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BOUNDED WEAK SOLUTIONS TO NONLINEAR ELLIPTIC EQUATIONS

ABDERRAHMANE EL HACHIMI, JAOUAD IGBIDA

Abstract. In this work, we are concerned with a class of elliptic problems with both absorption terms and critical growth in the gradient. We suppose that the data belong to $L^m(\Omega)$ with $m > n/2$ and we prove the existence of bounded weak solutions via L^∞-estimates. A priori estimates and Stampacchia's L^∞-regularity are our main ingredient.

1. Introduction

In this work, we intend to study the Dirichlet problem for some nonlinear elliptic equations whose model example is:

$$(P)\begin{cases} -\Delta u + a(x)u|u|^{r-1} = \beta(u)|\nabla u|^2 + f(x) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

where Ω is a bounded open set in $\mathbb{R}^N$, $N > 2$, Δ denotes the Laplace operator and β is a continuous nonincreasing real function , with $\beta \in L^1(\mathbb{R})$. The real function $a(x)$ is nonnegative and bounded in $L^\infty(\Omega)$. Under suitable conditions on the data, we shall study existence and regularity of solutions for problem (P).

These kind of problems have been treated in a large literature starting from the classical references [18] and [19]. Later, many works have been devoted to elliptic problems with lower order terms having quadratic growth with respect to the gradients (see e.g. [8], [9], [13], [15], [16], [17], [22] and the references therein).

The general problem (P), though being physically natural, does not seem to have been studied in the literature. So for special situations, in the case where $a = 0$, β is constant and $f = 0$, this equation may be considered as the stationary part of equation

$$u_t - \Delta u = \epsilon|\nabla u|^2 ,$$

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which appears in the physical theory of growth and roughening of surfaces. It is well known as the Kardar-Parisi-Zhang equation (see [16]). It presents also the viscosity approximation as $\epsilon \to +\infty$ of Hamilton-Jacobi equations from stochastic control theory (see [21]).

For the simpler case where $a = 0$, β is a constant (we can assume $\beta = 1$ without loss of generality) and $f \in L^{\frac{N}{2}}$; that is when (P) of the form

$$(1.1) \qquad \begin{aligned} -\Delta u &= |\nabla u|^2 + f(x) \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \end{aligned}$$

the problem has been studied in [17], where the change of variable $v = e^u - 1$ leads to the following problem

$$(1.2) \qquad \begin{aligned} -\Delta v &= f(x)(v+1) \text{ in } \Omega, \\ v &= 0 \text{ on } \partial\Omega. \end{aligned}$$

Then, provided that $f \in L^{\frac{N}{2}}$, it is proved there that (1.1) admits a unique solution in $W_0^{1,2}(\Omega)$.

In the case where $a = 0$, $f \in L^q$ with $q > \frac{N}{2}$, and β is a continuous nonnegative function satisfying supplementary conditions according to each situation, for instance $\beta(s) = \frac{1}{\sqrt{(1+s^2)^3}}$, or $\beta(s) = \frac{e^{|s|}}{(1+s^2)}$, a priori estimates have been proved in [1] and [6] to obtain existence and regularity results, while uniqueness have been shown in [2].

In this paper, to prove existence of bounded weak solutions for (P), we assume that $f \in L^m$, $m > \frac{N}{2}$, β is a continuous real function nonincreasing with $\beta \in L^1(\mathbb{R})$ and a is a nonnegative bounded real function. We shall obtain a solution by an approximating process. Using a priori estimates and Stampacchia's L^∞-regularity results we shall show that the approximated solutions converges to a solution of problem (P).

2. Preliminaries and main results

In this section, we present some notations and assumptions. We also recall some concepts and results which will be used in our further considerations. We will refer the reader to the corresponding references.

Throughout this paper Ω will denotes a bounded open set in $\mathbb{R}^N$ with $N > 2$. We denote by c a positive constant which may only depend on the parameters of our problem, its value my vary from line to line.

For $1 \leq q \leq N$ we denote $q^* = \frac{N\,q}{N-q}$. Moreover, we denote $N' = \frac{N}{N-1}$ and its Sobolev conjugate by $N_0 = \frac{N}{N-2}$.

For $k > 0$ we define the truncature at level $\pm k$ as

$$T_k(s) = min\{k, max\{s, -k\}\}.$$

We also consider $G_k(s) = s - T_k(s) = (|s| - k)^+ \, sign\,(s)$. We introduce $T_0^{1,2}(\Omega)$ as the set of all measurable functions $u : \Omega \to \mathbb{R}^N$ such that $T_k(u) \in W_0^{1,2}(\Omega)$ for all $k > 0$. We point out that $T_0^{1,2}(\Omega) \cap L^\infty(\Omega) = W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.

