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On the solvability of anti-periodic boundary value problems with impulse

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Abstract

In this paper, we are concerned with the existence of solutions for second order impulsive anti-periodic boundary value problem

$$\begin{cases} u''(t) + f(t, u(t), u'(t)) = 0, \quad t \neq t_k,\ t \in [0, T], \\ \triangle u(t_k) = I_k(u(t_k)), \quad k = 1, \cdots, m, \\ \triangle u'(t_k) = I_k^*(u(t_k)), \quad k = 1, \cdots, m, \\ u(0) + u(T) = 0,\ u'(0) + u'(T) = 0, \end{cases}$$

new criteria are established based on Schaefer's fixed-point theorem.

Keywords: Anti-periodic boundary value problem; Impulsive; Schaefer's fixed-point theorem

1. Introduction

In recent years, the solvability of the anti-periodic boundary value problems of first-order and second-order differential equations were studied by many authors, for example, we refer to [1-5] and the references therein. It should be noted that anti-periodic boundary value problems appear in physics in a variety of situations [6,7].

Impulsive differential equations, which arise in biology, physics, population dynamics, economics, etc., are a basic tool to study evolution processes that are subjected to abrupt in their states (see [8-12]). Recently, the existence results were extended to anti-periodic boundary value problems for first-order impulsive differential equations [13,14]. Very recently, Wang and Shen [15] investigated the anti-periodic boundary value problem for a class of second order differential equations by using Schauder's fixed point theorem and the lower and upper solutions method.

Inspired by [13-15], in this paper, we investigate the anti-periodic boundary value problem for second order impulsive nonlinear differential equations of the form

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$$\begin{cases} u''(t) + f(t, u(t), u'(t)) = 0, \quad t \in J_0 = J \setminus \{t_1, \cdots, t_m\}, \\ \triangle u(t_k) = I_k(u(t_k)), \quad k = 1, \cdots, m, \\ \triangle u'(t_k) = I_k^*(u(t_k)), \quad k = 1, \cdots, m, \\ u(0) + u(T) = 0, \; u'(0) + u'(T) = 0, \end{cases} \tag{1.1}$$

where $J = [0, T]$, $0 < t_1 < t_2 < \cdots < t_m < T$, $f : [0, T] \times R^2 \to R$ is continuous on $(t, x, y) \in J_0 \times R^2$, $f(t_k^+, x, y) := \lim_{t \to t_k^+} f(t, x, y)$, $f(t_k^-, x, y) := \lim_{t \to t_k^-} f(t, x, y)$ exist, $f(t_k^-, x, y) = f(t_k, x, y)$; $\triangle u(t_k) = u(t_k^+) - u(t_k^-)$, $\triangle u'(t_k) = u'(t_k^+) - u'(t_k^-)$; $I_k, I_k^* \in C(R, R)$.

To the best of the authors knowledge, no one has studied the existence of solutions for impulsive anti-periodic boundary value problem (1.1). The aim of this paper is to fill the gap in the relevant literatures.

The following Schaefer's fixed-point theorem is fundamental in the proof of our main results.

Lemma 1.1.[16] *(Schaefer) Let E be a normed linear space with $H : E \to E$ a compact operator. If the set*

$$S := \{x \in E | x = \lambda H x, \; for \; some \; \lambda \in (0, 1)\}$$

is bounded, then H has at least one fixed-point.

The paper is formulated as follows. In section 2, some definitions and lemmas are given. In section 3, we obtain a new existence theorem by using Schaefer's fixed point theorem, and uniqueness result by using Banach's fixed point theorem. In Section 4, an illustrative example is given to demonstrate the effectiveness of the obtained results.

2. Preliminaries

In order to define the concept of solution for (1.1), we introduce the following spaces of functions:

$PC(J) = \{u : J \to R : u$ is continuous for any $t \in J_0$, $u(t_k^+)$, $u(t_k^-)$ exist, and $u(t_k^-) = u(t_k), k = 1, \cdots, m\}$,

$PC^1(J) = \{u : J \to R : u$ is continuously differentiable for any $t \in J_0$, $u'(t_k^+)$, $u'(t_k^-)$ exist, and $u'(t_k^-) = u'(t_k), k = 1, \cdots, m\}$.

$PC(J)$ and $PC^1(J)$ are Banach space with the norms :

$\|u\|_{PC} = \sup_{t \in J} |u(t)|$,

and

$$\|u\|_{PC^1} = \max\{\|u\|_{PC}, \|u'\|_{PC}\}.$$

A solution to the impulsive BVP (1.1) is a function $u \in PC^1(J) \cap C^2(J_0)$ that satisfies (1.1) for each $t \in J$.

Consider the following impulsive BVP with $p \geq 0, q > 0$

$$\begin{cases} -u''(t) + pu'(t) + qu(t) = \sigma(t), & t \in J_0, \\ \triangle u(t_k) = I_k(u(t_k)), & k = 1, \cdots, m, \\ \triangle u'(t_k) = I_k^*(u(t_k)), & k = 1, \cdots, m, \\ u(0) + u(T) = 0, \ u'(0) + u'(T) = 0, \end{cases} \tag{2.1}$$

where $\sigma \in PC(J)$.

