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Existence Results for Impulsive Neutral Functional Differential Equations With State-Dependent Delay

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Abstract

In this article, we study the existence of mild solutions for a class of impulsive abstract partial neutral functional differential equations with state-dependent delay. The results are obtained by using Leray-Schauder Alternative fixed point theorem. Example is provided to illustrate the main result.

Keywords: Abstract Cauchy problem, impulsive neutral equations, state-dependent delay, semigroup of linear operators, unbounded delay.
AMS-Subject Classification: 34A37, 34K30, 34K40.

1 Introduction

The purpose of this article is to establish the existence of mild solutions for a class of impulsive abstract neutral functional differential equations with state-dependent delay described by the form

$$\frac{d}{dt}[x(t) + G(t, x_t)] = Ax(t) + F(t, x_{\rho(t,x_t)}), \quad t \in I = [0, a], \tag{1.1}$$

$$x_0 = \varphi \in \mathcal{B}, \tag{1.2}$$

$$\Delta x(t_i) = I_i(x_{t_i}), \quad i = 1, 2, ..., n, \tag{1.3}$$

where A is the infinitesimal generator of a compact C_0-semigroup of bounded linear operators $(T(t))_{t\geq 0}$ on a Banach space X; the function $x_s : (-\infty, 0] \to X$, $x_s(\theta) = x(s + \theta)$, belongs to some abstract phase space $\mathcal{B}$ described axiomatically; $0 < t_1 .... < t_n < a$ are prefixed numbers; $F, G : I \times \mathcal{B} \to X$, $\rho : I \times \mathcal{B} \to (-\infty, a]$, $I_i : \mathcal{B} \times X \to X$, $i = 1, 2, ..., n,$ are appropriate functions and $\Delta\xi(t)$ represents the jump of the function ξ at t, which is defined by $\Delta\xi(t) = \xi(t^+) - \xi(t^-)$.

Many evolution processes are characterized by the fact that at certain moments of time they experience a change of state abruptly. These processes are subject to short-term perturbations whose duration is negligible in comparison with the duration of the process.

Consequently, it is natural to assume that these perturbations act instantaneously, that is, in the form of impulses. It is known, for example, that many biological phenomena involving thresholds, bursting rhythm models in medicine and biology, optimal control models in economics, pharmacokinetics and frequency modulated systems, do exhibit impulsive effects. Thus impulsive differential equations, that is, differential equations involving impulse effects, appear as a natural description of observed evolution phenomena of several real world problems. For more details on this theory and its applications we refer the monographs of Bainov and Simeonov [3], Lakshmikantham et al. [16] and Samoilenko and Perestyuk [19], where numerous properties of their solutions are studied and detailed bibliographies are given.

Functional differential equations with state-dependent delay appear frequently in applications as model of equations and for this reason the study of this type of equations has received great attention in the last few years, see for instance [1, 4, 5, 8, 9, 10, 12, 13, 14, 20, 21] and the references therein. The literature related to impulsive partial functional differential equations with state-dependent delay is limited, to our knowledge, to the recent works [2, 11]. The study of impulsive partial neutral functional differential equations with state-dependent delay described in the general abstract form (1.1)-(1.3) is an untreated topic in the literature, and this fact, is the main motivation of our paper.

2 Preliminaries

Throughout this article, $A : D(A) \subset X \to X$ is the infinitesimal generator of a compact C_0-semigroup of linear operators $(T(t))_{t\geq 0}$ on a Banach space X and $\widetilde{M}$ is a positive constant such that $\| T(t) \| \leq \widetilde{M}$ for every $t \in I$. For background information related to semigroup theory, we refer the reader to Pazy [18].