For a measurable function u belonging to $T_0^{1,2}(\Omega)$, a gradient can be defined as a measurable function which is also denoted by ∇u and satisfies $\nabla T_k(u) = \nabla u\, \chi_{[|u|<k]}$ for all $k > 0$ (see e.g. [3]).

We are going to investigate the solution of the following nonlinear elliptic problem

$$(P)\begin{cases} -\Delta u + a(x)u|u|^{r-1} = \beta(u)|\nabla u|^2 + f(x) \text{ in } \Omega, \\ \qquad\qquad\qquad\qquad u|_{\partial\Omega} = 0, \end{cases}$$

where Ω denotes a bounded open set in $\mathbb{R}^N$ with $N > 2$. u denote a real function depending on x in $\mathbb{R}^N$.

We denote by γ the real function

$$\gamma(s) = \int_0^s \beta(\sigma)\, d\sigma. \tag{2.1}$$

We assume that $r > 1$, and that

$$f \in L^m(\Omega) \text{ with } m > \frac{N}{2}. \tag{2.2}$$

Both functions β and a have to satisfy certain structural assumptions which are described by:

(A) There exists a_0 such that $a \geq a_0 > 0$ a.e in Ω and $a \in L^\infty(\Omega)$.

(B) The real function β is continuous nonincreasing with $\beta \in L^1(\mathbb{R})$. Without loss of generality we assume $\beta(0) = 0$

By a weak solution of problem (P) we mean a function u, such that both functions $\beta(u)\,|\nabla u|^2$ and $a(x)u|u|^{r-1}$ are integrable, and the following equality holds

$$\int_\Omega \nabla u\nabla\phi + \int_\Omega a(x)u|u|^{r-1}\phi = \int_\Omega \beta(u)|\nabla u|^2\phi + \int_\Omega f\,\phi, \tag{2.3}$$

for any test function ϕ in $H_0^1(\Omega)\cap L^\infty(\Omega)$.

Theorem 2.1. *Let f in $L^m(\Omega)$, $m > \frac{N}{2}$ and $r > 1$. Then, under the assumptions (A) and (B) the problem (P) has at least one solution which belongs to $W^{1,2}(\Omega)\cap L^\infty(\Omega)$.*

□

3. Fundamental estimates

3.1. **Estimates on general problem.** In this sections we prove some basic estimate for regular elliptic problem. The main tools for proving theorem 2.1 are a priori estimates together with compactness arguments applied to a sequences of bounded approximating solutions. We shall study the nonlinear elliptic equation

$$-\Delta u = B(x,u,\nabla u) + f(x) \text{ in } \Omega, \tag{3.1}$$

under the assumption

$$u|_{\partial\Omega} = 0. \tag{3.2}$$

Where $B(x,s,\xi) = b(s,\xi) - a(x,s)\,|s|^{r-1}$; $a(.,.)$ and $b(.,.)$ are two functions satisfying the following hypothesis:

- (a_1) $a(x,s) : \Omega\times R \to R$ is measurable in $x \in R^N$ for any fixed $s\in R$ and continuous in s for a.e. x.
- (a_2) There exists a constant $c > 0$ such that for all s and almost every x
$$a(x,s) \geq a(x)\,s + c.$$
- (a_3) For any $\alpha > 0$ the function
$$a_\alpha(x) = \sup_{|s|\leq\alpha}\{a(x,s)\,|s|^{r-1}\}$$
is integrable over Ω.
- (b_1) $b(s,\xi) : R\times R^N \to R$ is measurable in $s\in R$ for any fixed $\xi\in R^N$ and continuous in ξ for a.e. s.

(b_2) The real function b satisfies

$$b\,(s,\xi) \leq \beta\,(s)\,|\xi|^2\ ,\ \text{for all } s \text{ and } \xi\ .$$

Let us note that if u is a weak solution of (3.1), then it satisfy the following equality

$$\int_\Omega \nabla u \nabla \psi = \int_\Omega B(x,u,\nabla u)\psi + \int_\Omega f\,\psi,$$

for all ψ in $H_0^1\,(\Omega) \cap L^\infty\,(\Omega)$.

