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This paper is devoted to fond memory of M.M. Dzhrbashjan

A Problem on the Zeros of the Mittag-Leffler Function and the Spectrum of a Fractional-Order Differential Operator

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Abstract. We carry out spectral analysis of one class of integral operators associated with fractional order differential equations that arises in mechanics. We establish a connection between the eigenvalues of these operators and the zeros of Mittag-Leffler type functions. We give sufficient conditions for complete nonselfadjointness and completeness the systems of the eigenvalues.

1.Introducion.

In [1] the spectral analysis of operators of the form

$$A_{\gamma}^{[\alpha,\beta]}u(x) = c_{\alpha}\int_0^x (x-t)^{\frac{1}{\alpha}-1}u(t)dt + c_{\beta,\gamma}\int_0^1 x^{\frac{1}{\beta}-1}(1-t)^{\frac{1}{\gamma}-1}u(t)dt.$$

was carried out [1]. Here $\alpha, \beta, \gamma, c_{\alpha}, c_{\beta,\gamma}$ are real numbers, and α, β, γ are positive. (G.M.Gubreev considered similar operators in the paper [2]). These operators arise in the study of boundary value problems for differential equations of fractional order (see [3] and references therein, in which corresponding Green functions are constructed).

In particular it was shown in [1] , that the operator

$$A^{\rho}u(x) = A_{\rho}^{[\rho,\rho]}u(x) = \frac{1}{\Gamma(\rho^{-1})}\int_0^x (x-t)^{\frac{1}{\rho}-1}u(x)dt - \frac{1}{\Gamma(\rho^{-1})}\int_0^1 x^{\frac{1}{\rho}-1}(1-t)^{\frac{1}{\rho}-1}u(t)dt$$

is almost non-self-conjugate (see [4] for $\rho > 1$), while for $0 < \rho < 1$, the kernel of the operator A^{ρ} is non-negative and the Fredholm spector of the operator A^{ρ} coincides with the set of roots of the whole function of Mittag-Leffler type

$$E_{\rho}(\lambda;\rho^{-1}) = \sum_{k=0}^{\infty}\frac{\lambda^k}{\Gamma(\rho^{-1}+k\rho^{-1})}.$$

From the above it follows, that all the eigennumbers of the operator A^ρ are complex for $\rho > 1$, while for $0 < \rho < 1$, the operator A^ρ has real eigennumbers (in fact, if $\frac{1}{2} < \rho < 1$ the set of real eigennumbers is finite), i.e. all the zeros of the function $E_\rho(\lambda; \rho^{-1})$ are complex for $\rho > 1$, while for $0 < \rho < 1$ the function $E_\rho(\lambda; \rho^{-1})$ has real zeros. This proves the assymption about the existence of a real zeros of the function $E_\rho(\lambda; \rho^{-1})$ for $\frac{1}{2} < \rho < 1$, as stated in the monograph [5 pg. 248].

The given paper is devoted as well to study the boundary value problems for the differential equations of fractional order and the accompanying them integrated operators of the form $A_\gamma^{[\alpha;\beta]}$.

In order to state the problems in concern we must mention some concepts from fractional calculus.

2.Background and Preliminary Results.

Let $f(x) \in L_1(0,1)$. Then, the function

$$\frac{d^{-\alpha}}{dx^{-\alpha}} f(x) \equiv \frac{1}{\Gamma(\alpha)} \int\limits_0^x (x-t)^{\alpha-1} f(t)dt \in L_1(0,1)$$

is called the fractional integral of order $\alpha > 0$ with starting point $x = 0$ [6]. And the function

$$\frac{d^{-\alpha}}{d(1-x)^{-\alpha}} f(x) \equiv \frac{1}{\Gamma(\alpha)} \int\limits_x^1 (t-x)^{\alpha-1} f(t)dt \in L_1(0,1)$$

is called the fractional integral of order $\alpha > 0$ with ending point $x = 1$ [6]. Here $\Gamma(\alpha)$ is Euler's gamma-function. It is clear that when $\alpha = 0$, we identify both the fractional integrals with the function $f(x)$. As we know [6], the function

$$g(x) \in L_1(0,1)$$

is called the fractional derivative of the function

$$f(x) \in L_1(0,1)$$

of order $\alpha > 0$ with starting point $x = 0$, if

$$f(x) = \frac{d^{-\alpha}}{dx^{-\alpha}} g(x).$$

Then denoting

$$g(x) = \frac{d^{\alpha}}{dx^{\alpha}} f(x),$$

we shall mean in the future, by

$$\frac{d^{\alpha}}{dx^{\alpha}},$$

the fractional integral when $\alpha < 0$ and, the fractional derivative when $\alpha > 0$. The fractional derivative

$$\frac{d^{\alpha}}{d(1-x)^{\alpha}}$$

of the function $f(x) \in L_1(0,1)$ of order $\alpha > 0$, with the ending point $x = 1$ is defined in a similar way.