To consider the impulsive condition (1.3), it is convenient to introduce some additional concepts and notations. We say that a function $u : [\sigma, \tau] \to X$ is a normalized piecewise continuous function on $[\sigma, \tau]$ if u is piecewise continuous and left continuous on $(\sigma, \tau]$. We denote by $\mathcal{PC}([\sigma, \tau]; X)$ the space formed by the normalized piecewise continuous functions from $[\sigma, \tau]$ into X. In particular, we introduce the space $\mathcal{PC}$ formed by all functions $u : [0, a] \to X$ such that u is continuous at $t \neq t_i, u(t_i^-) = u(t_i)$ and $u(t_i^+)$ exists, for all $i = 1, \cdots, n$. In this paper we always assume that $\mathcal{PC}$ is endowed with the norm $\| u \|_{\mathcal{PC}} = \sup_{s\in I} \| u(s) \|$. It is clear that $(\mathcal{PC}, \| \cdot \|_{\mathcal{PC}})$ is a Banach space.

To simplify the notations, we put $t_0 = 0$, $t_{n+1} = a$ and for $u \in \mathcal{PC}$ we denote by $\tilde{u}_i \in C([t_i, t_{i+1}]; X)$, $i = 0, 1, \cdots, n$, the function given by

$$\widetilde{u}_i(t) = \begin{cases} u(t), & \text{for } t \in (t_i, t_{i+1}], \\ u(t_i^+), & \text{for } t = t_i. \end{cases}$$

Moreover, for $B \subseteq \mathcal{PC}$ we denote by $\widetilde{B}_i$, $i = 0, 1, \cdots, n$, the set $\widetilde{B}_i = \{\tilde{u}_i : u \in B\}$.

Lemma 2.1 *A set $B \subseteq \mathcal{PC}$ is relatively compact in $\mathcal{PC}$ if, and only if, the set $\widetilde{B}_i$ is relatively compact in $C([t_i, t_{i+1}]; X)$, for every $i = 0, 1, \cdots, n$.*

In this work we will employ an axiomatic definition for the phase space $\mathcal{B}$ which is similar to those introduced in [15]. Specifically, $\mathcal{B}$ will be a linear space of functions mapping $(-\infty, 0]$ into X endowed with a seminorm $\| \cdot \|_{\mathcal{B}}$, and satisfies the following axioms:

(**A**) If $x : (-\infty, \sigma + b] \to X$, $b > 0$, is such that $x|_{[\sigma,\sigma+b]} \in \mathcal{PC}([\sigma, \sigma + b] : X)$ and $x_\sigma \in \mathcal{B}$, then for every $t \in [\sigma, \sigma + b]$ the following conditions hold:

(i) x_t is in $\mathcal{B}$,

(ii) $\| x(t) \| \leq H \| x_t \|_{\mathcal{B}}$,

(iii) $\| x_t \|_{\mathcal{B}} \leq K(t - \sigma) \sup\{\| x(s) \| : \sigma \leq s \leq t\} + M(t - \sigma) \| x_\sigma \|_{\mathcal{B}}$,

where $H > 0$ is a constant; $K, M : [0, \infty) \to [1, \infty)$, K is continuous, M is locally bounded, and H, K, M are independent of $x(\cdot)$.

(**B**) The space $\mathcal{B}$ is complete.

Example 2.1 The phase spaces $\mathcal{PC}_h(X)$, $\mathcal{PC}^0_g(X)$.

As usual, we say that $\varphi : (-\infty, 0] \to X$ is normalized piecewise continuous, if φ is left continuous and the restriction of φ to any interval $[-r, 0]$ is piecewise continuous.