We will now prove the following basic results. If u is a weak solution of (3.1) we denote $u_k = T_k(u)$. Then we have the following estimates:

Lemma 3.1. *Let u be a weak solution of problem (3.1). Then u satisfies*

$$||u||_{L^r(\Omega)} \leq c.$$

Proof. We consider for $m > 1$ the following function

$$\begin{array}{ll} \psi_m\,(s) = (m-1)\int_0^s \frac{1}{(1+t)^m} & if\ s \geq 0, \\ \psi_m\,(s) = -\psi_m\,(-s) & if\ s \leq 0. \end{array}$$

Taking $\psi_m\,(u)$ as test function in (3.1), where m is such that $0 < m-1 < r\ -1$, we obtain

$$\int_\Omega \nabla u\,\nabla \psi_m(u) = \int_\Omega B(x,u,\nabla u)\psi_m(u) + \int_\Omega f\psi_m\,(u)\,.$$

$$\int_\Omega \nabla u\,\nabla \psi_m(u) + \int_\Omega a(x,u)|u|^{r-1}\psi_m(u) \leq \int_\Omega b(u,\nabla u\,)\psi_m(u) + \int_\Omega f\psi_m\,(u)\,.$$

$$\int_\Omega \nabla u\,\nabla \psi_m(u) + \int_\Omega a(x)u|u|^{r-1}\psi_m(u) \leq \int_\Omega \beta\,(u)\,|\nabla u|^2\psi_m(u) + \int_\Omega f\psi_m\,(u)\,.$$

Then we have

$$\int_\Omega |\nabla u|^2\,\psi_m'(u) + \int_\Omega a(x)u|u|^{r-1}\psi_m(u) \leq \int_\Omega \beta\,(u)\,|\nabla u|^2\psi_m(u) + \int_\Omega f\psi_m\,(u)\,.$$

$$(m-1)\int_\Omega |\nabla u|^2\,\frac{1}{(1+|u|)^m} + \int_\Omega a(x)u|u|^{r-1}\psi_m(u) \leq \int_\Omega f\psi_m\,(u)\,.$$

$$(3.3)\quad (m-1)\int_\Omega |\nabla u|^2\,\frac{1}{(1+|u|)^m} + \int_\Omega a_0\,u\,|u|^{r-1}\psi_m(u) \leq \int_\Omega f\psi_m\,(u)\,.$$

Since $s|s|^{r-1}\psi(s)$ is nonnegative, then using the fact that

$$s|s|^{r-1}\psi_m(s) \geq |s|^r\psi_m(1),\quad (\psi_m(1) = 1-2^{-m+1}),\ \text{for } s > 1.$$

We get for all u

$$\int_\Omega |u|^r \leq \int_\Omega u|u|^{r-1}\frac{\psi_m(u)}{\psi_m(1)} + |\Omega|.$$

Then we obtain

$$\int_\Omega |u|^r \leq c.$$

□

Lemma 3.2. *Let u be a weak solution of problem (3.1). Then, for $N > 2$ and $r > 1$, one has*

$$\nabla u \in L^q(\Omega) \text{ for any } q, 1 \leq q < N' = \frac{N}{N-1}$$

and

$$u \in L^{q^*}(\Omega) \text{ where } q^* = \frac{qN}{N-q}.$$

Proof. Let $q \in [1, N'[$, where $N' = \frac{N}{N-1}$. We note that $N' \in]1, N[$.

We chose m such that $0 < m < m_0 = (N' - q)\frac{N-1}{N-q}$ and we use Hölder's inequality to obtain

(3.4)

$$\int_\Omega |\nabla u|^q dx \leq \left(\int_\Omega |\nabla u|^2 \frac{1}{(1+|u|)^m} dx\right)^{\frac{q}{2}} \left(\int_\Omega (1+|u|)^{m\frac{q}{2-q}} dx\right)^{\frac{2-q}{2}}.$$

Moreover $m < m_0$ is equivalent to $m\frac{q}{2-q} < q^*$, thus we get for any $\epsilon > 0$, that

$$(1+|u|)^{m\frac{q}{2-q}} \leq \epsilon|u|^{q^*} + c(\epsilon). \tag{3.5}$$

From (3.3) and (3.4), we obtain

$$\int_\Omega |\nabla u|^q dx \leq c_1 \left(\int_\Omega |u|^{q^*} dx\right)^{\frac{2-q}{2}} + c_2. \tag{3.6}$$

