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# A note on a nonlinear functional differential system with feedback control*

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**Abstract** In this note we apply Avery-Peterson multiple fixed point theorem to investigate the existence of multiple positive periodic solutions to the following nonlinear non-autonomous functional differential system with feedback control

$$\begin{cases} \frac{dx}{dt} = -r(t)x(t) + F(t, x_t, u(t-\delta(t))), \\ \frac{du}{dt} = -h(t)u(t) + g(t)x(t-\sigma(t)). \end{cases}$$

We prove the system above admits at least three positive periodic solutions under certain growth conditions imposed on $F$.

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## 1. Introduction

In this note, we obtain a new result for the existence of at least three positive $\omega$-periodic solutions of the following system with feedback control:

$$\begin{cases} \frac{dx}{dt} = -r(t)x(t) + F(t, x_t, u(t-\delta(t))), \\ \frac{du}{dt} = -h(t)u(t) + g(t)x(t-\sigma(t)). \end{cases} \tag{1}$$

where $\delta(t), \sigma(t) \in C(\mathbb{R}, \mathbb{R}), r(t), h(t), g(t) \in C(\mathbb{R}, (0, +\infty))$, all of the above functions are $\omega$-periodic functions and $\omega > 0$ is a constant. $F(t, x_t, z)$ is a function defined on $\mathbb{R} \times BC \times \mathbb{R}$ and $F(t+\omega, x_{t+\omega}, z) = F(t, x_t, z)$, where $BC$ denotes the Banach space of bounded continuous functions $\varphi : \mathbb{R} \to \mathbb{R}$ with the norm $|| \varphi|| = \sup_{\theta \in R} |\varphi(\theta)|$. If $x \in BC$, then $x_t \in BC$ for any $t \in \mathbb{R}$ is defined by $x_t(\tau) = x(t+\tau)$ for $\tau \in \mathbb{R}$.

Many special cases of (1) have been widely investigated by many authors, see, for example [10,8,4,6]. But most work is for autonomous. In real world, any biological

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or environmental parameters, however, are naturally subject to fluctuation in time, so it is reasonable to study the corresponding non-autonomous systems. System (1) includes many mathematical ecological models with feedback control, such as the multiplicative delay logistic model with feedback control studied in [7]. For more information about the population models concerned with (1), we refer to [12] and the references therein.

Considering the biological and environmental periodicity, such as seasonal effects of weather, food supplies, mating habits, we shall focus on the existence of periodic solutions with strictly positive components of (1). Recently, some authors have studied the existence of at least one and two positive periodic solutions of (1) (cf. [9,3,11]). The methods used in the references are mainly the continuation theorem of Gaines and Mawhin's coincidence degree theory [4] and the fixed point theorem in cones [5]. We note that in a recent paper [9], by using a fixed point theorem [1] involving three functionals on cones, Liu and Li studied problem (1) and obtained two positive periodic solutions with some monotonic properties, that is, $x(t) \geq 0$ is $\omega-$periodic and $x(t)e^{\int_0^t r(s)ds}$ is no decreasing on $[0, \omega]$. The main goal of this paper is to revisit problem (1) and obtain at least three positive periodic solutions of (1) by means of a three fixed-points theorem due to Avery and Peterson [2].

The paper is divided into three sections, including this section. In Section 2 we give some background knowledge and preliminary results. Section 3 is devoted to the proof of the main result.

## 2. Background material and results

For the reader's convenience, we shall summarize below a few concepts and results from the theory of cones in Banach spaces [5].

Let $E$ be a real Banach space and $P \subset E$ a cone. The map $\alpha$ is said to be a nonnegative continuous concave functional on the cone $P$ provided that $\alpha : P \to [0, \infty)$ is continuous and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in P$ and $0 \leq t \leq 1$. Similarly, we say the map $\gamma$ is a nonnegative continuous convex functional on cone $P$ provided that $\gamma : P \to [0, \infty)$ is continuous and

$$\gamma(tx + (1-t)y) \leq t\gamma(x) + (1-t)\gamma(y)$$

for all $x, y \in P$ and $0 \leq t \leq 1$.