Let $\{\gamma_k\}_0{}^n$ be any set of real numbers, satisfying the condition $0 < \gamma_j \leq 1, (j = 0, 1, .., n)$. We denote

$$\sigma_k = \sum_{j=0}^{k}; \mu_k = \sigma_k + 1 = \sum_{j=0}^{k} \gamma_j, (k = 0, 1, ..., n)$$

and we assume that

$$\frac{1}{\rho} = \sum_{j=0}^{n} \gamma_j - 1 = \sigma_n = \mu_n - 1 > 0.$$

Following [6], we consider the differential operators,

$$D^{(\sigma_0)} f(x) \equiv \frac{d^{-(1-\gamma_0)}}{dx^{-(1-\gamma_0)}} f(x),$$

which are, generally, of fractional order

$$D^{(\sigma_1)} f(x) \equiv \frac{d^{-(1-\gamma_1)}}{dx^{-(1-\gamma_1)}} \frac{d^{\gamma_0}}{dx^{\gamma_0}} f(x),$$

$$D^{(\sigma_n)} f(x) \equiv \frac{d^{-(1-\gamma_n)}}{dx^{-(1-\gamma_n)}} \frac{d^{\gamma_{n-1}}}{dx^{\gamma_{n-1}}} \cdots \frac{d^{\gamma_0}}{dx^{\gamma_0}}.$$

Here we note, that if $\gamma_0 = \gamma_1 = ... = \gamma_n = 1$ then obviously

$$D^{(\sigma_k)} f(x) = f^{(k)}(x), (k = 0, 1, 2, ..., n).$$

To demonstrate the basic ideas, we start by investigating most simple cases. Thus, we put, $\gamma_3 = \gamma_4 = ... = \gamma_n = 0$, and consider a problem which is an analogue of the well-known problem of Storm-Liouville. This problem which we call problem (A), as in [7], means the following:

In class

$$L_2(0,1)(or L_1(0,1))$$

find a nontrivial solution of the equation

$$D^{(\sigma_2)} y - [\lambda + q(x)]y = 0 \tag{1}$$

$x \in (0,1]$, satisfying the boundary conditions

$$\begin{cases} D^{(\sigma_0)}y\Big|_{x=0}\cos\alpha + D^{(\sigma_1)}y\Big|_{x=0}\sin\alpha = 0 \\ D^{(\sigma_0)}y\Big|_{x=1}\cos\beta + D^{(\sigma_1)}y\Big|_{x=1}\sin\beta = 0, \end{cases} \tag{2}$$

where $\lambda, \alpha, \beta; (Im\alpha = Im\beta = 0)$, are arbitrary parameters, $q(x) \in L_2(0,1)$. M.M.Dzhrbashchjan writes [7]: "questions of completeness of system of the eigenfunctions of problem (A), or a more delicate question, whether these functions form a basis in $L_2(0,1)$, are undoubtly interesting. But their solutions, apparently, are faced with significant analytical difficulties."

In [9] (see also [3]), it has been proved that the system of eigenfunctions of problem (A) is complete in $L_2(0,1)$, when $q(x) \equiv 0$.

Later, M.M.Malamud and his disciples [10] - [11] also have established the completeness of system of the eigenfunctions of similar problems the case of when $q(x)$ is an analytical function.

In this paper we give the complete solution of the problem on completeness of the system of eigenfunctions of problem (1)-(2) providing $q(x) > 0$. We shall consider various variants of equation (1). At $\gamma_0 = \gamma_1 = 1$, equation (1) is transformed into the equation

$$\frac{1}{\Gamma(1-\gamma_2)}\int_0^x \frac{u''(t)}{(x-t)^{\gamma_2}}dt - (\lambda + q(x))u(x) = 0, \tag{3}$$

which is called a fractional oscillational equation [5], and the operator $D^{(\sigma_2)}$ is called - the operator of fractional differentiation in Caputo sence [5]. At $\gamma_0 = \gamma_2 = 1$, the equation (1)is transformed into the equation

$$\frac{1}{\Gamma(1-\gamma_1)}\frac{d}{dx}\int_0^x \frac{u'(t)}{(x-t)^{\gamma_1}}dt - (\lambda + q(x))u(x) = 0. \tag{4}$$

The equation (4) as the modelling equation of fractional order $1 < \sigma < 2$ has been investigated (see [3] and references therein).

Theorem 1:*Let $q(x)$ be the half-limited function. Then the system of the eigenfunctions of problem (A)*

$$\frac{1}{\Gamma(1-\gamma_1)}\frac{d}{dx}\int_0^x \frac{u'(t)}{(x-t)^{\gamma_1}}dt - (q(x) + \lambda)u(t)dt = 0,$$

$u(0) = 0, u(1) = 0$ *is complete in* $L_2(0,1)$.

The proof of Theorem 1 is based on Lemma 1.

Lemma 1. *The operator S, induced by the differential expression*

$$l(u) = \frac{1}{\Gamma(1-\gamma_1)} \frac{d}{dx} \int_0^x \frac{u'(t)}{(x-t)^{\gamma_1}} dt - q(x)u(x),$$

and boundary conditions

$$u(0) = 0, u(1) = 0,$$

has an inverse operator.