For convenience, we set $I_k = I_k(u(t_k)), I_k^* = I_k^*(u(t_k))$,

$$r_1 := \frac{p + \sqrt{p^2 + 4q}}{2} > 0, \qquad r_2 := \frac{p - \sqrt{p^2 + 4q}}{2} < 0. \tag{2.2}$$

Lemma 2.1. *$u \in PC^1(J) \cap C^2(J_0)$ is a solution of (2.1) if and only if $u \in PC^1(J)$ is a solution of the impulsive integral equation*

$$u(t) = \int_0^T G(t,s)\sigma(s)ds + \sum_{k=1}^m [G(t,t_k)(-I_k^*) + W(t,t_k)I_k], \tag{2.3}$$

where

$$G(t,s) = \frac{1}{r_1 - r_2} \begin{cases} \frac{e^{r_2(t-s)}}{1+e^{r_2 T}} - \frac{e^{r_1(t-s)}}{1+e^{r_1 T}}, & 0 \leq s < t \leq T, \\ \frac{e^{r_1(T+t-s)}}{1+e^{r_1 T}} - \frac{e^{r_2(T+t-s)}}{1+e^{r_2 T}}, & 0 \leq t \leq s \leq T, \end{cases} \tag{2.4}$$

and

$$W(t,s) = \frac{1}{r_1 - r_2} \begin{cases} \frac{r_1 e^{r_2(t-s)}}{1+e^{r_2 T}} - \frac{r_2 e^{r_1(t-s)}}{1+e^{r_1 T}}, & 0 \leq s < t \leq T, \\ \frac{r_2 e^{r_1(T+t-s)}}{1+e^{r_1 T}} - \frac{r_1 e^{r_2(T+t-s)}}{1+e^{r_2 T}}, & 0 \leq t \leq s \leq T. \end{cases} \tag{2.5}$$

Proof. If $u \in PC^1(J) \bigcap C^2(J_0)$ is a solution of (2.1), setting

$$v(t) = u'(t) - r_2 u(t), \tag{2.6}$$

then by the first equation of (2.1) we have

$$v'(t) - r_1 v(t) = -\sigma(t), \quad t \neq t_k. \tag{2.7}$$

Multiplying (2.7) by $e^{-r_1 t}$ and integrating on $[0,t]$ and $(t_1, t]$, respectively, we get

$$e^{-r_1 t} v(t) - v(0) = -\int_0^t \sigma(s) e^{-r_1 s} ds, \quad 0 \leq t \leq t_1,$$

$$e^{-r_1 t} v(t) - e^{-r_1 t_1} v(t_1^+) = -\int_{t_1}^t \sigma(s) e^{-r_1 s} ds, \quad t_1 < t \leq t_2.$$

So

$$v(t) = e^{r_1 t}[v(0) - \int_0^t e^{-r_1 s}\sigma(s)ds + e^{-r_1 t_1}(I_1^* - r_2 I_1), \quad t_1 < t \le t_2.$$

In the same way, we can obtain that

$$v(t) = e^{r_1 t}\left[v(0) - \int_0^t e^{-r_1 s}\sigma(s)ds + \sum_{0<t_k<t} e^{-r_1 t_k}(I_k^* - r_2 I_k)\right], \quad t \in J, \tag{2.8}$$

where $v(0) = u'(0) - r_2 u(0)$. Multiplying (2.6) by $e^{-r_2 t}$ and integrating on $[0, t_k]$ and $(t_k, t]$ $(t_k < t \le t_{k+1})$, respectively, similar to the proof of (2.8), we have

$$u(t) = e^{r_2 t}\left[u(0) + \int_0^t v(s)e^{-r_2 s}ds + \sum_{0<t_k<t} e^{-r_2 t_k} I_k\right], \quad t \in J. \tag{2.9}$$

By some calculation, we get

$$\begin{aligned}\int_0^t v(s)e^{-r_2 s}ds &= \frac{1}{r_1 - r_2}\Bigg[v(0)(e^{(r_1-r_2)t} - 1) - \int_0^t (e^{(r_1-r_2)t} - e^{(r_1-r_2)s})\sigma(s)e^{-r_1 s}ds \\ &\quad + \sum_{0<t_k<t}\left(e^{(r_1-r_2)t} - e^{(r_1-r_2)t_k}\right)e^{-r_1 t_k}(I_k^* - r_2 I_k)\Bigg].\end{aligned} \tag{2.10}$$

Substituting (2.10) into (2.9), we obtain

$$\begin{aligned}u(t) &= \frac{1}{r_1 - r_2}\Big[(r_1 u(0) - u'(0))e^{r_2 t} + (u'(0) - r_2 u(0))e^{r_1 t} \\ &\quad + \int_0^t (e^{r_2(t-s)} - e^{r_1(t-s)})\sigma(s)ds + \sum_{0<t_k<t} e^{r_1(t-t_k)}(I_k^* - r_2 I_k) \\ &\quad - \sum_{0<t_k<t} e^{r_2(t-t_k)}(I_k^* - r_1 I_k)\Bigg], \qquad t \in [0, T],\end{aligned} \tag{2.11}$$

$$\begin{aligned}u'(t) &= \frac{1}{r_1 - r_2}\Big[r_2(r_1 u(0) - u'(0))e^{r_2 t} + r_1(u'(0) - r_2 u(0))e^{r_1 t} \\ &\quad + \int_0^t (r_2 e^{r_2(t-s)} - r_1 e^{r_1(t-s)})\sigma(s)ds + \sum_{0<t_k<t} r_1 e^{r_1(t-t_k)}(I_k^* - r_2 I_k) \\ &\quad - \sum_{0<t_k<t} r_2 e^{r_2(t-t_k)}(I_k^* - r_1 I_k)\Bigg], \qquad t \in [0, T].\end{aligned} \tag{2.12}$$