Let $\gamma$ and $\theta$ be nonnegative continuous convex functionals on $P$, $\alpha$ be a nonnegative continuous concave functional on $P$, and $\psi$ be a nonnegative continuous functional on $P$. Then for positive real numbers $a, b, c$ and $d$, we define the following convex sets:

$$
\begin{aligned}
&P(\gamma, d) = \{x \in P \mid \gamma(x) < d\},\\
&P(\gamma, \alpha, b, d) = \{x \in P \mid b \le \alpha(x), \gamma(x) \le d\},\\
&P(\gamma, \theta, \alpha, b, c, d) = \{x \in P \mid b \le \alpha(x), \theta(x) \le c, \gamma(x) \le d\},
\end{aligned}
$$

and a closed set $R(\gamma, \psi, a, d) = \{x \in P \mid a \le \psi(x), \gamma(x) \le d\}$.

The following fixed point theorem due to Avery and Peterson [2] is crucial in the proof of our main result.

**Theorem 2.1 [2].** *Let $P$ be a cone in a real Banach space $E$. Let $\gamma$ and $\theta$ be nonnegative continuous convex functionals on $P$, $\alpha$ be a nonnegative continuous concave functional on $P$, and $\psi$ be a nonnegative continuous functional on $P$ satisfying $\psi(\lambda x) \le \lambda\psi(x)$ for $0 \le \lambda \le 1$, such that for some positive numbers $M$ and $d$,*

$$\alpha(x) \le \psi(x) \quad \text{and} \quad \|x\| \le M\gamma(x) \tag{$*$}$$

*for all $x \in \overline{P(\gamma, d)}$. Suppose $T : \overline{P(\gamma, d)} \to \overline{P(\gamma, d)}$ is completely continuous and there exist positive numbers $a, b,$ and $c$ with $a < b$ such that*

*(S1) $\{x \in P(\gamma, \theta, \alpha, b, c, d) \mid \alpha(x) > b\} \ne \emptyset$ and $\alpha(Tx) > b$ for $x \in P(\gamma, \theta, \alpha, b, c, d)$;*

*(S2) $\alpha(Tx) > b$ for $x \in P(\gamma, \alpha, b, d)$ with $\theta(Tx) > c$;*

*(S3) $0 \notin R(\gamma, \psi, a, d)$ and $\psi(Tx) < a$ for $x \in R(\gamma, \psi, a, d)$ with $\psi(x) = a$.*

*Then $T$ has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\gamma, d)}$, such that*

$$
\begin{gathered}
\gamma(x_i) \le d \quad \text{for } i = 1, 2, 3, \quad b < \alpha(x_1),\\
a < \psi(x_2), \quad \text{with} \quad \alpha(x_2) < b, \quad \text{and } \psi(x_3) < a.
\end{gathered}
$$

In order to apply Theorem 2.1 to establish the existence of multiple positive solutions of system (1), we shall define an operator on a cone in a suitable Banach space. To this end, we first transform system (1) into a single equation. By integrating the latter equation in system (1) from $t$ to $t + \omega$, we obtain

$$u(t) = \int_t^{t+\omega} k(t, s)g(s)x(s - \sigma(s))ds := (\Phi x)(t), \tag{2}$$

where

$$k(t, s) = \frac{\exp\{\int_t^s h(v)dv\}}{\exp\{\int_0^\omega h(v)dv\} - 1}.$$

When $x$ is an $\omega$-periodic function, it is easy to see that $k(t+\omega, s+\omega) = k(t,s), u(t+\omega) = u(t)$, and

$$n := \frac{\exp\{-\int_0^\omega h(v)dv\}}{\exp\{\int_0^\omega h(v)dv\} - 1} \leq k(t,s) \leq \frac{\exp\{\int_0^\omega h(v)dv\}}{\exp\{\int_0^\omega h(v)dv\} - 1} := m \qquad (**)$$

for$(t,s) \in \mathbb{R}^2$, where $m, n$ are positive constants.

Therefore, the existence problem of $\omega$-periodic solution of system (1) is equivalent to that of $\omega$-periodic solution of the following equation:

$$\frac{dx}{dt} = -r(t)x(t) + F(t, x_t, (\Phi x)(t - \delta(t))). \qquad (3)$$

In what follows, we shall adopt the following notations for convenience: $\mathbb{R} = (-\infty, \infty), \mathbb{R}^+ = [0, \infty), I = [0, \omega], f^* := \max_{t \in I} f(t), f_* := \min_{t \in I} f(t)$, where $f$ is a continuous positive periodic function with period $\omega$. $BC(X,Y)$ denotes the space of bounded continuous functions $\varphi : X \to Y$.

Throughout the paper we always assume that

(H$_1$) $F(t, x_t, (\Phi x)(t - \delta(t)))$ is a continuous function of $t$ for each $x \in BC(\mathbb{R}, \mathbb{R}^+)$, and $F(t, x_t, (\Phi x)(t - \delta(t))) \geq 0$ for $(t, x) \in \mathbb{R} \times BC(\mathbb{R}, \mathbb{R}^+)$, where $\Phi$ is defined as (2).