Proof of Lemma 1: Lemma 1 will be proved if we can show, that problem (A), at $q(x) > 0$ has the unique trivial solution $u(0) = 0$.To prove the last statement we do the following. We multiply both sides of the equation

$$\frac{1}{\Gamma(1-\gamma_1)} \frac{d}{dx} \int_0^x \frac{u'(t)}{(x-t)^{\gamma_1}} dt = q(x)u(x)$$

by $(x-t)^{\gamma_1}$, and integrate from 0 to x. We obtain

$$\frac{1}{\Gamma(1-\gamma_1)} \int_0^x (x-t)^{\gamma_1} \frac{d}{dt} \int_0^t \frac{u'(\xi)}{(t-\xi)^{\gamma_1}} d\xi dt = \int_0^x (x-t)^{\gamma_1} q(t)u(t)dt.$$

Let's calculate the integral

$$Iu = \frac{1}{\Gamma(1-\gamma_1)} \int_0^x (x-t)^{\gamma_1} (\frac{d}{dt} \int_0^t \frac{u'(\xi)}{(t-\xi)^{\gamma_1}} d\xi) dt.$$

We have

$$Iu = \frac{1}{\Gamma(1-\gamma_1)} \int_0^x (x-t)^{\gamma_1} d(\int_0^t \frac{u'(\xi)}{(t-\xi)^{\gamma_1}} d\xi)$$

$$= \frac{1}{\Gamma(1-\gamma_1)} (x-t)^{\gamma_1} \int_0^t \frac{u'(\xi)}{(t-\xi)^{\gamma_1}} d\xi \Big|_0^x - \frac{1}{\Gamma(1-\gamma_1)} \int_0^x [(x-t)^{\gamma_1}]' \int_0^t \frac{u'(\xi)}{(t-\xi)^{\gamma_1}} d\xi dt$$

$$= -x^{\gamma_1} D^{(\sigma_1)} u \Big|_{t=0} - \frac{\gamma_1}{\Gamma(1-\gamma_1)} \int_0^x (x-t)^{\gamma_1 - 1} (\int_0^t \frac{u'(\xi)}{(t-\xi)^{\gamma_1}} d\xi) dt$$

$$= -x^{\gamma_1} D^{(\sigma_1)} u \Big|_{t=0} - \frac{\gamma_1}{\Gamma(1-\gamma_1)} \int_0^x u'(\xi) (\int_\xi^x (x-t)^{\gamma_1 - 1} (t-\xi)^{-\gamma_1} dt) d\xi$$

$$= x^{\gamma_1} D^{(\sigma_1)} u\Big|_{t=0} - \frac{\Gamma(\gamma_1)\gamma_1}{\Gamma(1)} \int_0^x u^{'}(\xi) d\xi.$$

Thus

$$Iu = -x^{\gamma_1} D^{(\sigma_1)} u\Big|_{t=0} - c[u(x) - u(0)].$$

Therefore

$$cu(x) = x^{\gamma_1} D^{(\sigma_1)} u\Big|_{t=0} + \int_0^x (x-1)^{\gamma_1} q(t) u(t) dt,$$

$c = \Gamma(1+\gamma_1)$. Hence, the general solution of the equation

$$\frac{1}{\Gamma(1-\gamma_1)} \frac{d}{dx} \int_0^x \frac{u^{'}(t)}{(t-\xi)^{\gamma_1}} d\xi - q(x)u = 0,$$

satisfying the condition $u(0) = 0$, has the form

$$u(x) = cx^{\gamma_1} + \int_0^x (x-t)^{\gamma_1} q(t) u(t) dt = cx^{\gamma} + Au(x).$$

The last equation leads, to:

$$u(x) = c(I-A)^{-1} x^{\gamma_1} = c(x^{\gamma_1} + Ax^{\gamma_1} + A^2 x^{\gamma_1} + ... + A^n x^{\gamma_1} + ...).$$

Since

$$Ax^{\gamma_1} = \int_0^x (x-t)^{\gamma_1} q(t) t^{\gamma_1} dt,$$

thus

$$A^2 x^{\gamma_1} = A(Ax^{\gamma_1}) = \int_0^x (x-t)^{\gamma_1} q(t) (\int_0^t (t-\xi)^{\gamma_1} q(\xi) \xi^{\gamma_1} d\xi) dt.$$

Consequently, the kernel of the operator A^2 is equal to

$$K_2(t;s) = \int_s^t (t-\tau)^{\gamma_1} q(\tau) (\tau - s)^{\gamma_1} q(s) d\tau.$$

The kernel of the operator A^n is defined by the known formula

$$K_n(t;s) = \int_s^t K(t;s) K_{n-1}(\tau;s) d\tau.$$

Since, the kernels of the operators

$$A, A^2, ..., A^n$$

are positive, the function

$$u(x) = c(x^{\gamma_1} + Ax^{\gamma_1} + A^2x^{\gamma_1} + ... + A^n x^{\gamma_1} + ...),$$

at $x = 1$, cannot be equal to 0, which proves Theorem 2. Now we give the complete proof of Theorem 1.

Proof of Theorem 1: The proof of Theorem 1, is based on V.B.Lidskii's known theorem

(Theorem of V.N. Lidskii: Let in present operator C be a dissipative, and trace class. Then, system of eigenfunctions and adjoint functions of the operator C, is complete in the domain of the operator C.) [12]. According to V.B.Lidskii's [12] theorem, it is enough to establish, that the operator S^{-1} is dissipative and trace class.