In view of $u(0)+u(T)=0$, $u'(0)+u'(T)=0$, we have

$$u'(0)-r_2u(0)=\frac{1}{1+e^{r_1T}}\left[\int_0^T e^{r_1(T-s)}\sigma(s)ds-\sum_{0<t_k<T}e^{r_1(T-t_k)}(I_k^*-r_2I_k)\right], \tag{2.13}$$

$$r_1u(0)-u'(0)=\frac{1}{1+e^{r_2T}}\left[-\int_0^T e^{r_2(T-s)}\sigma(s)ds+\sum_{0<t_k<T}e^{r_2(T-t_k)}(I_k^*-r_1I_k)\right]. \tag{2.14}$$

Substituting (2.13) and (2.14) into (2.11), by routine calculation, we can get (2.3).

Conversely, if u is a solution of (2.3), then direct differentiation of (2.3) gives $-u''(t)=\sigma(t)-pu'(t)-qu(t)$, $t\neq t_k$. Moreover, we obtain $\triangle u|_{t=t_k}=I_k(u(t_k))$, $\triangle u'|_{t=t_k}=I_k^*(u(t_k))$, $u(0)+u(T)=0$ and $u'(0)+u'(T)=0$. Hence, $u\in PC^1(J)\cap C^2(J_0)$ is a solution of (2.1) □

Define a mapping $A: PC^1(J)\to PC^1(J)$ by

$$Au(t)=\int_0^T G(t,s)[f(s,u(s),u'(s))+pu'(s)+qu(s)]ds$$
$$+\sum_{k=1}^m[G(t,t_k)(-I_k^*)+W(t,t_k)I_k],\quad t\in[0,T]. \tag{2.15}$$

In view of Lemma 2.1, we easily know that u is a fixed point of operator A if and only if u is a solution to the impulsive boundary value problem (1.1).

Lemma 2.2 *If $u\in PC^1(J)$ and $u(0)+u(T)=0$, then*

$$\|u\|_{PC}\leq\frac{1}{2}\left(\int_0^T|u'(s)|ds+\sum_{k=1}^m|\triangle u(t_k)|\right).$$

Proof. Since $u\in PC^1(J)$, we have

$$u(t)=u(0)+\sum_{0<t_k<t}\triangle u(t_k)+\int_0^t u'(s)ds. \tag{2.16}$$

Set $t=T$, we obtain from $u(0)+u(T)=0$ that

$$u(0)=-\frac{1}{2}\left(\sum_{k=1}^m\triangle u(t_k)+\int_0^T u'(s)ds\right). \tag{2.17}$$

Substituting (2.17) into (2.16), we get

$$|u(t)|=\left|\frac{1}{2}\left(\int_0^t u'(s)ds-\int_t^T u'(s)ds\right)+\frac{1}{2}\left(\sum_{0<t_k<t}\triangle u(t_k)-\sum_{t\leq t_k}\triangle u(t_k)\right)\right|$$
$$\leq\frac{1}{2}\left(\int_0^t|u'(s)|ds+\int_t^T|u'(s)|ds\right)+\frac{1}{2}\left(\sum_{0<t_k<t}|\triangle u(t_k)|+\sum_{t\leq t_k}|\triangle u(t_k)|\right)$$

$$= \frac{1}{2}\left(\int_0^T |u'(s)|ds + \sum_{k=1}^{m} |\triangle u(t_k)|\right).$$

The proof is complete. $\square$

It is easy to check that

$$|G(t,s)| \le \frac{1}{r_1 - r_2}\left(\frac{1}{1+e^{r_2 T}} - \frac{1}{1+e^{r_1 T}}\right) := G_1. \tag{2.18}$$

Since

$$\frac{\partial}{\partial t}G(t,s) = \frac{1}{r_1 - r_2}\begin{cases} \frac{-r_1 e^{r_1(t-s)}}{1+e^{r_1 T}} + \frac{r_2 e^{r_2(t-s)}}{1+e^{r_2 T}}, & 0 \le s < t \le T \\ \frac{r_1 e^{r_1(T+t-s)}}{1+e^{r_1 T}} + \frac{-r_2 e^{r_2(T+t-s)}}{1+e^{r_2 T}}, & 0 \le t \le s \le T, \end{cases}$$

and

$$\frac{\partial}{\partial t}W(t,s) = \frac{1}{r_1 - r_2}\begin{cases} \frac{r_1 r_2 e^{r_2(t-s)}}{1+e^{r_2 T}} - \frac{r_2 r_1 e^{r_1(t-s)}}{1+e^{r_1 T}}, & 0 \le s < t \le T, \\ \frac{r_1 r_2 e^{r_1(T+t-s)}}{1+e^{r_1 T}} - \frac{r_1 r_2 e^{r_2(T+t-s)}}{1+e^{r_2 T}}, & 0 \le t \le s \le T, \end{cases}$$

we obtain by $r_1 \ge -r_2 > 0$ that

$$\left|\frac{\partial}{\partial t}G(t,s)\right| \le \frac{r_1}{r_1 - r_2}\begin{cases} \frac{e^{r_1(t-s)}}{1+e^{r_1 T}} + \frac{e^{r_2(t-s)}}{1+e^{r_2 T}}, & 0 \le s < t \le T \\ \frac{e^{r_1(T+t-s)}}{1+e^{r_1 T}} + \frac{e^{r_2(T+t-s)}}{1+e^{r_2 T}}, & 0 \le t \le s \le T, \end{cases}$$