Let $g : (-\infty, 0] \to [1, \infty)$ be a continuous, nonincreasing function with $g(0) = 1$, which satisfies the conditions (g-1), (g-2) of [15]. This means that $\lim_{\theta\to-\infty} g(\theta) = \infty$ and that the function $G(t) := \sup_{-\infty<\theta\leq-t} \frac{g(t+\theta)}{g(\theta)}$ is locally bounded for $t \geq 0$. Next, we modify slightly the definition of the spaces C_g, C^0_g in [15]. We denote by $\mathcal{PC}_g(X)$ the space formed by the normalized piecewise continuous functions φ such that $\frac{\varphi}{g}$ is bounded on $(-\infty, 0]$ and by $\mathcal{PC}^0_g(X)$ the subspace of $\mathcal{PC}_g(X)$ formed by the functions φ such that $\frac{\varphi(\theta)}{g(\theta)} \to 0$ as $\theta \to -\infty$. It is easy to see that $\mathcal{PC}_g(X)$ and $\mathcal{PC}^0_g(X)$ endowed with the norm $\| \varphi \|_{\mathcal{B}} := \sup_{\theta\leq0} \frac{\|\varphi(\theta)\|}{g(\theta)}$, are phase spaces in the sense considered in this work. Moreover, in these cases $K(s) \equiv 1$ for $s \geq 0$.

Example 2.2 The phase space $\mathcal{PC}_r \times L^2(g\,, X)$.

Let $1 \leq p < \infty$, $0 \leq r < \infty$ and $g(\cdot)$ be a nonnegative Borel measurable function on $(-\infty, r)$ which satisfies the conditions (g-5)-(g-6) in the terminology of [15]. Briefly, this means that $g(\cdot)$ is locally integrable on $(-\infty, -r)$ and that there exists a nonnegative and locally bounded function G on $(-\infty, 0]$ such that $g(\xi + \theta) \leq G(\xi)g(\theta)$ for all $\xi \leq 0$ and $\theta \in (-\infty, -r) \setminus N_\xi$, where $N_\xi \subseteq (-\infty, -r)$ is a set with Lebesgue measure 0.

Let $\mathcal{B} := \mathcal{PC}_r \times L^p(g; X)$, $r \geq 0, p > 1$, be the space formed of all classes of functions $\varphi : (-\infty, 0] \to X$ such that $\varphi |_{[-r,0]} \in \mathcal{PC}([-r, 0], X)$, $\varphi(\cdot)$ is Lebesgue-measurable on $(-\infty, -r]$ and $g\|\varphi\|^p$ is Lebesgue integrable on $(-\infty, -r]$. The seminorm in $\| \cdot \|_{\mathcal{B}}$ is defined by

$$\| \varphi \|_{\mathcal{B}} := \sup_{\theta\in[-r,0]} \| \varphi(\theta) \| + \left(\int_{-\infty}^{-r} g(\theta) \| \varphi(\theta) \|^p \, d\theta \right)^{1/p}.$$

Proceeding as in the proof of [15, Theorem 1.3.8] it follows that $\mathcal{B}$ is a phase space which satisfies the axioms (**A**) and (**B**). Moreover, for $r = 0$ and $p = 2$ this space coincides with $C_0 \times L^2(g,\, X)$, $H = 1$; $M(t) = G(-t)^{\frac{1}{2}}$ and $K(t) = 1 + \left(\int_{-t}^{0} g(\tau)d\tau\right)^{\frac{1}{2}}$, for $t \geq 0$.

Remark 2.1 In retarded functional differential equations without impulses, the axioms of the abstract phase space $\mathcal{B}$ include the continuity of the function $t \to x_t$, see [15, 7] for details. Due to the impulsive effect, this property is not satisfied in impulsive delay systems and, for this reason, has been unconsidered in our description of $\mathcal{B}$.

Remark 2.2 Let $\varphi \in \mathcal{B}$ and $t \leq 0$. The notation φ_t represents the function defined by $\varphi_t(\theta) = \varphi(t+\theta)$. Consequently, if the function $x(\cdot)$ in axiom (**A**) is such that $x_0 = \varphi$, then $x_t = \varphi_t$. We observe that φ_t is well defined for $t < 0$ since the domain of φ is $(-\infty, 0]$. We also note that in general $\varphi_t \notin \mathcal{B}$; consider, for example, functions of the type $x^\mu(t) = (t-\mu)^{-\alpha}\mathcal{X}_{(\mu,0]}$, $\mu > 0$, where $\mathcal{X}_{(\mu,0]}$ is the characteristic function of $(\mu, 0]$, $\mu < -r$ and $\alpha p \in (0,1)$, in the space $\mathcal{PC}_{\mathbf{r}} \times \mathbf{L^p}(\mathbf{g}; \mathbf{X})$.