Let $u(t)$ be any function from the domain of $D(S) \subset L_2(0,1)$. Then

$$Re(Su, \overline{u}) = \frac{1}{\Gamma(1-\gamma_1)} Re \int_0^1 (\frac{d}{dx} \int_0^x \frac{u'(t)}{(x-t)^{\gamma_1}} dt + q(x))\overline{u}(x)dx$$

$$= \frac{1}{\Gamma(1-\gamma)} Re \int_0^1 (\frac{d}{dx} \int_0^x \frac{u'(t)}{(x-t)^{\gamma_1})} dt \overline{u}(x)dx + \frac{1}{\Gamma(1-\gamma_1)} Re \int_0^1 q(x)u(x)\overline{u}(x)dx.$$

Since

$$\frac{1}{\Gamma(1-\gamma_1)} Re \int_0^1 \frac{d}{dx} \int_0^x \frac{u'(t)}{(x-t)^{\gamma}} dt)\overline{u}(x)dx$$

$$= \frac{1}{\Gamma(1-\gamma_1)} Re \int_0^1 \overline{u}(x)dx \int_0^x \frac{u'(\xi)}{(x-\xi)^{\gamma_1}} d\xi$$

$$= \frac{1}{\Gamma(1-\gamma_1)} Re[\overline{u}(x) \int_0^x \frac{u'(\xi)}{(x-\xi)^{\gamma_1}} d\xi \Big|_0^1$$

$$- \int_0^1 \{\int_0^t \frac{u'(\xi)}{(t-\xi)^{\gamma_1}} d\xi\} \overline{u'}(\xi)dx]$$

$$= -\frac{1}{\Gamma(1-\gamma_1)} Re \left(\int_0^x \frac{u'(\xi)}{(x-\xi)^{\gamma_1}} d\xi, \overline{u'}(x) \right).$$

We take advantage of the Matsaev-Polant's theorem [13, 14]

(Theorem of Matsaev-Polant: Let A be a dissipative operator. Then, the values of the form $(A^\nu f; f)$, $(0 \le \nu \le 1)$, lie in the angle $0 \le \arg\lambda \le \pi\nu$ (p. 481)).

Theorem of Matsaev-Polant states, that the value of the form $(\frac{d^{-\alpha}}{dx^{-\alpha}}f, f)$ lies in the angle $|argz| < \frac{\pi\alpha}{2}$. So, it follows from the theorem of Matsaev-Polant that the operator

$$Tu = \begin{cases} \frac{1}{\Gamma(1-\gamma_1)}\frac{d}{dx}\int\limits_0^x \frac{u'(t)}{(x-t)^{\gamma_1}}dt \\ u(0) = 0, u(1) = 0 \end{cases}$$

is not only dissipative, but also sectorial; the values of the form (Su, u) lies in the angle $|argz| \le \frac{\pi\gamma_1}{2}$. Now the dissipativity of the operator S implies the dissipativity of the operator S^{-1}. Next we establish, that the operator S^{-1} is trace class. Let $\lambda_1, \lambda_2, ..., \lambda_n, ..$ and $\mu_1, \mu_2, ..., \mu_n...$ be the eigenvalues of the operators S and T correspondingly, numerated in the non-decreasing order of modulus. Then [8] we know, that $|\lambda_n - \mu_n| \le ||q(x)||$. Now, taking into account that the eigennumbers of the operator T coincide, with zeros of the function $E_\rho(\lambda; \rho^{-1})([1], [8])$ and the asymptotic of zeros of the Mittag-Leffler function

$$E_\rho(z; \mu) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu + k\rho^{-1})}, \rho > 0$$

is well studied (see [15] p.142), we obtain, that the operator S^{-1} has a finite spectral trace, i.e. the operator S^{-1} is trace class.

Since the operator S^{-1} is dissipative and is trace class according to V.B.Lidskii's theorem, the operator S^{-1} has the complete system of eigenfunctions in $L_2(0;1)$.

Corollary 1: *All eigenvalues of the operator S lie in the angle $|argz| \le \frac{\pi\gamma_1}{2}$; in particular all zeros of the function $E_\rho(z; \rho^{-1})$ also lie in the angle $|argz| \le \frac{\pi\gamma_1}{2}$.*

In a similar way we can investigate any variants of problem (A). We show how to do it. We consider the Dirichlet problem for the fractional oscillational equation, i.e. problem $(\widetilde{A_1})$

$$\frac{1}{\Gamma(1-\gamma_2)}\int\limits_0^x \frac{u''(t)}{(x-t)^{\gamma_2}}dt - (q(x) + \lambda)u = 0$$

$$u(0) = 0, u(1) = 0.$$

Theorem $1'$. *The system of eigenfunctions of problem $(\widetilde{A})$ is complete in $L_2(0;1)$.*

The proof of Theorem $1'$ as in the case with Theorem 1, is based on Lemma $1'$.

Lemma $1'$. *The operator, induced by the differential expression*

$$\widetilde{\ell}(u) = \frac{1}{\Gamma(1-\gamma_2)} \int\limits_0^x \frac{u''(t)}{(x-t)^{\gamma_2}} dt - (q(x)+\lambda)u$$

and boundary conditions

$$u(0) = 0, u(1) = 0$$

has an inverse operator.