$$|W(t,s)| \le \frac{r_1}{r_1 - r_2}\begin{cases} \frac{e^{r_1(t-s)}}{1+e^{r_1 T}} + \frac{e^{r_2(t-s)}}{1+e^{r_2 T}}, & 0 \le s < t \le T \\ \frac{e^{r_1(T+t-s)}}{1+e^{r_1 T}} + \frac{e^{r_2(T+t-s)}}{1+e^{r_2 T}}, & 0 \le t \le s \le T, \end{cases}$$

and

$$\left|\frac{\partial}{\partial t}W(t,s)\right| \le \frac{-r_1 r_2}{r_1 - r_2}\begin{cases} \frac{e^{r_2(t-s)}}{1+e^{r_2 T}} + \frac{e^{r_1(t-s)}}{1+e^{r_1 T}}, & 0 \le s < t \le T, \\ \frac{e^{r_1(T+t-s)}}{1+e^{r_1 T}} + \frac{e^{r_2(T+t-s)}}{1+e^{r_2 T}}, & 0 \le t \le s \le T. \end{cases}$$

Let $h(x) = Be^{r_1 x} + Ce^{r_2 x}$, for $x \in [0,T]$, where B, C are two nonnegative constants. Obviously, $h''(x) \ge 0$, that is, h is a convex function on $[0,T]$. Thus,

$$h(x) \le \max\{h(0), h(T)\} = \max\{B + C, Be^{r_1 T} + Ce^{r_2 T}\}. \tag{2.19}$$

By (2.19), we easily obtain that

$$\left|\frac{\partial}{\partial t}G(t,s)\right| \le \frac{r_1}{r_1 - r_2}\left(\frac{e^{r_1 T}}{1+e^{r_1 T}} + \frac{e^{r_2 T}}{1+e^{r_2 T}}\right) := G_2, \quad |W(t,s)| \le G_2, \tag{2.20}$$

and

$$\left|\frac{\partial}{\partial t}W(t,s)\right| \le \frac{-r_1 r_2}{r_1 - r_2}\left(\frac{e^{r_1 T}}{1+e^{r_1 T}} + \frac{e^{r_2 T}}{1+e^{r_2 T}}\right) := G_3. \tag{2.21}$$

3.Main results

Throughout this section, we assume that

(H$_1$) $3\int_0^T c(t)dt + 6pT < 2$;

(H$_2$) There exist constant $0 < \eta < 1$ and functions $a, b, c, h \in C(J, [0, +\infty))$ such that

$$|f(t, u, v)| \leq a(t)|u| + b(t)|u|^{\eta} + c(t)|v| + h(t);$$

(H$_3$) There exist nonnegative constants $\alpha_k, \beta_k, \gamma_k, \delta_k$ $(k = 1, 2, \cdots, m)$ such that

$$|I_k(u)| \leq \alpha_k|u| + \beta_k, \quad |I_k^*(u)| \leq \gamma_k|u| + \delta_k, \quad k = 1, \cdots m.$$

For convenience, let

$$\begin{aligned} a_1 &= \frac{3\left(\int_0^T a(t)dt + 2qT + \sum_{i=1}^m \gamma_i\right)}{2 - 3\int_0^T c(t)dt - 6pT}, \quad a_2 = \frac{3\int_0^T b(t)dt}{2 - 3\int_0^T c(t)dt - 6pT}, \\ a_3 &= \frac{3\left(\int_0^T h(t)dt + \sum_{i=1}^m \delta_i\right)}{2 - 3\int_0^T c(t)dt - 6pT}. \end{aligned} \tag{3.1}$$

Theorem 3.1. *Suppose that* (H$_1$) $-$ (H$_3$) *hold. Further assume that*

$$\frac{b_1 T}{2(2 - c^*)} + \sqrt{\frac{b_1 T}{2(2 - c^*)}} \sum_{i=1}^m \alpha_i + \frac{m}{4}\sum_{i=1}^m \alpha_i^2 < 1, \tag{3.2}$$

where

$$b_1 = \int_0^T a(t)dt + \frac{1}{2}\int_0^T c(t)dt + \sum_{i=1}^m [(p + a_1)\alpha_i + \gamma_i],$$

a_1 *as in* (3.1) *and* $c^* = \max_{t\in J} c(t) < 2$. *Then BVP* (1.1) *has at least one solution.*

Proof. It is easy to check by Arzela-Ascoli theorem that the operator A is completely continuous. Assume that u is a solution of the equation

$$u = \lambda A u, \quad \lambda \in (0, 1).$$

Then,

$$\begin{aligned} u''(t) &= \lambda(Au)''(t) = \lambda[-f(t, u(t), u'(t)) - pu'(t) - qu(t) + p(Au)'(t) + q(Au)(t)] \\ &= -\lambda f(t, u(t), u'(t)) - p(\lambda - 1)u'(t) - q(\lambda - 1)u(t), \end{aligned} \tag{3.3}$$

$$\begin{aligned} -u(t)u''(t) &= \lambda u(t) f(t, u(t), u'(t)) + p(\lambda - 1)u(t)u'(t) + q(\lambda - 1)u^2(t) \\ &\leq \lambda u(t) f(t, u(t), u'(t)) + p(\lambda - 1)u(t)u'(t). \end{aligned} \tag{3.4}$$