Since $q < N$, then from Sobolev's inequality, we have

$$||u - \widetilde{u}||_{L^{q^*}(\Omega)} \leq ||\nabla u||_{L^q(\Omega)},$$

where

$$\widetilde{u} := mes(\Omega)^{-1} \int_\Omega u(x)\, dx.$$

This implies that

$$\left(\int_\Omega |u|^{q^*} dx\right)^{1/q^*} \leq \left(\int_\Omega |\nabla u|^q dx\right)^{1/q} + |\Omega|^{\frac{1}{q^*}} ||\widetilde{u}||_r. \tag{3.7}$$

From lemma 3.1, we have for $r > 1$ that

$$||\widetilde{u}||_r^r \le \frac{1}{|\Omega|}\int_\Omega |u|^r dx \le c.$$

Using (3.6) and (3.7), we deduce that

$$||u\,||_{L^{q^*}(\Omega)} \le \frac{1}{2}\left(\int_\Omega |u|^{q^*} dx\right)^{\frac{2-q}{2q}} + c.$$

Now, since $N > 2$, $\frac{2-q}{2q} < \frac{1}{q^*} = \frac{N-q}{qN}$, we obtain

$$||u\,||_{L^{q^*}(\Omega)} \le c.$$

$\square$

Lemma 3.3. *For a weak solution of (3.1).The following estimates*

$$\int_{[|u|>k]} |\nabla u|^2 dx \le c\int_\Omega |f\,G_k\,(u)\,|, \tag{3.8}$$

and

$$\int_\Omega |\nabla u_k|^2 dx \le c\int_\Omega |f\,u_k|, \tag{3.9}$$

hold for all $k > 0$.

Proof. We define the following functions

$$\varphi_{k,h}\,(s) = G_k\,(T_h\,(s))\,,$$

$$\psi_{k,h}\,(s) = e^{\gamma(T_k(s))}\varphi_{k,h}\,(s)\,.$$

Taking $\psi_{k,h}\,(u)$ as a test function in (3.1), we obtain

$$\int_\Omega \beta(u_k)\psi_{k,h}(u)\nabla u\nabla u_k + \int_\Omega e^{\gamma(u_k)}\nabla u\nabla u_k\varphi_{k,h}(u)$$
$$= \int_\Omega B(x,u,\nabla u)\psi_{k,h}(u) + \int_\Omega f\,\psi_{k,h}(u).$$

$$\int_\Omega \beta(u_k)\psi_{k,h}(u)\nabla u\nabla u_k + \int_\Omega e^{\gamma(u_k)}\nabla u\nabla u_k\varphi_{k,h}(u) + \int_\Omega a(x,u)|u|^{r-1}\psi_{k,h}(u)$$
$$\le \int_\Omega b(u,\nabla u)\psi_{k,h}(u) + \int_\Omega f\,\psi_{k,h}(u).$$

(3.10)
$$\int_\Omega \beta(u_k)\psi_{k,h}(u)\nabla u\nabla u_k+\int_\Omega e^{\gamma(u_k)}\nabla u\nabla u_k\varphi_{k,h}(u)+\int_\Omega a(x)u|u|^{r-1}\psi_{k,h}(u)$$
$$\leq \int_\Omega \beta(u)|\nabla u|^2\psi_{k,h}(u)+\int_\Omega f\,\psi_{k,h}(u).$$

We note that
$$\int_\Omega \beta(u_k)\psi_{k,h}(u)\nabla u\nabla u_k=\int_{[u<k]}\beta(u)e^{\gamma(u)}\varphi_{k,h}(u)|\nabla u|^2.$$

Applying monotone convergence theorem, we have
$$\lim_{h\to+\infty}\int_\Omega \beta(u_k)\varphi_{k,h}(u)\nabla u\nabla u_k=\int_\Omega \beta(u)e^{\gamma(u)}G_k(u)|\nabla u|^2,$$
$$\lim_{h\to+\infty}\int_\Omega \beta(u)|\nabla u|^2\psi_{k,h}(u)=\int_\Omega \beta(u)|\nabla u|^2e^{\gamma(u)}G_k(u).$$

Letting h tend to infinity in (3.10) and applying Lebesgue's dominated convergence theorem, we obtain
$$\int_\Omega \beta(u)|\nabla u|^2e^{\gamma(u)}G_k(u)+\int_\Omega e^{\gamma(u)}\nabla u\nabla G_k(u)+\int_\Omega a(x)u|u|^{r-1}e^{\gamma(u)}G_k(u)$$
$$\leq \int_\Omega \beta(u)|\nabla u|^2e^{\gamma(u)}G_k(u)+\int_\Omega f\,e^{\gamma(u)}G_k(u).$$
Then, we obtain
$$\int_\Omega e^{\gamma(u)}\nabla u\nabla G_k(u)\leq\int_\Omega f\,G_k(u).$$
Therefore, we have
$$\int_{[|u|>k]}|\nabla G_k(u)|^2dx\leq c\int_\Omega |f\,G_k\,(u)\,|,$$
which implies that (3.8) is satisfied.

