T_0-\varepsilon} \mathscr{S}\left(T_0, \tau_k^+\right) c_k.$$

Thus, combined with (4.5), we have

$$\begin{aligned}
\big\|P_\varepsilon(\bar{x}) - P(\bar{x})\big\| \le\ & \left\| \int_0^{T_0-\varepsilon} \mathscr{S}(T_0, \theta) f\left(\theta, x(\theta, \bar{x}), \int_0^\theta g(\theta, s, x(s, \bar{x}))ds\right) d\theta \right. \\
& \left. - \int_0^{T_0} \mathscr{S}(T_0, \theta) f\left(\theta, x(\theta, \bar{x}), \int_0^\theta g(\theta, s, x(s, \bar{x}))ds\right) d\theta \right\| \\
& + \left\| \sum_{0 \le \tau_k < T_0-\varepsilon} \mathscr{S}\left(T_0, \tau_k^+\right) c_k - \sum_{0 \le \tau_k < T_0} \mathscr{S}\left(T_0, \tau_k^+\right) c_k \right\| \\
\le\ & \int_{T_0-\varepsilon}^{T_0} \|\mathscr{S}(T_0, \theta)\| \left\| f\left(\theta, x(\theta, \bar{x}), \int_0^\theta g(\theta, s, x(s, \bar{x}))ds\right) \right\| d\theta \\
& + M_{T_0} \sum_{T_0-\varepsilon \le \tau_k < T_0} \|c_k\| \quad \text{for} \quad \tau_k \in [T_0-\varepsilon, T_0) \\
\le\ & M_{T_0} M_f (1 + M_g T_0)(1 + \rho)\varepsilon + M_{T_0} \sum_{T_0-\varepsilon \le \tau_k < T_0} \|c_k\| \quad \text{for} \quad \tau_k \in [T_0-\varepsilon, T_0).
\end{aligned}$$

If there are no impulsive points between $[T_0-\varepsilon, T_0]$, it is clear that $\sum_{T_0-\varepsilon \le \tau_k < T_0} \|c_k\|$ being zero.

As a result, it is showing that the set K can be approximated to an arbitrary degree of accuracy by a precompact set K_ε. Hence K itself is precompact set in X. That is, P takes a bounded set into a precompact set in X. As a result, P is a compact operator. □

In order to use Leray-Schauder fixed pointed theorem to examine the operator P has a fixed point, we have to make the assumptions [F2] and [G2] a little stronger as following.

[F2′]: There exist constant $N_f > 0$ and $0 < \lambda < 1$ such that

$$\|f(t,x,y)\| \leq N_f\left(1 + \|x\|^{\lambda} + \|y\|^{\lambda}\right) \text{ for all } x, y \in X.$$

[G2′]: There exists a positive constant $N_g > 0$ and $0 < \lambda < 1$ such that

$$\|g(t,s,x)\| \leq N_g(1 + \|x\|^{\lambda}) \text{ for all } x \in X.$$

Now, we can give the main results in this paper.

Theorem 4.2: Assumptions [A1], [A2], [A3], [B], [C], [F1], [F2′], [F3], [G1], [G2′], [G3] hold. Then system (4.1) has a T_0-periodic PC-mild solution on $[0, +\infty)$.

Proof. By virtue of (5) of Lemma 2.3, $\mathscr{S}(T_0, 0)$ is a compact operator on infinite dimensional space X. Thus, $\mathscr{S}(T_0, 0) \neq \alpha I$, $\alpha \in \mathbb{R}$. Then, there exists $\beta > 0$ such that $\left\|[\sigma\mathscr{S}(T_0,0) - I]\bar{x}\right\| \geq \beta\|\bar{x}\|$ for $\sigma \in [0,1]$. In fact, define $\Pi_\sigma = I - \sigma\mathscr{S}(T_0,0)$, $\sigma \in [0,1]$, and $\Pi_\sigma : [0,1] \to \pounds_b(X)$ and $h(\sigma) = \|\Pi_\sigma\| : [0,1] \to \mathbb{R}^+$. It is obvious that $h \in C([0,1];\mathbb{R}^+)$. Thus, there exist $\sigma_* \in [0,1]$ and $\beta > 0$ such that

$$h(\sigma_*) = \min\{h(\sigma) \mid \sigma \in [0,1]\} \geq \beta > 0.$$

If not, there exits $\bar{\sigma} \in [0,1]$ such that $h(\bar{\sigma}) = 0$. We can assert that $\bar{\sigma} \neq 0$ unless $h(\bar{\sigma}) = 1$. Thus, for $\bar{\sigma} \in (0,1]$,

$$\mathscr{S}(T_0,0) = \frac{1}{\bar{\sigma}} I \quad \text{where} \quad \frac{1}{\bar{\sigma}} \geq 1,$$

which is a contradiction with $\mathscr{S}(T_0,0) \neq \alpha I$, $\alpha \in \mathbb{R}$.