Integrating (3.3) from 0 to T, we get that

$$u'(T) - u'(0) = \int_0^T u''(t)dt + \sum_{i=1}^m I_i^*$$

$$= -\lambda \int_0^T f(t, u(t), u'(t))dt - p(\lambda - 1) \int_0^T u'(t)dt - q(\lambda - 1) \int_0^T u(t)dt + \sum_{i=1}^m I_i^*. \tag{3.5}$$

In view of $u'(0) + u'(T) = 0$, we obtain by (3.5) that

$$|u'(0)| \le \frac{1}{2} \int_0^T |f(t, u(t), u'(t))|dt + p \int_0^T |u'(t)|dt + q \int_0^T |u(t)|dt + \frac{1}{2} \sum_{i=1}^m |I_i^*|. \tag{3.6}$$

Integrating (3.3) from 0 to t, we obtain that

$$u'(t) - u'(0) = \int_0^t u''(s)ds + \sum_{0<t_i<t} I_i^*$$

$$= -\lambda \int_0^t f(s, u(s), u'(s))ds - p(\lambda - 1) \int_0^t u'(s)ds - q(\lambda - 1) \int_0^t u(s)ds + \sum_{0<t_i<t} I_i^*. \tag{3.7}$$

From (3.6), (3.7) and assumptions (H_2), (H_3), we have

$$|u'(t)| \le |u'(0)| + \int_0^T |f(s, u(s), u'(s))|ds + 2p \int_0^T |u'(s)|ds + 2q \int_0^T |u(s)|ds + \sum_{i=1}^m |I_i^*|$$

$$\le \frac{3}{2} \int_0^T (a(t)|u(t)| + b(t)|u(t)|^\eta + c(t)|u'(t)| + h(t))dt + 3p \int_0^T |u'(t)|dt$$

$$+3q \int_0^T |u(t)|dt + \frac{3}{2} \sum_{i=1}^m (\gamma_i \|u\|_{PC} + \delta_i)$$

$$\le \frac{3}{2} \left(\|u\|_{PC} \int_0^T a(t)dt + \|u\|_{PC}^\eta \int_0^T b(t)dt + \|u'\|_{PC} \int_0^T c(t)dt + \int_0^T h(t)dt \right)$$

$$+3pT\|u'\|_{PC} + 3qT\|u\|_{PC} + \frac{3}{2} \sum_{i=1}^m (\gamma_i \|u\|_{PC} + \delta_i),$$

that is,

$$\|u'\|_{PC} \le \left(\frac{3}{2} \int_0^T a(t)dt + 3qT + \frac{3}{2} \sum_{i=1}^m \gamma_i \right) \|u\|_{PC}$$

$$+\|u\|_{PC}^\eta \int_0^T b(t)dt + \left(\frac{3}{2} \int_0^T c(t)dt + 3pT \right) \|u'\|_{PC} + \frac{3}{2} \int_0^T h(t)dt + \frac{3}{2} \sum_{i=1}^m \delta_i.$$

Thus, in view of assumption (H_1), we have

$$\|u'\|_{PC} \le a_1 \|u\|_{PC} + a_2 \|u\|_{PC}^\eta + a_3, \tag{3.8}$$

where a_1, a_2, a_3 are as in (3.1). Integrating (3.4) from 0 to T, we get that

$$-\int_0^T u(t)u''(t)dt \le \lambda \int_0^T u(t)f(t,u(t),u'(t))dt + p(\lambda-1)\int_0^T u(t)u'(t)dt. \tag{3.9}$$

In view of $u(0)+u(T)=0$, $u'(0)+u'(T)=0$, we have

$$\begin{aligned}
\int_0^T u(t)u'(t)dt &= \frac{1}{2}\int_0^T d(u^2(t)) \\
&= \frac{1}{2}\left[\int_0^{t_1} d((u'(t))^2) + \int_{t_1}^{t_2} d(u^2(t)) + \cdots + \int_{t_m}^T d(u^2(t))\right] \\
&= \frac{1}{2}\left[(u^2(t_1-0)-u^2(0)) + (u^2(t_2-0)-u^2(t_1+0)) + \cdots + (u^2(T)-u^2(t_m+0))\right] \\
&= \frac{1}{2}\left[(u^2(t_1-0)-u^2(t_1+0)) + (u^2(t_2-0)-u^2(t_2+0)) + \cdots + (u^2(t_m-0)-u^2(t_m+0))\right] \\
&= \frac{1}{2}\left[-(u(t_1-0)+u(t_1+0))I_1 - (u(t_2-0)+u(t_2+0))I_2 - \cdots - (u(t_m-0)+u(t_m+0))I_m\right],
\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}
\int_0^T u(t)u''(t)dt &= \int_0^T u(t)d(u'(t)) \\
&= \int_0^{t_1} u(t)d(u'(t)) + \int_{t_1}^{t_2} u(t)d(u'(t)) + \cdots + \int_{t_m}^T u(t)d(u'(t)) \\
&= u(t)u'(t)|_0^{t_1} - \int_0^{t_1}(u'(t))^2dt + u(t)u'(t)|_{t_1}^{t_2} - \int_{t_1}^{t_2}(u'(t))^2dt + \cdots + u(t)u'(t)|_{t_m}^T - \int_{t_m}^T (u'(t))^2dt \\
&= u(t_1-0)u'(t_1-0) - u(0)u'(0) + u(t_2-0)u'(t_2-0) - u(t_1+0)u'(t_1+0) \\
&\quad + \cdots + u(T)u'(T) - u(t_n+0)u'(t_n+0) - \int_0^T (u'(t))^2dt \\
&= u(t_1-0)u'(t_1-0) - u(t_1+0)u'(t_1+0) + \cdots \\
&\quad + u(t_n-0)u'(t_m-0) - u(t_m+0)u'(t_m+0) - \int_0^T (u'(t))^2dt \\
&= u(t_1-0)u'(t_1-0) - u(t_1-0)u'(t_1+0) + u(t_1-0)u'(t_1+0) - u(t_1+0)u'(t_1+0) \\
&\quad + \cdots + u(t_m-0)u'(t_m-0) - u(t_m-0)u'(t_m+0) \\
&\quad + u(t_m-0)u'(t_m+0) - u(t_m+0)u'(t_m+0) - \int_0^T (u'(t))^2dt
\end{aligned}$$