To prove the second assertion, let us take
$$\phi_{k,h}=e^{\gamma(u_k)}u_h$$
as a test function in (3.1), we obtain
$$\int_\Omega \beta(u_k)\phi_{k,h}(u)\nabla u\nabla u_k+\int_\Omega e^{\gamma(u_k)}\nabla u\nabla u_k$$
$$=\int_\Omega B(x,u,\nabla u)\phi_{k,h}(u)+\int_\Omega f\,\phi_{k,h}(u).$$

$$\int_\Omega \beta(u_k)\phi_{k,h}(u)\nabla u\nabla u_k + \int_\Omega e^{\gamma(u_k)}\nabla u\nabla u_k + \int_\Omega a(x,u)|u|^{r-1}\phi_{k,h}(u) \leq \int_\Omega \beta(u)|\nabla u|^2\phi_{k,h}(u) + \int_\Omega f\,\phi_{k,h}(u).$$

(3.11)

$$\int_\Omega \beta(u_k)\phi_{k,h}(u)\nabla u\nabla u_k + \int_\Omega e^{\gamma(u_k)}\nabla u\nabla u_k + \int_\Omega a(x)u|u|^{r-1}\phi_{k,h}(u) \leq \int_\Omega \beta(u)|\nabla u|^2\phi_{k,h}(u) + \int_\Omega f\,\phi_{k,h}(u).$$

The monotone Convergence Theorem yields

$$\lim_{h\to+\infty}\int_\Omega \beta(u_k)\phi_{k,h}(u)\nabla u\nabla u_k = \int_\Omega \beta(u)e^{\gamma(u)}u_k|\nabla u|^2,$$

$$\lim_{h\to+\infty}\int_\Omega \beta(u)|\nabla u|^2\phi_{k,h}(u) = \int_\Omega \beta(u)|\nabla u|^2 e^{\gamma(u)}u_k.$$

Now, applying Lebesgue's dominated convergence theorem in (3.11), we obtain

$$\int_\Omega \beta(u)e^{\gamma(u)}|\nabla u|^2 T_k(u) + \int_\Omega e^{\gamma(u)}\nabla u\nabla u_k + \int_\Omega a(x)u|u|^{r-1}e^{\gamma(u)}u_k \leq \int_\Omega \beta(u)|\nabla u|^2 e^{\gamma(u)}u_k + \int_\Omega f\,e^{\gamma(u)}u_k.$$

After simplifications we have

$$\int_\Omega e^{\gamma(u)}\nabla u\nabla u_k \leq \int_\Omega f\,e^{\gamma(u)}u_k.$$

Therefore, we get

$$\int_{[|u|<k]} e^{\gamma(u)}|\nabla u|^2 dx \leq c\int_\Omega |f\,u_k|.$$

Finally, by Fatou's lemma, we deduce that

$$\int_\Omega |\nabla u_k)|^2 dx \leq c\int_\Omega |f\,u_k|.$$

□

Lemma 3.4. *There exists a constant c such that the solution of problem (3.1) satisfies*

$$\int_{[|u|\geq k]} |b(u,\nabla u)| \leq c.$$

Proof. Let us consider

$$\varphi_k(s) = \gamma\left(G_k(s) + k\, sign(s)\right) - \gamma\left(k\, sign(s)\right),$$
$$\psi_{k,h}(s) = \varphi_k(T_h(s)).$$

Taking $e^{\gamma(u_h)}\psi_{k,h}(u)$ as test function in (3.1), we obtain

$$\int_\Omega \beta(u_h)e^{\gamma(u_h)}\psi_{k,h}(u)\nabla u\nabla u_k + \int_\Omega e^{\gamma(u_h)}\nabla u\nabla\psi_{k,h}(u)$$
$$\leq \int_\Omega B(x,u,\nabla u)e^{\gamma(u_h)}\psi_{k,h}(u) + \int_\Omega f\, e^{\gamma(u_h)}\psi_{k,h}(u).$$

$$\int_\Omega \beta(u_h)e^{\gamma(u_h)}\psi_{k,h}(u)\nabla u\nabla u_k + \int_\Omega e^{\gamma(u_h)}\nabla u\nabla\psi_{k,h}(u) + \int_\Omega a(x,u)|u|^{r-1}e^{\gamma(u_h)}\psi_{k,h}(u)$$
$$\leq \int_\Omega \beta(u)|\nabla u|^2 e^{\gamma(u_h)}\psi_{k,h}(u) + \int_\Omega f\, e^{\gamma(u_h)}\psi_{k,h}(u).$$