(H$_2$) For any $C > 0$ and $\varepsilon > 0$, there exists $\mu > 0$ such that for $\gamma, \xi \in BC$, $\|\gamma\| \leq C$, $\|\xi\| \leq C$, $\|\gamma - \xi\| < \mu$ and for $0 \leq s \leq \omega$ imply

$$|F(s, \gamma_s, (\Phi\gamma)(s - \delta(s))) - F(s, \xi_s, (\Phi\xi)(x - \delta(s)))| < \varepsilon.$$

By integrating equation (3) from $t$ to $t + \omega$, we have

$$x(t) = \int_t^{t+\omega} G(t,s) F(s, x_s, (\Phi x)(s - \delta(s))) ds,$$

where

$$G(t,s) = \frac{\exp\{\int_t^s r(v)dv\}}{\exp\{\int_0^\omega r(v)dv\} - 1}. \qquad (4)$$

It is clear that $G(t+\omega, s+\omega) = G(t,s)$ for all $(t,s) \in \mathbb{R}^2$ and

$$p := \frac{\exp\{-\int_0^\omega r(v)dv\}}{\exp\{\int_0^\omega r(v)dv\} - 1} \leq G(t,s) \leq \frac{\exp\{\int_0^\omega r(v)dv\}}{\exp\{\int_0^\omega r(v)dv\} - 1} := q$$

for all $s \in [t, t+\omega]$, where $p, q$ are positive constants.

We denote by $\beta := \frac{p}{q}$. In order to use Theorem 2.1, we let $E$ be the set $E = \{x \in C(\mathbb{R}, \mathbb{R}) : x(t+\omega) = x(t), t \in \mathbb{R}\}$ with the norm $\|x\| = \sup_{t \in I} |x(t)|$; then $E \subset BC$ is a Banach space. Also we define $P$ as $P = \{x \in E : x(t) \geq \beta\|x\|, t \in I\}$. One may readily verify that $P$ is a cone in $E$.

Define an operator $T : P \to P$ by

$$(Tx)(t) = \int_t^{t+\omega} G(t,s)F(s,x_s,(\Phi x)(s-\delta(s)))ds \tag{5}$$

for $x \in P, t \in \mathbb{R}$, where $G(t,s)$ is defined by (4) and $\Phi$ is defined by (2). Under the conditions ($H_1$), ($H_2$), we known that $T : P \to P$ is well defined, completely continuous [9], and each positive $\omega$-periodic solution of (1) is a fixed point of the operator $T$ on $P$.

From now on, we fix $0 \le \eta \le l \le \omega, \tau \in \mathbb{R}$ and let the nonnegative continuous concave functional $\alpha$, the nonnegative continuous convex functionals $\theta, \gamma$, and the nonnegative continuous functional $\psi$ be defined on the cone $P$ by

$$\begin{aligned}
\alpha(x) &= \min_{\eta \le t \le l} |x_t(\tau)|, \\
\psi(x) &= \theta(x) = \max_{0 \le t \le \omega} |x_t(\tau)| = \max_{0 \le t \le \omega} |x(t)|, \\
\gamma(x) &= \max_{0 \le t \le \omega} |(\Phi x)(t-\delta(t))|.
\end{aligned}$$

The following lemma is useful in the proof of our main result.

**Lemma 2.1**. *For $x \in P$, there exists a constant $M > 0$ such that*

$$\max_{0 \le t \le \omega} |x_t(\tau)| \le M \max_{0 \le t \le \omega} |(\Phi x)(t-\delta(t))|.$$

*Proof.* For $x \in P$, we have

$$\begin{aligned}
\max_{0 \le t \le \omega} |(\Phi x)(t-\delta(t))| &= \max_{0 \le t \le \omega} (\Phi x)(t-\delta(t)) \\
&= \max_{0 \le t \le \omega} \int_{t-\delta(t)}^{t-\delta(t)+\omega} k(t-\delta(t),s)g(s)x(s-\sigma(s))ds \\
&= \max_{0 \le t \le \omega} \int_0^{\omega} k(t-\delta(t),s)g(s)x(s-\sigma(s))ds \\
&\ge \beta\|x\| \max_{0 \le t \le \omega} \int_0^{\omega} k(t-\delta(t),s)g(s)ds = L\beta \max_{0 \le t \le \omega} |x_t(\tau)|,
\end{aligned} \tag{6}$$

where $L := \max_{0 \le t \le \omega} \int_0^{\omega} k(t-\delta(t),s)g(s)ds$. Setting $M := \frac{1}{L\beta}$, we complete the proof. □