$$= -u(t_1-0)I_1^* - u'(t_1+0)I_1 - \cdots - u(t_m-0)I_n^* - u'(t_m+0)I_m - \int_0^T (u'(t))^2 dt. \quad (3.11)$$

Substituting (3.10) and (3.11) into (3.9), we obtain by $(\mathrm{H}_2), (\mathrm{H}_3)$ and (3.8) that

$$\begin{aligned}
\int_0^T (u'(t))^2 dt &\le \lambda \int_0^T u(t)f(t,u(t),u'(t))dt - u(t_1-0)I_1^* - u'(t_1+0)I_1 - \cdots - u(t_m-0)I_m^* \\
&\quad - u'(t_m+0)I_m + \frac{p(1-\lambda)}{2}\left[(u(t_1-0)+u(t_1+0))I_1 + \cdots + (u(t_m-0)+u(t_m+0))I_m\right] \\
&\le \lambda \int_0^T u(t)f(t,u(t),u'(t))dt + \|u\|_{PC}\sum_{i=1}^m |I_i^*| + \|u'\|_{PC}\sum_{i=1}^m |I_i| + p(1-\lambda)\|u\|_{PC}\sum_{i=1}^m |I_i| \\
&\le \lambda \int_0^T u(t)f(t,u(t),u'(t))dt + \|u\|_{PC}\sum_{i=1}^m (p|I_i| + |I_i^*|) + \|u'\|_{PC}\sum_{i=1}^m |I_i| \\
&\le \int_0^T (a(t)u^2(t) + b(t)|u(t)|^{1+\eta} + \frac{1}{2}c(t)(u^2(t) + (u'(t))^2) + h(t))dt \\
&\quad + \sum_{i=1}^m [p(\alpha_i\|u\|_{PC} + \beta_i) + \gamma_i\|u\|_{PC} + \delta_i]\|u\|_{PC} + \sum_{i=1}^m (\alpha_i\|u\|_{PC} + \beta_i)\|u'\|_{PC} \\
&\le \|u\|_{PC}^2 \int_0^T a(t)dt + \|u\|_{PC}^{1+\eta}\int_0^T b(t)dt + \frac{1}{2}\|u\|_{PC}^2\int_0^T c(t)dt + \frac{c^*}{2}\int_0^T (u'(t))^2 dt \\
&\quad + \int_0^T h(t)dt + \sum_{i=1}^m (p\alpha_i + \gamma_i)\|u\|_{PC}^2 + \sum_{i=1}^m ((p\beta_i + \delta_i)\|u\|_{PC} \\
&\quad + a_1\sum_{i=1}^m \alpha_i\|u\|_{PC}^2 + a_2\sum_{i=1}^m \alpha_i\|u\|_{PC}^{1+\eta} + a_2\sum_{i=1}^m \beta_i\|u\|_{PC}^{\eta} + \sum_{i=1}^m (a_1\beta_i + a_3\alpha_i)\|u\|_{PC} + a_3\sum_{i=1}^m \beta_i.
\end{aligned}$$

Thus,

$$\int_0^T (u'(t))^2 dt \le \frac{1}{1-c^*/2}\Big[b_1\|u\|_{PC}^2 + b_2\|u\|_{PC}^{1+\eta} + b_3\|u\|_{PC} + b_4\|u\|_{PC}^{\eta} + b_5\Big], \quad (3.12)$$

where

$$\begin{aligned}
b_1 &= \int_0^T a(t)dt + \frac{1}{2}\int_0^T c(t)dt + \sum_{i=1}^m ((p+a_1)\alpha_i + \gamma_i), \quad b_2 = \int_0^T b(t)dt + a_2\sum_{i=1}^m \alpha_i, \\
b_3 &= \sum_{i=1}^m ((p+a_1)\beta_i + a_3\alpha_i + \delta_i), \quad b_4 = a_2\sum_{i=1}^m \beta_i, \quad b_5 = \int_0^T b(t)dt + a_3\sum_{i=1}^m \beta_i.
\end{aligned} \quad (3.13)$$