By Theorem 4.1, for fixed $\bar{x} \in X$, the Cauchy problem (4.2) corresponding to the initial value $x(0) = \bar{x}$ has the PC-mild solution $x(\cdot, \bar{x})$. By Lemma 4.3, the operator P defined by (4.5), is compact.

According to Leray-Schauder fixed point theory, it suffices to show that the set $\{\bar{x} \in X \mid \bar{x} = \sigma P\bar{x}, \sigma \in [0,1]\}$ is a bounded subset of X. In fact, let $\bar{x} \in \{\bar{x} \in X \mid \bar{x} = \sigma P\bar{x}, \sigma \in [0,1]\}$, we have

$$\begin{aligned}
\beta\|\bar{x}\| &\leq \left\|[\sigma\mathscr{S}(T_0,0) - I]\bar{x}\right\| \\
&= \sigma\int_0^{T_0} \|\mathscr{S}(T_0,\theta)\| \left\| f\left(\theta, x(\theta,\bar{x}), \int_0^{\theta} g(\theta,s,x(s,\bar{x}))ds\right)\right\| d\theta + \sigma \sum_{0\leq\tau_k<T_0} \left\|\mathscr{S}\left(T_0,\tau_k^+\right)\right\| \|c_k\|.
\end{aligned}$$

By assumptions [F2′] and [G2′],

$$\begin{aligned}
\|\bar{x}\| &\leq \frac{\sigma}{\beta}\int_0^{T_0} \|\mathscr{S}(T_0,\theta)\| \left\| f\left(\theta, x(\theta,\bar{x}), \int_0^{\theta} g(\theta,s,x(s,\bar{x}))ds\right)\right\| d\theta + \frac{\sigma}{\beta} \sum_{0\leq\tau_k<T_0} \left\|\mathscr{S}\left(T_0,\tau_k^+\right)\right\| \|c_k\| \\
&\leq \frac{\sigma}{\beta} M_{T_0}\left((N_f + N_g T_0)T_0 + N_f\int_0^{T_0} \|x(\theta,\bar{x})\|^{\lambda} d\theta + N_f N_g T_0 \int_0^{T_0} \|x_{s,\bar{x}}\|_B^{\lambda} d\theta + \sum_{0\leq\tau_k<T_0} \|c_k\|\right).
\end{aligned}$$

For $t \in [0, T_0]$, we obtain

$$
\begin{aligned}
&\|x(t,\bar{x})\| \\
\leq\ & M_{T_0}\|\bar{x}\| + M_{T_0}\left((N_f + N_g T_0)T_0 + N_f\int_0^t \|x(\theta,\bar{x})\|^\lambda d\theta + N_f N_g T_0 \int_0^t \|x_{s,\bar{x}}\|_B^\lambda d\theta + \sum_{0\leq \tau_k < t}\|c_k\|\right) \\
\leq\ & \frac{\sigma}{\beta}M_{T_0}^2\left((N_f + N_g T_0)T_0 + N_f\int_0^{T_0} \|x(\theta,\bar{x})\|^\lambda d\theta + N_f N_g T_0 \int_0^{T_0} \|x_{s,\bar{x}}\|_B^\lambda d\theta + \sum_{0\leq \tau_k < T_0}\|c_k\|\right) \\
& + M_{T_0}\left((N_f + N_g T_0)T_0 + N_f\int_0^t \|x(\theta,\bar{x})\|^\lambda d\theta + N_f N_g T_0 \int_0^t \|x_{s,\bar{x}}\|_B^\lambda d\theta + \sum_{0\leq \tau_k < t}\|c_k\|\right) \\
\leq\ & \left(\frac{\sigma}{\beta}M_{T_0} + 1\right) M_{T_0}\left((N_f + N_g T_0)T_0 + \sum_{0\leq \tau_k < T_0}\|c_k\|\right) + M_{T_0}N_f\int_0^t \|x(\theta,\bar{x})\|^\lambda d\theta \\
& + \frac{\sigma}{\beta}M_{T_0}^2 N_f\int_0^{T_0} \|x(\theta,\bar{x})\|^\lambda d\theta + M_{T_0}N_f N_g T_0\int_0^t \|x_{s,\bar{x}}\|_B^\lambda d\theta + \frac{\sigma}{\beta}M_{T_0}^2 N_f N_g T_0\int_0^{T_0} \|x_{s,\bar{x}}\|_B^\lambda d\theta.
\end{aligned}
$$

By Lemma 3.2, there exists $M^* > 0$ such that

$$\|x(t,\bar{x})\| \leq M^* \text{ for } t \in [0, T_0].$$

This implies that $\|x(0,\bar{x})\| = \|\bar{x}\| \leq M^*$ for all $\bar{x} \in \big\{\bar{x} \in X \mid \bar{x} = \sigma P\bar{x}, \sigma \in [0,1]\big\}$.