Additional terminologies and notations used in this paper are standard in functional analysis. In particular, for Banach spaces $(Z, \|\cdot\|_Z)$, $(W, \|\cdot\|_W)$, the notation $\mathcal{L}(Z,W)$ stands for the Banach space of bounded linear operators from Z into W and we abbreviate to $\mathcal{L}(Z)$ whenever $Z = W$. Moreover, $B_r(x, Z)$ denotes the closed ball with center at x and radius $r > 0$ in Z, and for a bounded function $\xi : I \to Z$ and $0 \leq t \leq a$ we employ the notation $\|\xi\|_{Z,t}$ for

$$\|\xi(\theta)\|_{Z,t} = \sup\{\|\xi(s)\|_Z : s \in [0,t]\}. \tag{2.4}$$

We will simply write $\|\xi\|_t$ when no confusion arises. In particular, if $M(\cdot), K(\cdot)$ are the functions in axiom (**A**), then $M_a = \sup_{t\in I} M(t)$ and $K_a = \sup_{t\in I} K(t)$.

This paper has four sections. In Section 3 we establish the existence of mild solutions for system (1.1)-(1.3). Section 4 is reserved for examples.

To conclude the current section, we recall the following well-known result.

Theorem 2.1 *[6, Theorem 6.5.4].* **(Leray-Schauder Alternative)** *Let D be a closed convex subset of a Banach space Z and assume that $0 \in D$. Let $\Gamma : D \to D$ be a completely continuous map. Then, either the set $\{z \in D : z = \lambda\Gamma(z),\ 0 < \lambda < 1\}$ is unbounded or the map Γ has a fixed point in D.*

3 Existence Results

In this section we discuss the existence of mild solutions for the abstract system (1.1)-(1.3). To prove our results we always assume that $\varphi \in \mathcal{B}$ and that $\rho : I \times \mathcal{B} \to (-\infty, a]$ is a continuous function and $(Y, \|\cdot\|_Y)$ is a Banach space continuously included in X. Additionally, we introduce the following conditions.

- $\mathbf{H_1}$ For every $y \in Y$, the function $t \to T(t)y$ is continuous from $[0,\infty)$ into Y. Moreover, $T(t)(Y) \subset D(A)$ for every $t > 0$ and there exists a positive function $\gamma \in L^1([0,a])$ such that $\|AT(t)\|_{\mathcal{L}(Y;X)} \leq \gamma(t)$, for every $t \in I$.

- $\mathbf{H_2}$ Let $\mathcal{R}(\rho^-) = \{\rho(s,\psi) : (s,\psi) \in I \times \mathcal{B},\ \rho(s,\psi) \leq 0\}$. The function $t \to \varphi_t$ is well defined from $\mathcal{R}(\rho^-)$ into $\mathcal{B}$ and there exists a continuous and bounded function $J^\varphi : \mathcal{R}(\rho^-) \to \mathbb{R}$ such that $\| \varphi_t \|_{\mathcal{B}} \leq J^\varphi(t) \| \varphi \|_{\mathcal{B}}$ for every $t \in \mathcal{R}(\rho^-)$.

$\mathbf{H_3}$ The function $F : I \times \mathcal{B} \to X$ satisfies the following conditions:

(i) Let $x : (-\infty, a] \to X$ be such that $x_0 = \varphi$ and $x|_I \in \mathcal{PC}$. The function $t \to F(t, x_{\rho(t,x_t)})$ is measurable on I and the function $t \to F(s, x_t)$ is continuous on $\mathcal{R}(\rho^-) \cup I$ for every $s \in I$.

(ii) For each $t \in I$, the function $F(t, \cdot) : \mathcal{B} \to X$ is continuous.