By Lemma 2.2 and (3.12), we have

$$\|u\|_{PC}^2 \le \frac{1}{4}\left(\int_0^T |u'(t)|dt\right)^2 + \frac{1}{2}\int_0^T |u'(t)|dt\sum_{i=1}^m |I_i| + \frac{1}{4}\left(\sum_{i=1}^m |I_i|\right)^2$$

$$\leq \frac{T}{4}\int_0^T (u'(t))^2dt + \frac{\sqrt{T}}{2}\left(\int_0^T (u'(t))^2dt\right)^{1/2}\sum_{i=1}^m |I_i| + \frac{1}{4}m\sum_{i=1}^m |I_i|^2$$

$$\leq \frac{T}{2(2-c^*)}\left[b_1\|u\|_{PC}^2 + b_2\|u\|_{PC}^{1+\eta} + b_3\|u\|_{PC} + b_4\|u\|_{PC}^{\eta} + b_5\right]$$

$$+\sqrt{\frac{T}{2(2-c^*)}}\left[b_1\|u\|_{PC}^2 + b_2\|u\|_{PC}^{1+\eta} + b_3\|u\|_{PC} + b_4\|u\|_{PC}^{\eta} + b_5\right]^{1/2}\sum_{i=1}^m(\alpha_i\|u\|_{PC}+\beta_i)$$

$$+\frac{m}{4}\sum_{i=1}^m(\alpha_i^2\|u\|_{PC}^2 + 2\alpha_i\beta_i\|u\|_{PC} + \beta_i^2).$$

It follows from the above inequality and condition (3.2) that there exists $M_1 > 0$ such that $\|u\|_{PC} \leq M_1$. Thus, we get by (3.8) that

$$\|u'\|_{PC} \leq a_1M_1 + a_2M_1^{\eta} + a_3 := M_2. \tag{3.14}$$

Thus, $\|u\|_{PC^1} \leq \max\{M_1, M_2\}$. It follows from Lemma 1.1 that BVP (1.1) has at least one solution. The proof is complete. □

Corollary 3.2. *Assume that* $(\mathrm{H}_1), (\mathrm{H}_2)$ *hold. Suppose that there exist nonnegative constants* β_k, δ_k $(k = 1, 2, \cdots, m)$ *such that*

(H_4) $|I_k(u)| \leq \beta_k, \quad |I_k^*(u)| \leq \delta_k, \quad k = 1, \cdots m,$

holds. Further assume that

$$\int_0^T a(t)dt + \frac{1}{2}\int_0^T c(t)dt < 2(2-c^*), \tag{3.15}$$

where $c^* = \max_{t\in J} c(t) < 2$. *Then BVP* (1.1) *has at least one solution.*

Proof. Set $\alpha_k = \gamma_k = 0$, $k = 1, 2, \cdots, m$. Then (H_3) reduces to (H_4), and (3.1) reduces to (3.15). So, by Theorem 3.1, we know that Corollary 3.2 holds.

Theorem 3.3. *Suppose that there exist constants* K_1, K_2, *and* L_k, l_k $(k = 1, 2, \cdots, m)$ *such that*

$$|f(t,u,v) - f(t,x,y)| \leq K_1|u-x| + K_2|v-y|, \quad \forall u, v, x, y \in \mathbf{R},$$

and

$$|I_k(u) - I_k(v)| \leq L_k|u-v|, \quad |I_k^*(u) - I_k^*(v)| \leq l_k|u-v|, \quad \forall u, v \in \mathbf{R}.$$

Moreover suppose that

$$(K_1+K_2+p+q)T\max\{G_1,G_2\}+\sum_{k=1}^{m}(\max\{G_1,G_2\}L_k+\max\{G_2,G_3\}l_k)<1, \quad (3.16)$$

where G_1, G_2 and G_3 are as in (2.18), (2.20) and (2.21), respectively, then BVP (1.1) has a unique solution.

Proof. From (2.15), we have

$$|Au(t)-Av(t)|=\left|\int_0^T G(t,s)[(f(s,u(s),u'(s))-f(s,v(s),v'(s))+p(u'(s)-v'(s))\right.$$

$$\left.+q(u(s)-v(s))]ds+\sum_{k=1}^{m}[G(t,t_k)(I_k^*(v(t_k))-I_k^*(u(t_k)))+W(t,t_k)(I_k(u(t_k))-I_k(v(t_k)))]\right|$$

$$\le\int_0^T|G(t,s)|[|f(s,u(s),u'(s))-f(s,v(s),v'(s)|+p|u'(s)-v'(s)|$$

$$+q|u(s)-v(s)|]ds+\sum_{k=1}^{m}[|G(t,t_k)||I_k^*(v(t_k))-I_k^*(u(t_k))|+|W(t,t_k)||I_k(u(t_k))-I_k(v(t_k))|]$$

$$\le\int_0^T|G(t,s)|[K_1|u(s)-v(s)|+K_2|u'(s)-v'(s)|+p|u'(s)-v'(s)|$$

$$+q|u(s)-v(s)|]ds+\sum_{k=1}^{m}[|G(t,t_k)|L_k|u(t_k))-v(t_k))|+|W(t,t_k)|l_k|u(t_k))-v(t_k))|]$$

$$\le(K_1+K_2+p+q)\|u-v\|_{PC^1}\int_0^T|G(t,s)|ds+\sum_{k=1}^{m}[|G(t,t_k)|L_k+|W(t,t_k)|l_k]\|u-v\|_{PC^1}$$

$$\le[(K_1+K_2+p+q)TG_1+\sum_{k=1}^{m}(G_1L_k+G_2l_k)]\|u-v\|_{PC^1}.$$

$$|(Au)'(t)-(Av)'(t)|=\left|\int_0^T\frac{\partial}{\partial t}G(t,s)[(f(s,u(s),u'(s))-f(s,v(s),v'(s))+p(u'(s)-v'(s))\right.$$

$$\left.+q(u(s)-v(s))]ds+\sum_{k=1}^{m}[\frac{\partial}{\partial t}G(t,t_k)(I_k^*(v(t_k))-I_k^*(u(t_k)))+\frac{\partial}{\partial t}W(t,t_k)(I_k(u(t_k))-I_k(v(t_k)))]\right|$$