Thus, by Leray-Schauder fixed pointed theory, there exits $x_0 \in X$ such that $Px_0 = x_0$. By Lemma 4.2, we know that the PC-mild solution $x(\cdot, x_0)$ of Cauchy problem (4.2) corresponding to the initial value $x(0) = x_0$, is just T_0-periodic. Therefore $x(\cdot, x_0)$ is a T_0-periodic PC-mild solution of system (4.1). □

5. An Example

Consider the following non-autonomous integrodifferential periodic population evolution equation with periodic impulsive perturbations

$$
(5.1)\quad
\begin{cases}
\frac{\partial}{\partial t}x(r,t) + \sin t\frac{\partial}{\partial r}x(r,t) = -0.2\sin t x(r,t) + x^{\frac{2}{3}}(r,t) \\
\qquad\qquad + \int_0^t \psi(s)(1+\sin(t-s))\sqrt{3x^{\frac{2}{3}}(r,s)+2}ds, \\
\qquad\qquad r \in \Omega = (0, r_m),\ t > s \text{ and } t, s \in (0, 2\pi] \setminus \{\frac{1}{2}\pi, \pi, \frac{3}{2}\pi\}, \\
\Delta x(r,\tau_i) = x(r,\tau_i^+) - x(r,\tau_i^-) = \begin{cases} 0.05Ix(r,\tau_i), i = 1, \\ -0.05Ix(r,\tau_i), i = 2, \\ 0.05Ix(r,\tau_i), i = 3, \end{cases} \quad r \in \Omega, \tau_i = \frac{i}{2}\pi, i = 1,2,3, \\
x(r,0) = x_0(r), \quad r \in \Omega, \\
x(0,t) = \varphi_0(t), \quad t \in (0, 2\pi) \setminus \{\frac{1}{2}\pi, \pi, \frac{3}{2}\pi\}, \\
x(r,0) = x(r,2\pi).
\end{cases}
$$

where t denotes time, r denotes age, r_m is the highest age ever attained by individuals of the population, $x(r,t)$ is called age density function, $x_0(r)$ is an initial age density of the people, $\varphi_0(t)$ is the absolute infant fertility rate of population.

Set $X = L^2(0, r_m)$. Let $A(t)$ be defined by

$$(A(t)\varphi)(r) = \sin t\Big(- \frac{d}{dr}\varphi(r) - 0.2\varphi(r)\Big), \text{ for arbiartry } \varphi \in D(A(t))$$

where

$$D(A(t)) = \{\varphi \mid \varphi,\ A(t)\varphi \in L^2(0, r_m);\ \varphi(0) = \varphi_0\}.$$

It is well known that operator $A \equiv -\frac{d}{dr}\varphi(r) - 0.2\varphi(r)$ with $t > r_m$ is the infinitesimal generator of a compact semigroup $\{T(t), t \geq 0\}$ on X with domain $D(A) = \{\varphi \mid \varphi,\ A\varphi \in L^2(0, r_m);\ \varphi(0) = \varphi_0\}$. Thus, one can immediately obtain that $\{A(t), t > r_m\}$ can determine a compact, 2π-periodic evolutionary process $\{U(t, \theta),\ t \geq \theta \geq 0\}$.

Define $x(\cdot)(r) = x(r, \cdot)$, $\sin(\cdot)(r) = \sin(r, \cdot)$, $f\left(\cdot, x(\cdot), \int_0^{\cdot} g(\cdot, s, x)ds\right)(r) = x^{\frac{2}{3}}(\cdot)(r) + \int_0^t \psi(s)(1 + \sin(\cdot - s))\sqrt{3x^{\frac{2}{3}}(\cdot) + 2}ds(r)$ where $\psi(\cdot + 2\pi) = \psi(\cdot) \in L^1_{loc}([0, +\infty); X)$, $\int_0^{2\pi} \psi(s)(1 + \sin(t - s))\sqrt{3x^{\frac{2}{3}}(t) + 2}ds = 0$, and

$$B_i = \begin{cases} 0.05I, i = 1, \\ -0.05I, i = 2, \\ 0.05I, i = 3. \end{cases}$$

Thus we formulate (5.1) as the following abstract integrodifferential impulsive periodic system

$$(5.2) \qquad \begin{cases} \dot{x}(t) = A(t)x(t) + f\left(t, x, \int_0^t g(t, s, x)ds\right), & t \in (0, 2\pi] \setminus \{\frac{1}{2}\pi, \pi, \frac{3}{2}\pi\}, \\ \Delta x\left(\frac{i}{2}\pi\right) = B_i x\left(\frac{i}{2}\pi\right), & i = 1, 2, 3, \\ x(0) = x(2\pi). \end{cases}$$

It satisfies all the assumptions given in Theorem 4.2, our results can be used to system (5.1). That is, system (5.1) has a 2π-periodic PC-mild solution $x_{2\pi}(r, \cdot) \in PC_{2\pi}\left([0 + \infty); L^2(0, r_m)\right)$, where

$$PC_{2\pi}([0, +\infty); L^2(0, r_m)) \equiv \left\{x \in PC\left([0, +\infty); L^2(0, r_m)\right) \mid x(t) = x(t + 2\pi), t \geq 0\right\};$$

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