(iii) There exists an integrable function $m : I \to [0, \infty)$ and a continuous nondecreasing function $W : [0, \infty) \to (0, \infty)$ such that

$$\| F(t, \psi) \| \quad \leq \quad m(t)W(\| \psi \|_{\mathcal{B}}), \qquad (t, \psi) \in I \times \mathcal{B}.$$

$\mathbf{H_4}$ The function G is Y- valued, $G : I \times \mathcal{B} \to Y$ is continuous and there exist a positive constants c_1, c_2 such that $\|G(t, \psi)\|_Y \leq c_1 \|\psi\|_{\mathcal{B}} + c_2, \ \forall \ \ (t, \psi) \in I \times \mathcal{B}$.

$\mathbf{H_5}$ The function G is Y- valued, $G : I \times \mathcal{B} \to Y$ is continuous and there exists $L_G > 0$ such that

$$\|G(t, \psi_1) - G(t, \psi_2)\|_Y \leq L_G \|\psi_1 - \psi_2\|_{\mathcal{B}}, \quad (t, \psi_i) \in I \times \mathcal{B}, \ i = 1, 2.$$

$\mathbf{H_6}$ The maps I_i are completely continuous and there are positive constants c_i^j, $j = 1, 2$, such that $\|I_i(\psi)\| \leq c_i^1 \|\psi\|_{\mathcal{B}} + c_i^2, i = 1, 2, ..., n$, for every $\psi \in \mathcal{B}$.

$\mathbf{H_7}$ The functions $I_i : \mathbb{R} \times \mathcal{B} \to X$ are continuous and there are positive constants $L_i, i = 1, 2, ..., n$, such that

$$\|I_i(\psi_1) - I_i(\psi_2)\| \leq L_i \|\psi_1 - \psi_2\|_{\mathcal{B}}, \quad \psi_j \in \mathcal{B}, \quad j = 1, 2, \ i = 1, 2, ..., n.$$

$\mathbf{H_8}$ Let $S(a) = \{x : (-\infty, a] \to X : x_0 = 0; x|_I \in \mathcal{PC}\}$ endowed with the norm of uniform convergence on I and $y : (-\infty, a] \to X$ be the function defined by $y_0 = \varphi$ on $(-\infty, 0]$ and $y(t) = T(t)\varphi(0)$ on I. Then, for every bounded set $Q \subset S(a)$, the set of functions $\{t \to G(t, x_t + y_t) : x \in Q\}$ is equicontinuous on I.

Remark 3.3 The condition (**$\mathbf{H_2}$**) is frequently satisfied by functions that are continuous and bounded. In fact, assume that the space of continuous and bounded functions $C_b((-\infty, 0], X)$ is continuously included in $\mathcal{B}$. Then, there exists L > 0 such that

$$\| \psi_t \|_{\mathcal{B}} \leq L \frac{\sup_{\theta \leq 0} \| \psi(\theta) \|}{\| \psi \|_{\mathcal{B}}} \| \psi \|_{\mathcal{B}}, \qquad t \leq 0, \, \psi \neq 0, \, \psi \in C_b((-\infty, 0] : X).$$

It is easy to see that the space $C_b((-\infty, 0], X)$ is continuously included in $\mathcal{PC}_g(X)$ and $\mathcal{PC}_g^0(X)$. Moreover, if $g(\cdot)$ verifies (g-5)-(g-6) in [15] and $g(\cdot)$ is integrable on $(-\infty, -r]$, then the space $C_b((-\infty, 0], X)$ is also continuously included in $\mathcal{PC}_r \times L^p(g; X)$. For complementary details related this matter, see Proposition 7.1.1 and Theorems 1.3.2 and 1.3.8 in [15].

Motivated by general semigroup theory, we adopt the following concept of mild solution.