Proof of Lemma $1'$. Lemma $1'$ will be proved if we show that problem $(\widetilde{A_1})$, for $q(x) > 0$, has the unique trivial solution $u(x) = 0$. To prove the uniqueness of the trivial solution, we multiply both sides of the equation

$$\frac{1}{\Gamma(1-\gamma_2)} \int\limits_0^x \frac{u''(t)}{(x-t)^{\gamma_2}} dt = (q(x)u(x))$$

by $(x-t)^{\gamma_1}$ and then integrate from 0 to x. Then we get

$$\frac{1}{\Gamma(1-\gamma_2)} \int\limits_0^x (x-t)^{\gamma_2} \int\limits_0^t (\frac{u''(\xi)}{(t-\xi)^{\gamma_2}} d\xi) dt = \int\limits_0^x (x-t)^{\gamma_2} q(t) u(t) dt.$$

Using the Dirichlet permutation formulas, we transform the integral

$$\frac{1}{\Gamma(1-\gamma_2)} \int\limits_0^x (x-t)^{\gamma_2} (\int\limits_0^t \frac{u''(\xi)}{(t-\xi)^{\gamma_2}} d\xi) dt.$$

Clearly,

$$\frac{1}{\Gamma(1-\gamma^2)} \int\limits_0^x (x-t)^{\gamma_2} (\int\limits_0^t \frac{u''(\xi)}{(t-\xi)^{\gamma_2}} d\xi) dt$$

$$= \frac{1}{\Gamma(1-\gamma_2)} \int\limits_0^x u''(\xi) \{ \int\limits_\xi^x (x-t)^{\gamma_1} (t-\xi)^{-\gamma_1} dt \} d\xi$$

$$= \frac{\Gamma(1+\gamma_2)\Gamma(1-\gamma_2)}{\Gamma(1-\gamma_2)} \int\limits_0^x u''(\xi)(x-\xi) d\xi$$

$$= u'(0)\Gamma(1+\gamma_1)x + u'(x)\Gamma(1+\gamma_2) - \Gamma(1+\gamma_2)u(0).$$

Thus, the general solution of the equation

$$\frac{1}{\Gamma(1-\gamma_2)}\int_0^x \frac{u''(t)}{(x-t)^{\gamma_2}}dt - q(x)u(x) = 0,$$

satisfying the condition $u(0) = 0$, has the form

$$u(x) = cx + \int_0^x (x-t)^{\gamma_1} q(t)u(t)dt = cx + \widetilde{A}u.$$

Consequently

$$u(x) = c(I - \widetilde{A})^{-1}x = c(x + \widetilde{A}x + \widetilde{A}^2 x + ...).$$

Since $q(x) > 1$ it is clear that $y(1) \neq 0$, and from here follows the proof of Lemma $1'$. Now we give the proof of Theorem $1'$. We take advantage of V.B.Lidskii's theorem [12] again. We will prove, that the operator $\widetilde{S}$ is dissipative. Let $u(x)$ be any function from the domain of the operator S $D(S) \subset L_2(0,1)$. The function

$$u_\varepsilon(x) = \begin{cases} v(x), x \in [0,\varepsilon] \\ u(x), x \in [\varepsilon, 1] \end{cases},$$

where $\varepsilon > 0$, and the function $v(x)$, is such, that

$$v'(x)|_{x=0} = 0$$

and is summable with square in $[0;\varepsilon]$.

It is possible to show, that

$$\lim_{\varepsilon \to 0}(\widetilde{S}u_\varepsilon; \overline{u}_\varepsilon) = (\widetilde{S}u, \overline{u}).$$

For this purpose it is enough to take in account [15] (p. 573 2) the formula

$$\frac{d^\alpha}{dx^\alpha}f(x) = \frac{f'(0)}{\Gamma(1-\alpha)}x^{-\alpha} + \frac{1}{\Gamma(1-\alpha)}\int_0^x (x-t)^{-\alpha} f'(t)dt.$$

Here

$$f'(x) \in L(0,1), (0 < \alpha < 1).$$

After that, repeating the proof of Theorem 1 word by word, we obtain, that the values of the form $(\widetilde{S}u, \overline{u})$ lie in the angle $|argz| \leq \frac{\pi\gamma_2}{2}$. the existence of the kernel in operator $\widetilde{S}^{-1}$ can be established in a way similar to that of the existence of the kernel in operator S^{-1}. For this purpose it is enough to remember, that number λ_j is the eigenvalue of the operator $\widetilde{T}$ only in the case, when λ_j is a zero of the function $E_\rho(\lambda, 2)$. Thus the operator $(\widetilde{S}^{-1})$, when it is trace class and dissipative, has a complete system of eigenfunctions.

Corollary 1". *All eigenvalues of the operator $\widetilde{S}$ lie in the angle $|argz| \leq \frac{\pi\gamma_2}{2}$. In particular, all zeros of the function $E_\rho(z;2)$ also lie in the angle $|argz| \leq \frac{\pi\gamma_2}{2}$.*

By means of operators $A_\gamma^{[\alpha,\beta]}$ it is possible to obtain statements similar to Lemmas 1 and 1′, in a different way. We formulate and prove the following statement. For simplicity we put $||q(x)|| \leq \frac{2}{\Gamma(1+\gamma_1)}$.

Theorem 4: *The operator S is invertible and compact, and*

$$||S^{-1}||_{L_2} \leq \frac{1}{\Gamma(1-\gamma_1) - ||q(x)||}.$$

Proof of Theorem 4 can be obtained from the following known theorem.

Theorem [16] (p. 249): *Let T and A be any operators from X into Y. We assume that T^{-1} exists and belongs to $B(Y,X)$. We also assume,*

$$||A|| \leq a||u|| + b||Tu||$$

$$u \in D(T),$$

where the constants a and b satisfy the inequality $a||T^{-1}||+b<1$. Then the operator $S = T+A$ will be closed and may be invertible, and $S^{-1} \in B(Y,X)$ and

$$||S^{-1}|| \leq \frac{||T^{-1}||}{1-a||T^{-1}||-b},$$

$$||S^{-1} - T^{-1}|| \leq \frac{||T^{-1}||(a||T^{-1}||+b)}{1-a||T^{-1}||-b.}$$

Furthermore, if the operator T^{-1} is compact,then S^{-1} is compact too.