By Lemma 2.1, the functionals defined above satisfy relations

$$\alpha(x) \le \theta(x) = \psi(x), \quad \forall x \in P. \tag{7}$$

Moreover, for each $x \in P$, there holds

$$\|x\| = \max_{0 \le t \le \omega} |x_t(\tau)| \le \frac{\max_{0 \le t \le \omega} |(\Phi x)(t-\delta(t))|}{L\beta} = M\gamma(x). \tag{8}$$

We also find that $\psi(\lambda x)=\lambda\psi(x)$ for $\forall\lambda\in[0,1]$ and $\forall x\in P$. Therefore, the condition $(*)$ of Theorem 2.1 is satisfied.

## 3. Main result

Denote by $\lambda_\omega=\max\limits_{0\le t\le\omega}\int_0^\omega k(t-\delta(t),s)g(s)ds\times\max\limits_{0\le t\le\omega}\int_0^\omega G(t-\sigma(t),s)ds$, and

$$N=\min_{\eta\le t\le l}\int_0^\omega G(t+\tau,s)ds,\quad D=\max_{0\le t\le\omega}\int_0^\omega G(t+\tau,s)ds,\quad L=\max_{0\le t\le\omega}\int_0^\omega k(t,s)g(s)ds.$$

Our main result is the following

**Theorem 3.1**. *Let $0<a<b<\frac{b}{\beta}\le\frac{d}{L}$, and suppose that $F$ satisfies the following conditions:*

*(C1) $F(t,u,v)\le d/\lambda_\omega$, for $(t,u,v)\in[0,\omega]\times[0,Md]\times[0,d]$;*

*(C2) $F(t,u,v)>b/N$, for $(t,u,v)\in[\eta,l]\times[b,b/\beta]\times[0,d]$;*

*(C3) $F(t,u,v)<a/D$, for $(t,u,v)\in[0,\omega]\times[0,a]\times[0,d]$.*

*Then equation (3) admits at least three positive solutions $x_1,x_2$ and $x_3$ such that $\max\limits_{0\le t\le\omega}|(\Phi x_i)(t-\delta(t))|\le d; b<\min\limits_{\eta\le t\le l}|x_1(t+\tau)|,\ a<\max\limits_{0\le t\le\omega}|x_2(t+\tau)|$ with $\min\limits_{\eta\le t\le l}|x_2(t+\tau)|<b$, and $\max\limits_{0\le t\le\omega}|x_3(t+\tau)|<a$.*

*Proof.* Problem (1) has a solution $x=x(t)$ if and only if $x$ solves the operator equation $x=Tx$. Thus we set out to verify that the operator $T$ satisfies the Avery-Peterson fixed point theorem which will prove the existence of three fixed points of $T$ which satisfy Theorem 2.1.

For $x\in\overline{P(\gamma,d)}$, there is $\gamma(x)=\max\limits_{0\le t\le\omega}|(\Phi x)(t-\delta(t))|\le d$. By Lemma 2.1, we have $\max\limits_{0\le t\le\omega}|x_t(\tau)|=\max\limits_{0\le t\le\omega}x(t+\tau)\le Md$. Then condition (C1) implies $F(t,x_t(\tau),(\Phi x)(t-\delta(t))\le d/\lambda_\omega$. On the other hand, for $x\in\overline{P(\gamma,d)}$, there is $Tx\in P$ and

$$\begin{aligned}(Tx)(t-\sigma(t)) &= \int_{t-\sigma(t)}^{t-\sigma(t)+\omega}G(t-\sigma(t),s)F(s,x_s,(\Phi x)(s-\delta(s)))ds\\ &= \int_0^\omega G(t-\sigma(t),s)F(s,x_s,(\Phi x)(s-\delta(s)))ds\\ &\le \max_{0\le t\le\omega}\int_0^\omega G(t-\sigma(t),s)ds\times\frac{d}{\lambda_\omega},\end{aligned}$$

we have

$$\gamma(Tx)=\max_{0\le t\le\omega}|(\Phi(Tx))(t-\delta(t))|$$

$$
\begin{aligned}
&= \max_{0\le t\le \omega}\int_{t-\delta(t)}^{t-\delta(t)+\omega} k(t-\delta(t),s)g(s)(Tx)(s-\sigma(s))ds\\
&= \max_{0\le t\le \omega}\int_{0}^{\omega} k(t-\delta(t),s)g(s)(Tx)(s-\sigma(s))ds\\
&\le \max_{0\le t\le \omega}\int_{0}^{\omega} k(t-\delta(t),s)g(s)ds\times \max_{0\le t\le \omega}\int_{0}^{\omega} G(t-\sigma(t),s)ds\frac{d}{\lambda_\omega}\le d.
\end{aligned}
$$

Therefore, $T:\overline{P(\gamma,d)}\to\overline{P(\gamma,d)}$.