(3.12)
$$\int_\Omega \beta(u_h)e^{\gamma(u_h)}\psi_{k,h}(u)\nabla u\nabla u_k + \int_\Omega e^{\gamma(u_h)}\nabla u\nabla\psi_{k,h}(u) + \int_\Omega a(x)u|u|^{r-1}e^{\gamma(u_h)}\psi_{k,h}(u)$$
$$\leq \int_\Omega \beta(u)|\nabla u|^2 e^{\gamma(u_h)}\psi_{k,h}(u) + \int_\Omega f\, e^{\gamma(u_h)}\psi_{k,h}(u).$$

We note that

$$\int_\Omega \beta(u_h)e^{\gamma(u_h)}\psi_{k,h}(u)\nabla u\nabla u_k = \int_{[u<h]} \beta(u)e^{\gamma(u)}\varphi_{k,h}(u)|\nabla u|^2.$$

From Monotone Convergence Theorem, we have

$$\lim_{h\to+\infty}\int_\Omega \beta(u_h)e^{\gamma(u_h)}\nabla u\nabla u_k\psi_{k,h}(u) = \int_\Omega \beta(u)|\nabla u|^2 e^{\gamma(u)}\varphi_k(u).$$

Letting h tend to infinity in (3.12) and applying Lebesgue's dominated convergence theorem, we obtain

$$\int_\Omega \beta(u)e^{\gamma(u)}|\nabla u|^2\varphi_k(u) + \int_\Omega e^{\gamma(u)}\nabla u\nabla\varphi_k(u) + \int_\Omega a(x)u|u|^{r-1}e^{\gamma(u)}\varphi_k(u)$$
$$\leq \int_\Omega \beta(u)|\nabla u|^2 e^{\gamma(u)}\varphi_k(u) + \int_\Omega f\, e^{\gamma(u)}\varphi_k(u).$$

Then, we have

$$\int_\Omega e^{\gamma(u)}\nabla u\nabla\varphi_k(u) \leq \int_\Omega f\, e^{\gamma(u)}\varphi_k(u) + c,$$

which yields

$$\int_{[|u|<h]} \varphi_k'(u)|\nabla u|^2 \le c \int_\Omega f\,\varphi_k(u) \;\text{ for all } k>0.$$

Therefore from Fatou's lemma, we obtain

$$\int_\Omega \varphi_k'(u)|\nabla u|^2 \le c \int_\Omega f\,\varphi_k(u) \;\text{ for all } k>0,$$

and then we have

$$\int_\Omega |\beta(u)|\chi_{[|u|\ge k]}|\nabla u|^2 \le c.$$

Finally, this implies that

$$\int_{[|u|\ge k]} |b(u,\nabla u)| \le c. \tag{3.13}$$

$\square$

3.2. **Estimates on the approximating solutions.** This section is devoted to study the limiting process of the approximating problem. We consider the following sequence of problems which we denote by (P_n) :

$$-\Delta u_n = B_n(x,u_n,\nabla u_n) + f_n(x) \;\text{ in } \Omega, \tag{3.14}$$

under the assumption

$$u_n|_{\partial\Omega} = 0.$$

Where $B_n(x,u_n,\nabla u_n) = b_n(u_n,\nabla u_n) - a_n(x,u_n)|u_n|^{r-1}$, a_n and b_n are two sequences of functions defined by

$$a_n(x,s) = a(x)T_n(s), \;\; f_n = T_n(f) \text{ and } b_n(s,\xi) = T_n(\beta(s))|\xi|^2.$$

From standard result by Leray and Lions (see e.g. [19]) there exist weak solutions, for problem(3.14), which we denote by $u_n \in H_0^1(\Omega)\cap L^\infty(\Omega)$ satisfying for all $v \in H_0^1(\Omega)\cap L^\infty(\Omega)$

$$\int_\Omega \nabla u_n \nabla v = \int_\Omega B_n(x,u_n,\nabla u_n)v + \int_\Omega f_n\,v.$$