Definition 3.1 *A function $x : (-\infty, a] \to X$ is called a mild solution of the abstract Cauchy problem (1.1)-(1.3) if $x_0 = \varphi$; $x_{\rho(s,x_s)} \in \mathcal{B}$ for every $s \in I$; the function $t \to AT(t-s)G(s, x_s)$ is integrable on $[0, t)$, for every $t \in [0, a]$; and*

$$x(t) = T(t)[\varphi(0) + G(0, \varphi)] - G(t, x_t) - \int_0^t AT(t-s)G(s, x_s)ds + \int_0^t T(t-s)F(s, x_{\rho(s,x_s)})ds + \sum_{0<t_i<t} T(t-t_i)I_i(x_{t_i}),\ t \in I.$$

Remark 3.4. Let $x(\cdot)$ be a function as in axiom $(\mathbf{A})$. Let us mention that the conditions $(\mathbf{H_1}), (\mathbf{H_4}), (\mathbf{H_5})$ are linked to the integrability of the function $s \to AT(t-s)G(s, x_s)$. In general, except for the trivial case in which A is a bounded linear operator, the operator function $t \to AT(t)$ is not integrable over I. However, if condition $(\mathbf{H_1})$ holds and G satisfies either assumption $(\mathbf{H_4})$ or $(\mathbf{H_5})$, then it follows from Bochner's criterion and the estimate

$$\begin{aligned} \|AT(t-s)G(s, x_s)\| &\leq \|AT(t-s)\|_{\mathcal{L}(Y;X)} \|G(s, x_s)\|_Y \\ &\leq \gamma(t-s) \sup_{s \in I} \|G(s, x_s)\|_Y, \end{aligned}$$

that $s \to AT(t-s)G(s, x_s)$ is integrable over $[0, t)$, for every $t \in I$.

In the next Lemma, M_a, K_a are defined using the notation introduced in (2.4).

Lemma 3.1 *[13, Lemma 2.1] Let $x : (-\infty, a] \to X$ be a function such that $x_0 = \varphi$ and $x|_I \in \mathcal{PC}$. Then*

$$\| x_s \|_{\mathcal{B}} \leq (M_a + J_0^\varphi) \| \varphi \|_{\mathcal{B}} + K_a \sup\{ \| x(\theta) \|; \theta \in [0, \max\{0, s\}] \}, \quad s \in \mathcal{R}(\rho^-) \cup I,$$

where $J_0^\varphi = \sup_{t \in \mathcal{R}(\rho^-)} J^\varphi(t)$.

Theorem 3.1 *Let conditions $(\mathbf{H_1}) - (\mathbf{H_3}), (\mathbf{H_5})$ and $(\mathbf{H_7})$ be hold. If*

$$K_a \Big[L_G \left(1 + \int_0^a \gamma(s)ds\right) + \widetilde{M} \liminf_{\xi \to \infty^+} \frac{W(\xi)}{\xi} \int_0^a m(s)ds + \widetilde{M} \sum_{i=1}^n L_i \Big] < 1, \tag{3.1}$$

then there exists a mild solution of (1.1)-(1.3).

Proof: Consider the space $Y = \{u \in \mathcal{PC} : u(0) = \varphi(0)\}$ endowed with the norm $\|u\|_a = \sup_{s \in I} \|u(s)\|$, and define the operator $\Gamma : Y \to Y$ by

$$\begin{aligned} \Gamma x(t) &= T(t)[\varphi(0) + G(0, \varphi)] - G(t, \bar{x}_t) - \int_0^t AT(t-s)G(s, \bar{x}_s)ds \\ &\quad + \int_0^t T(t-s)F(s, \bar{x}_{\rho(s,\bar{x}_s)})ds + \sum_{0<t_i<t} T(t-t_i)I_i(\bar{x}_{t_i}), \quad t \in I, \end{aligned}$$

where $\bar{x} : (-\infty, a] \to X$ is such that $\bar{x}_0 = \varphi$ and $\bar{x} = x$ on I. From our assumptions it is easy to see that $\Gamma x \in \mathcal{PC}$.