To check condition (S1) of Theorem 2.1, we take $x(t)=b/\beta$. It is easy to see that $x(t)=b/\beta\in P(\gamma,\theta,\alpha,b,b/\beta,d)$ and $\alpha(x)=b/\beta>b$, and so $\{x\in P(\gamma,\theta,\alpha,b,b/\beta,d)|\ \alpha(x)>b\}\neq\emptyset$.

Hence, for $x\in P(\gamma,\theta,\alpha,b,b/\beta,d)$, there is

$$
\min_{\eta\le t\le l}|x_t(\tau)|\ge b,\quad \max_{0\le t\le\omega}|x_t(\tau)|\le b/\beta,\quad \max_{0\le t\le\omega}|(\Phi x)(t-\delta(t))|\le d
$$

such that $b\le x_t(\tau)\le b/\beta, 0\le(\Phi x)(t-\delta(t))\le d, t\in[\eta,l]$.

Thus, by condition (C2) of this theorem, we have $F(t,x_t,(\Phi x)(t-\delta(t)))>b/N$ and

$$
\begin{aligned}
\alpha(Tx)=\min_{\eta\le t\le l}|(Tx)(t+\tau)| &= \min_{\eta\le t\le l}\int_{t+\tau}^{t+\tau+\omega} G(t+\tau,s)F(s,x_s,(\Phi x)(s-\delta(s)))ds\\
&> \frac{b}{N}\min_{\eta\le t\le l}\int_0^\omega G(t+\tau,s)ds=b,
\end{aligned}
$$

i.e. $\alpha(Tx)>b$ for all$x\in P(\gamma,\theta,\alpha,b,b/\beta,d)$. This show that condition (S1) of Theorem 2.1 is satisfied.

Secondly, with (7), (8), we have $\alpha(Tx)\ge\beta\theta(Tx)>\beta\times\frac{b}{\beta}=b$ for all $x\in P(\gamma,\alpha,b,d)$ with $\theta(Tx)>\frac{b}{\beta}$. Thus, condition (S2) of Theorem 2.1 is satisfied.

Finally, we show that condition (S3) of Theorem 2.1 also holds. Clearly, as $\psi(0)=0<a$, there holds $0\notin R(\gamma,\psi,a,d)$. Suppose that $x\in R(\gamma,\psi,a,d)$ with $\psi(x)=a$. This implies that for $t\in[0,\omega]$, there is $\max\limits_{0\le t\le\omega}|x_t(\tau)|=a,\ \max\limits_{0\le t\le\omega}|(\Phi x)(t-\delta(t))|\le d$. Hence,

$$
0\le x_t(\tau)\le a,\ 0\le(\Phi x)(t-\delta(t))\le d,\ t\in[0,\omega],
$$

and by the condition (C3) of this theorem,

$$
\begin{aligned}
\psi(Tx)=\max_{0\le t\le\omega}|(Tx)_t(\tau)| &= \max_{0\le t\le\omega}\int_{t+\tau}^{t+\tau+\omega} G(t+\tau,s)F(s,x_s,(\Phi x)(s-\delta(s)))ds\\
&< \frac{a}{D}\max_{0\le t\le\omega}\int_0^\omega G(t+\tau,s)ds=a.
\end{aligned}
$$

So, the condition (S3) of Theorem 2.1 is satisfied. On the other hand, for $x\in P$, (7), (8) holds. Therefore, an application of Theorem 2.1 implies (3) has at least three positive solutions. □

**Remark 3.1.** Under the assumptions ($H_1$), ($H_2$) and the following assumption

($H_0$) $mg^* > \frac{1}{\omega}$, $ng_* < \frac{1}{\omega}$,

where $m, n$ are given in ($**$) and $g^* := \max_{t \in I} g(t)$, $g_* := \min_{t \in I} g(t)$ are defined as in Section 2, Liu and Li [9] obtained two positive periodic solutions for problem (1). But in this paper we completely remove assumption ($H_0$). Moreover, we obtain at least three positive periodic solutions for (1) with more general nonlinearity $F$.

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