$$\le\int_0^T\left|\frac{\partial}{\partial t}G(t,s)\right|[K_1|u(s)-v(s)|+K_2|u'(s)-v'(s)|+p|u'(s)-v'(s)|$$

$$+q|u(s)-v(s)|]ds+\sum_{k=1}^{m}\left[\left|\frac{\partial}{\partial t}G(t,t_k)\right|L_k|u(t_k))-v(t_k))|+\left|\frac{\partial}{\partial t}W(t,t_k)\right|l_k|u(t_k))-v(t_k))|\right]$$

$$\le(K_1+K_2+p+q)\|u-v\|_{PC^1}\int_0^T\left|\frac{\partial}{\partial t}G(t,s)\right|ds+\sum_{k=1}^{m}\left[\left|\frac{\partial}{\partial t}G(t,t_k)\right|L_k\right.$$

$$+\left|\frac{\partial}{\partial t}W(t,t_k)\right| l_k\Big] \|u-v\|_{PC^1}$$

$$\leq [(K_1+K_2+p+q)TG_2+\sum_{k=1}^{m}(G_2L_k+G_3l_k)]\|u-v\|_{PC^1}.$$

Hence,

$$\|Au-Av\|_{PC}\leq [(K_1+K_2+p+q)TG_1+\sum_{k=1}^{m}(G_1L_k+G_2l_k)]\|u-v\|_{PC^1},$$

$$\|(Au)'-(Av)'\|_{PC}\leq [(K_1+K_2+p+q)TG_2+\sum_{k=1}^{m}(G_2L_k+G_3l_k)]\|u-v\|_{PC^1}.$$

Thus, we obtain

$$\|Au-Av\|_{PC^1}\leq [(K_1+K_2+p+q)T\max\{G_1,G_2\}$$
$$+\sum_{k=1}^{m}(\max\{G_1,G_2\}L_k+\max\{G_2,G_3\}l_k)]\|u-v\|_{PC^1}.$$

In view of (3.16) and Banach fixed point theorem, A has a unique fixed point. The proof is complete. □

4. Example

In this section, we give an example to illustrate the effectiveness of our results.

Example 4.1. Consider the problem

$$\begin{cases} u''(t)+u(t)\sin^2 t-e^t u^{1/2}(t)+\frac{1}{2}u'(t)\cos^2 t+t^2+\sin 2t=0, \quad t\in[0,\frac{\pi}{2}]\setminus\{\frac{1}{3},\frac{2}{3}\}, \\ \triangle u\left(\frac{1}{3}\right)=\sin\left(u\left(\frac{1}{3}\right)\right), \quad \triangle u\left(\frac{2}{3}\right)=\cos\left(u\left(\frac{2}{3}\right)\right), \\ \triangle u'\left(\frac{1}{3}\right)=1+\frac{1}{2}\cos^2\left(u\left(\frac{1}{3}\right)\right), \quad \triangle u'\left(\frac{2}{3}\right)=\frac{1}{3}\sin\left(u\left(\frac{2}{3}\right)\right), \\ u(0)+u\left(\frac{\pi}{2}\right)=0,\ u'(0)+u'\left(\frac{\pi}{2}\right)=0, \end{cases} \tag{4.1}$$

Let $f(t,u,v)=u(t)\sin^2 t-e^t u^{1/2}(t)+\frac{v}{2}\cos^2 t+t^2+\sin 2t$, $I_1(u)=\sin u$, $I_2(u)=\cos u$, $I_1^*(u)=1+\frac{1}{2}\cos^2 u$, $I_2^*(u)=\frac{1}{3}\sin u$, $T=\frac{\pi}{2}$, $J=[0,\frac{\pi}{2}]$.

It is easy to show that

$$|I_1(u)|\leq 1, \quad |I_2(u)|\leq 1, \quad |I_1^*(u)|\leq \tfrac{3}{2}, \quad |I_2^*(u)|\leq \tfrac{1}{3},$$

and

$$|f(t,u,v)|\leq a(t)|u|+b(t)|u|^{1/2}+c(t)|v|+h(t),$$

where

$$a(t) = \sin^2 t, \quad b(t) = e^t, \quad c(t) = \tfrac{1}{2}\cos^2 t, \quad h(t) = t^2 + \sin 2t. \tag{4.2}$$

Thus, (H_2) and (H_4) hold. Let $p = 0$ and $q = \frac{1}{4}$, we have by (3.1) and (4.2) that

$$a_1 = \frac{12\pi}{16 - 3\pi}, \quad a_2 = \frac{24(e^{\pi/2} - 1)}{16 - 3\pi}, \quad a_3 = \frac{\pi^3 + 68}{16 - 3\pi},$$

and

$$3\int_0^T c(t)dt + 6pT = \frac{3}{2}\int_0^{\pi/2} \cos^2 t dt = \frac{3\pi}{8} < 2,$$

which implies that (H_1) holds. Moreover, we see that

$$b_1 = \int_0^{\pi/2} a(t)dt + \frac{1}{2}\int_0^{\pi/2} c(t)dt = \frac{5}{16}\pi < 3 = 2(2 - c^*),$$

where $c^* = \max_{t \in J} c(t) = \frac{1}{2}$. Thus, (3.15) holds. So, all the conditions of Corollary 3.2 are satisfied. By Corollary 3.2, anti-period boundary value problem (4.1) has at least one solution.

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