Proof of Theorem 2. Operator S, in our case, has the form

$$Su = \begin{cases} \frac{1}{\Gamma(1-\gamma_1)} \frac{d}{dx} \int\limits_0^x \frac{u'(t)}{(x-t)^{\gamma_1}} dt + q(x)u(x) \\ u(0) = 0, u(1) = 0 \end{cases}$$

It is possible to represent the operator S in the form $Su = Tu + Au$ where Tu is not a disturbed operator,

$$Tu = \begin{cases} \frac{1}{\Gamma(1-\gamma_1)} \frac{d}{dx} \int\limits_0^x \frac{u'(t)}{(x-t)^{\gamma_1}} dt \\ u(0) = 0, u(1) = 0 \end{cases}$$

and Au is the corresponding disturbance

$$Au = \begin{cases} q(x)u(x) \\ u(0) = 0, u(1) = 0 \end{cases} .$$

As it was already noted the operator

$$A^\rho u(t) = \frac{1}{\Gamma(\rho^{-1})}\left[\int_0^x (x-t)^{\frac{1}{\rho}-1}u(t)dt - \int_0^1 x^{\frac{1}{\rho}-1}(1-t)^{\frac{1}{\rho}-1}u(t)dt\right],$$

is the inverse of the operator T when $\frac{1}{\rho} = 1+\gamma_1$ [3], i.e. $T^{-1}u = A^\rho u$ and it is obvious that

$$||A^\rho u|| \leq \frac{2}{\Gamma(\rho^{-1})} = \frac{2}{\Gamma(1+\gamma_1)},$$

$$||Au|| \leq ||q(x)||||u|| \leq \frac{2}{\Gamma(1+\gamma_1)}$$

Now

$$||S^{-1}|| \leq \frac{\frac{2}{\Gamma(1+\gamma_1})}{1-||q(x)||}\frac{2}{\Gamma(1+\gamma_1)} = \frac{2}{\Gamma(1+\gamma_1) - 2||q(x)||}$$

In the proof of Lemma 1, it is certainly essential that $q(x)$ is positive. In the proof of the Theorem 2, $q(x)$ is not required to be positive. In exactly the same way we can obtain similar results for the problem $(\widetilde{A})$, assosiated with problem (A). To state problem $(\widetilde{A})$ we define

$$\widetilde{\sigma}_k = \sum_{j=0}^{k} \gamma_{2-j} - 1,$$

$$\widetilde{\mu}_k = \widetilde{\sigma}_k + 1 = \sum_{j=0}^{k} \gamma_{2-j}$$

$(k = 0, 1, 2)$,

$$D_1^{(\widetilde{\sigma_0})} f(x) \equiv \frac{d^{-(1-\gamma_2)}}{d(1-x)^{-(1-\gamma_2)}} f(x),$$

$$D_1^{(\widetilde{\sigma_1})} f(x) \equiv -\frac{d^{-(1-\gamma_1)}}{d(1-x)^{-(1-\gamma_1)}} \frac{d^{\gamma_2}}{d(1-x)^{\gamma_2}} f(x),$$

$$D_1^{(\widetilde{\sigma_2})} f(x) \equiv \frac{d^{-(1-\gamma_0)}}{d(1-x)^{-(1-\gamma_0)}} \frac{d^{\gamma_1}}{d(1-x)^{\gamma_1}} \frac{d^{\gamma_2}}{d(1-\gamma)^{\gamma_2}} f(x).$$

And now, problem $(\widetilde{A})$ may be put as follows. In class $L_2(0,1)$ (or $L_1(0,1)$) we find nontrivial solution of the equation

$$D^{(\widetilde{\sigma_2})}z - \{\lambda + q(x)\}z = 0, x \in [0,1),$$

satisfying the boundary conditions

$$D_1^{(\widetilde{\sigma_0})}z\Big|_{x=0}\cos\alpha + D_1^{(\widetilde{\sigma_1})}z\Big|_{x=0}\sin\alpha = 0$$

$$D_1^{(\widetilde{\sigma_0})}z\Big|_{x=0}\cos\beta + D_1^{(\widetilde{\sigma_1})}z\Big|_{x=0}\sin\beta = 0.$$

the associated problem gives essentially, new results, in the case when the order of the fractional differential equation is less than one. To this case we shall devote separate paper. To show, how to transfer the obtained results on a case of the differential equations of order higher than two, we consider the following problem.