It yields that,

$$\int_\Omega \nabla u_n \nabla v + \int_\Omega a_n(x,u_n)|u_n|^{r-1}v = \int_\Omega b_n(u_n,\nabla u_n)v + \int_\Omega f_n\,v. \tag{3.15}$$

Hence the previous result of the precedent section can be applied. Using lemma (3.3) we deduce that there exist a constant c such that

$$\int_\Omega |\nabla G_k(u_n)|^2 dx \leq c \int_\Omega |f\, G_k\,(u)\,|.$$

Applying Hölder's inequality, we obtain

$$\int_\Omega |\nabla G_k(u_n)|^2 dx \leq c\, ||f\,||_{L^m(\Omega)} \left(\int_\Omega |G_k\,(u_n)\,|^{m'} \right)^{\frac{1}{m'}}.$$

Using Sobolev's imbedding theorem, we obtain for $\bar{N} = \frac{2N}{N-2}$ that

$$\left(\int_\Omega |e^{G_k(u_n)}|^{\bar{N}} \right)^{\frac{2}{\bar{N}}} \leq c\, ||f\,||_{L^m(\Omega)} \left(\int_\Omega |G_k\,(u_n)\,|^{m'} \right)^{\frac{1}{m'}}.$$

Denoting $A_K = \{|u| \geq k\}$, we get

$$\left(\int_\Omega |G_k(u_n)|^{\bar{N}} \right)^{\frac{2}{\bar{N}}} \leq c\, |A_K|^{\frac{1}{m'} - \frac{1}{\bar{N}}} \left(\int_\Omega |G_k\,(u_n)\,|^{\bar{N}} \right)^{\frac{1}{\bar{N}}}.$$

Thus

$$\left(\int_\Omega |G_k(u_n)|^{\bar{N}} \right)^{\frac{1}{\bar{N}}} \leq c\, |A_K|^{\frac{1}{m'} - \frac{1}{\bar{N}}}.$$

In this stage by using Stampacchia's L^∞-regularity procedure (see [24]) we obtain that u_n is bounded uniformly in $L^\infty(\Omega)$. That is

$$||u_n||_{L^\infty(\Omega)} \leq c,$$

where $c > 0$ is a constant that only depends on the parameters of the problem.

Using lemma 3.3 we obtain

$$\int_\Omega |\nabla u_n|^2 dx \leq c, \tag{3.16}$$

that is u_n is bounded in $H_0^1(\Omega)$.

Afterwards we consider $\lambda > \max\{|\beta(s)|; |s| \leq k\}$. Then

$$\begin{aligned} \int_\Omega |b_n(u_n, \nabla u_n)| &= \int_{\Omega\cap[|u_n|<k]} |b_n(u_n, \nabla u_n)| + \int_{\Omega\cap[|u_n|\geq k]} |b_n(u_n, \nabla u_n)| \\ &\leq \lambda \int_\Omega |\nabla T_k(u_n)|^2 + \int_{[|u_n|\geq k]} |b_n(u_n, \nabla u_n)|. \end{aligned}$$

From (3.16) and lemma (3.4) we obtain

$$\int_\Omega |b_n(u_n, \nabla u_n)|dx \leq c, \tag{3.17}$$

that is $b_n(u_n, \nabla u_n)$ equi-integrable.

Lemma 3.5. *Let u_n a sequence of functions satisfying (3.15). Then*

$$\lim_{n,m\to+\infty} \int_\Omega |\nabla u_n - \nabla u_m|dx = 0. \tag{3.18}$$

Proof. We consider $A_\epsilon^{m,n} = \{|u_n - u_m| \leq \epsilon\} \cap \Omega$.

We apply the weak formulation (3.15) successively to u_n and u_m and substitute v by the function defined by $\xi = \inf(u_n - u_m, \epsilon)$ if $u_n \geq u_m$ and $\xi = -\inf(u_m - u_n, \epsilon)$ if $u_n \leq u_m$.

After substraction, we obtain

$$\int_{A_\epsilon^{m,n}} |\nabla u_n - \nabla u_m|^2 dx \leq \epsilon(\int_\Omega |f_n + B_n(x, u_n, \nabla u_n)|dx + \int_\Omega |f_m + B_m(x, u_m, \nabla u_m)|dx).$$

The equi-integrability of f_n and $B_n(x, u_n, \nabla u_n)$ gives

$$\int_{A_\epsilon^{m,n}} |\nabla u_n - \nabla u_m|^2 dx \leq \epsilon\, c. \tag{3.19}$$

Let us now observe that by Hölder's inequality, we have

$$\int_\Omega |\nabla u_n - \nabla u_m|dx \leq c \left(\int_{A_\epsilon^{m,n}} |\nabla u_n - \nabla u_m|^2 dx\right)^{\frac{1}{2}} + \int_{\bar{A}_\epsilon^{m,n}} |\nabla u_n - \nabla u_m|dx; \tag{3.20}$$

where $\bar{A}_\epsilon^{m,n} = \{|u_n - u_m| \geq \epsilon\} \cap \Omega$.