In class $L_2(0,1)$ we find non-trivial solution of the equation

$$D^{(\sigma_3)}y - \{\lambda + q(x)y = 0, \}(7)$$

satisfying the conditions

$$D^{(\sigma_0)}y|_{x=0} = 0; D^{(\sigma_1)}y|_{x=0} = 0; D^{(\sigma_0)}y|_{x=1} = 0; (8)$$

We put

$$\gamma_0 = \gamma_2 = \gamma_3 = 1.$$

$$\sigma_3 = \frac{1}{\rho} = \gamma_0 + \gamma_1 + \gamma_2 + \gamma_3 - 1 = 2 + \gamma_1$$

In this case problem $(7) - (8)$ will be rewritten as follows

$$\frac{1}{\Gamma(1-\gamma_1)}\frac{d^2}{dx^2}\int_0^x \frac{y'(t)}{(x-t)^{\gamma_1}} - \{\lambda + q(x)\}y = 0(7')$$

$$y(0) = 0; D^{(\sigma_1)}y|_{x=0} = 0; y(1) = 0(8')$$

We know from [3], that the operator, inverse to the operator induced by differential expression

$$ly = \frac{1}{\Gamma(1-\gamma_1)}\frac{d^2}{dx^2}\int_0^x \frac{y'(t)}{(x-t)^{\gamma_1}}dt$$

and satisfying to boundary conditions $(8')$ is equal to the operator A^ρ for $1 < \rho < 2$ (in paper [1], the eigennumbers and the eigenvalues of the expression for A^ρ at every A^ρ.)Boundary-value problems for the equations of order higher than 2 are investigated not enough. (there is M.M.Malamud's remarkable paper [10] in this topic). Therefore the author would like to dwell on this problem in more details.

Theorem 3.*Let* $q(x) \equiv 0$, *then the number* λ_j *is the eigenvalue of problem* $(7') - (8')$, *iff* λ *is a zero of the function* $E_\rho(\lambda; \rho^{-1})$, *and functions*

$$x^{\frac{1}{\rho}-1} E_\rho(\lambda_j x^{\frac{1}{\rho}}; \frac{1}{\rho})$$

are eigenfunctions of this problem.

Proof: The general solution of the equation $(7')$ satisfying the conditions

$$y(0) = 0, D^{(\sigma_1)} y|_{x=0} = 0; D^{(\sigma_2)} y|_{x=0} = y_2^0$$

obviously satisfies the equation

$$y(x;\lambda) = \frac{y_2^0}{\Gamma(2+\gamma_1)} x^{1+\gamma_1} + \frac{\lambda}{\Gamma(\frac{1}{\rho})} \int_0^x (x-t)^{\frac{1}{\rho}-1} y(t;\lambda) dt; \; (9)$$

We write out the solution of the equation (9) according to the known formula [[15] p.123];

$$y(x;\lambda) = \frac{y_2^0}{\Gamma(2+\gamma_1)} x^{1+\gamma_1} + \lambda \int_0^x (x-t)^{\frac{1}{\rho}-1} E_\rho(\lambda (x-t)^{\frac{1}{\rho}}; \frac{1}{\rho}) (\frac{y_2^0}{\Gamma(1+\gamma_1)} t^{2+\gamma_1}) dt$$

and calcculate the integral

$$\frac{1}{\Gamma(1+\gamma_1)} \int_0^x (x-t)^{\frac{1}{\rho}-1} E_\rho(\lambda (x-t)^{\frac{1}{\rho}}; \frac{1}{\rho}) t^{1+\gamma_1} dt$$

according to M.M.Dzhrbashchjan's formula [6 p.121]

$$\int_0^l x^{\alpha-1} E_\rho(\lambda x^{\frac{1}{\rho}}; \alpha)(l-x)^{\beta-1} E_\rho(\lambda^* (l-x)^{\frac{1}{\rho}}; \beta) dx$$

$$= \frac{\lambda E_\rho(l^{\frac{1}{\rho}}\lambda; \alpha+\beta) - \lambda^* E_\rho(l^{\frac{1}{\rho}}\lambda^*; \alpha+\beta)}{\lambda - \lambda^*} l^{\alpha+\beta-1}$$

$$(\alpha > 0, \beta > 0)$$

$$\frac{1}{\Gamma(\rho^{-1})} \int_0^x (x-t)^{\frac{1}{\rho}-1} E_\rho(\lambda (x-t)^{\frac{1}{\rho}}; \frac{1}{\rho}) t^{\frac{1}{\rho}-1} dt$$

$$= E_\rho(x^{\frac{1}{\rho}}; \frac{2}{\rho})x^{\frac{2}{\rho}-1}.$$

From here we get

$$y(x;\lambda) = \frac{{y_2}^0}{\Gamma(\rho-1)}x^{\frac{1}{\rho}-1} + \lambda y_2^0 E_\rho(\lambda x^{\frac{1}{\rho}}; \frac{2}{\rho})x^{\frac{2}{\rho}-1}$$

$$= y_2^0 x^{\frac{1}{\rho}-1}[\frac{1}{\Gamma(\rho^{-1}})) + \lambda x^{\frac{1}{\rho}} E_\rho(\lambda x^{\frac{1}{\rho}}; \frac{2}{\rho})]$$

$$= y_2^0 x^{\frac{1}{\rho}-1} E_\rho(\lambda\ x^{\frac{1}{\rho}}; \frac{1}{\rho}).$$

Thus, we have proved, that the general solution of equation $(7')$, satisfying

$$y(o) = 0; D^{(\sigma_1)}y|_{x=0} = 0; D^{(\sigma_2)}y|_{x=0} = y_2^0,$$

has the form

$$y(x;\lambda) = y_2^0 x^{\frac{1}{\rho}-1} E_\rho(\lambda x^{\frac{1}{\rho}}; \frac{1}{\rho}).$$

From here it follows that the number λ is an eigennumber of problem $(7') - (8')$when and only , when

$$y(1;\lambda) = y_2^0 E_\rho(\lambda; \frac{1}{\rho}) = 0,$$

i.e. eigennumbers of problem $(7') - (8')$ coincide with zero of the function $E_\rho(\lambda x^{\frac{1}{\rho}}; \frac{1}{\rho})$, and eigenfunctions have the form

$$E_\rho(\lambda x^{\frac{1}{\rho}}; \frac{1}{\rho}),$$

which proves Theorem 3. It is interesting how all this has a precise junction with the case of $\frac{1}{2} < \rho < 1$, which we investigated above.