Since $\lim\limits_{n,m\to+\infty} \int_{\bar{A}_\epsilon^{m,n}} |\nabla u_n - \nabla u_m|d = 0$ (the measure of $\bar{A}_\epsilon^{m,n}$ tends to 0 for n, m tending to $+\infty$), then from (3.19) and (3.20), we have

$$\lim_{n,m\to+\infty} \int_\Omega |\nabla u_n - \nabla u_m|dx = 0.$$

□

4. Existence and regularity results

Let u_n be the solution of the approximating problem. Then for all $\psi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, we have

$$\int_\Omega \nabla u_n \nabla \psi = \int_\Omega B_n(x, u_n, \nabla u_n)\psi + \int_\Omega f_n\, \psi.$$

$$\int_\Omega \nabla u_n \nabla \psi + \int_\Omega a_n(x, u_n)|u_n|^{r-1}\psi = \int_\Omega b_n(u_n, \nabla u_n)\psi + \int_\Omega f_n\, \psi.$$

From the construction of f_n we have

$$f_n \to f \quad \text{in } L^1(\Omega) \text{ for } n \text{ tending to } +\infty.$$

From lemma (3.2) the solution u_n is bounded independently on n in $W^{1,q}(\Omega)$, for any q, $1 \le q < q_0$. Then, up to a subsequence ,that we denote again by u_n, there exist $u \in W^{1,q}(\Omega)$, for any q, $1 \le q < q_0$, such that u_n converge to u weakly in $W^{1,q}(\Omega)$, for any q, $1 \le q < q_0$. From Rellich-Kondrachov's theorem we have the almost every where convergence in Ω. That is

$$u_n \to u \text{ weakly in } W^{1,q}(\Omega) \text{ for any } q, 1 \le q < q_0.$$

$$u_n \to u \text{ almost every where in } \Omega\,. \tag{4.1}$$

$$a_n(x, u_n) \to a(x, u) \text{ almost every where in } \Omega.$$

Taking into account the equi-integrability of u_n in $L^r(\Omega)$, it follows that of $a_n(x, u_n)|u_n|^{r-1}$ in $L^1(\Omega)$. Hence, we have

$$a_n(x, u_n)|u_n|^{r-1} \to a(x, u)|u|^{r-1} \text{ in } L^1(\Omega)\,. \tag{4.2}$$

From lemma 3.5 we have up to a subsequence u_n, that

$$\nabla u_n \to \nabla u \text{ almost every where in } \Omega\,. \tag{4.3}$$

Since ∇u_n is bounded in $L^q(\Omega)$ for any $q, 1 \le q < N'$, we have

$$\nabla u_n \to \nabla u \text{ in } L^q(\Omega) \text{ for any } q, 1 \le q < N',$$

and then we conclude that

$$\Delta u_n \to \Delta u \text{ in } L^1(\Omega)\,.$$

Now, we have from (4.1) and (4.3) that

$$\beta_n(u_n)|\nabla u_n|^2 \to \beta(u)|\nabla u|^2 \text{ almost every where in } \Omega.$$

$$b_n(u_n, \nabla u_n) \to b(u, \nabla u) \text{ almost every where in } \Omega.$$

$$B(x, u_n, \nabla u_n) \to B(x, u, \nabla u) \text{ almost every where in } \Omega.$$

From (3.17) we obtain that

$$b_n(u_n, \nabla u_n) \to \beta(u)|\nabla u|^2 \text{ in } L^1(\Omega),$$

and from (4.2), that

$$B(x, u_n, \nabla u_n) \to B(x, u, \nabla u) \text{ in } L^1(\Omega).$$

Which conclude to the desired convergence result.

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Abderrahmane El Hachimi UFR Mathématiques Appliquées et Industrielles, Faculté des sciences, b. P. 20, El Jadida, Maroc
E-mail address: aelhachi@yahoo.fr

Jaouad Igbida UFR Mathématiques Appliquées et Industrielles, Faculté des sciences, B. P. 20, El Jadida, Maroc
E-mail address: jigbida@yahoo.fr