Theorem 4. *The operator $D^{\frac{1}{\rho}}$ induced by the differential expression*

$$\frac{1}{\Gamma(1-\gamma_0)}\frac{d^2}{dx^2}\int\limits_0^x \frac{y'(t)}{(x-t)^\gamma}dt + q(x)y$$

and the boundary conditions $(8')$ is dissipative.

Proof:

$$(D^{\frac{1}{\rho}}y, \overline{y}) = \left(\frac{1}{\Gamma(1-\gamma_1)}\frac{d^2}{dx^2}\int\limits_0^x \frac{y'(t)}{(x-t)^{\gamma_1}}dt, \overline{y(x)}\right) + (q(x)y, \overline{y})$$

$$= \frac{1}{\Gamma(1-\gamma)} \int_0^1 \left(\int_0^x \frac{y'(t)}{(x-t)^\gamma} dt \right)'' \overline{y(x)dx}$$

$$+(q(x)y, \overline{y}) = \frac{1}{\Gamma(1-\gamma)} \int_0^1 \overline{y} d \left(\int_0^x \frac{y'(t)}{(x-t)^{\gamma_0}} dt \right)' + (q(x)y, \overline{y})$$

$$= \frac{1}{\Gamma(1-\gamma)} \left[\overline{y}(x) \int_0^x \frac{y'(t)}{(x-t)^\gamma} dt \right] \Bigg|_a^b$$

$$= \frac{1}{\Gamma(1-\gamma)} \int_0^1 \frac{d}{dx} \int_0^x \frac{y'(t)}{(x-t)\gamma_1} \overline{y}'(x) dx + (q(x)y, \overline{y})$$

$$= -\left(\frac{d\gamma}{dx^\gamma} z, \overline{z} \right) + (q(x)y, \overline{y}),$$

where $z = y'$. Since $z \in {A_0}^\gamma[0,1]$ (the set of all functions $z(x)$, having absolutely continuous on [0,1] fractional integral of order $1-\gamma$ with the starting point 0 and the ending point x, which vanishs at $x=0$), therefore according to Tamarkin's theorem ([15] p.574) there is a unique function $u \in L(0,1)$ such, that $z(x) = \frac{d^{-(1-\gamma)}}{dx^{-(1-\gamma)}} u$. Hence

$$\left(\frac{d^\gamma}{dx^\gamma} z, \overline{z} \right) = \left(u, \frac{d^{-(1-\gamma)}}{dx^{-(1-\gamma)}} \overline{u} \right),$$

and the values of the last form, according to Matsaev-Polant's theorem, mentioned above lays in the corresponding sector. From this follows the proof of our theorem. Now having a dissipativity, we can prove completeness of the system of the eigenfunctions of problem $(7') - (8')$, reverse as above, if there is an operator, inverse to the operator $D^{\frac{1}{\rho}} u$, and equal to

$$D^{\frac{1}{\rho}} u = \begin{cases} \frac{1}{\Gamma(1-\gamma)} \frac{d^2}{dx^2} \int_0^x \frac{u'(t)}{(x-t)^\gamma} dt, 0 < \gamma < 1 \\ u(0) = 0;\, D^{(\sigma_1)} u\big|_{x=0} = 0; u(1) = 0 \end{cases}$$

The operator A^ρ at $\frac{1}{\rho} = 2 + \gamma$ is the inverse to the operator $D^{\frac{1}{\rho}}$ [3]. And in general, the operator inverse to the operator, generated by the differential expression

$$lu = \frac{1}{\Gamma(1-\gamma)} \frac{d^{n-1}}{dx^{n-1}} \int_0^x \frac{u'(t)}{(x-t)^\gamma} dt, (\gamma_0 = 1, \gamma_1 = \gamma, \gamma_2 = 1, ..., \gamma_n = 1$$

and natural boundary conditions

$$u(0) = 0;\, D^{(\sigma_1)}u\Big|_{x=0} = 0, ..., D^{(\sigma_{n-1})}u\Big|_{x=0} = 0, u(1) = 0$$

is equal to A^{ρ} at $\frac{1}{\rho} = n - 1 + \gamma$. This can be checked directly as in [3] (it is necessary to show, that $A^{\rho}D^{\frac{1}{\rho}}u = u$; and $D^{\frac{1}{\rho}}A^{\rho}u = u$.) And now, just as was done in [3], one can investigate, by means of operators of kind $A_{\gamma}^{\alpha,\beta}$, the questions of simplicity of the corresponding eigenvalues and marking out the domains where these eigenvalues do not exist(or, equally, the questions of simplicity of zeros of the corresponding functions of Mittag-Leffler's type and domains in the complex plane where these zeros do not exist).

Finally, the next publication will be devoted to questions of existence of basis, the constructed systems of the eigenfunctions and the proof of the oscillating property of the operator ${A_{\rho}}^{[\rho,\rho]}$ of cases $0 < \rho < 1/2$. It is absolutely new class of the oscillating operators different from the operators of M.G.Krein [17].

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