We claim that there exists $r > 0$ such that $\Gamma(B_r(0,Y)) \subset B_r(0,Y)$. If we assume this property is false, then for every $r > \| \varphi \|$ there exist $x^r \in B_r(0,Y)$ and $t^r \in I$ such that $r < \| \Gamma x^r(t^r) \|$. Then, by using Lemma 3.1 we find that

$$
\begin{aligned}
r &< \| \Gamma x^r(t^r) \| \\
&\leq \widetilde{M}H \| \varphi \|_{\mathcal{B}} + \| T(t^r)G(0,\varphi) - G(t^r,\varphi) \| + \| G(t^r, \overline{(x^r)}_{t^r}) - G(t^r,\varphi) \| \\
&\quad + \widetilde{M} \int_0^{t^r} m(s) W(\| \overline{x^r}_{\rho(s,\overline{(x^r)}_s)} \|_{\mathcal{B}}) ds + \int_0^{t^r} \|AT(t^r - s)\|_{\mathcal{L}(Y;X)} \|G(s, \overline{(x^r)}_s) - G(s,\varphi)\| ds \\
&\quad + \int_0^{t^r} \|AT(t^r - s)\|_{\mathcal{L}(Y;X)} \|G(s,\varphi)\| ds + \widetilde{M} \sum_{i=1}^{n} (L_i \| \overline{x}_{t_i} \|_{\mathcal{B}} + \| I_i(0) \|) \\
&\leq \widetilde{M}H \| \varphi \|_{\mathcal{B}} + \| T(t^r)G(0,\varphi) - G(t^r,\varphi) \| + L_G(K_a r + (M_a + 1)\|\varphi\|) \\
&\quad + L_G(K_a r + (M_a + 1)\|\varphi\|) \int_0^a \gamma(s) ds + \|G(s,\varphi)\|_a \int_0^a \gamma(s) ds \\
&\quad + \widetilde{M} W\left((M_a + J_0^{\varphi}) \| \varphi \|_{\mathcal{B}} + K_a r\right) \int_0^{t^r} m(s) ds \\
&\quad + \widetilde{M} \sum_{i=1}^{n} (L_i (K_a r + M_a \|\varphi\|) + \| I_i(0) \|),
\end{aligned}
$$

and hence

$$
1 \leq K_a \left[L_G \left(1 + \int_0^a \gamma(s) ds \right) + \widetilde{M} \liminf_{\xi \to \infty} \frac{W(\xi)}{\xi} \int_0^a m(s) ds + \widetilde{M} \sum_{i=1}^{n} L_i \right],
$$

which is contrary to our assumption.

Let $r > 0$ be such that $\Gamma(B_r(0,Y)) \subset B_r(0,Y)$. Next, we will prove that Γ is a condensing map on $B_r(0,Y)$. Consider the decomposition $\Gamma = \Gamma_1 + \Gamma_2$ where

$$
\begin{aligned}
\Gamma_1 x(t) &= T(t)[\varphi(0) + G(0,\varphi)] - G(t, \bar{x}_t) - \int_0^t AT(t-s) G(s, \bar{x}_s) ds \\
&\quad + \sum_{0 < t_i < t} T(t - t_i) I_i(\bar{x}_{t_i}), \quad t \in I, \\
\Gamma_2 x(t) &= \int_0^t T(t-s) F(s, \bar{x}_{\rho(s,\bar{x}_s)}) ds, \qquad t \in I.
\end{aligned}
$$

Proceeding as in the proof of [11, Theorem 3.1] we can conclude that Γ is continuous and that Γ_2 is completely continuous. Moreover, from the estimate

$$
\|\Gamma_1 u - \Gamma_1 v\|_{\mathcal{PC}} \leq K_a \Big[L_G \left(1 + \int_0^a \gamma(s) ds \right) + \widetilde{M} \sum_{i=1}^{n} L_i \Big] \|u - v\|_{\mathcal{PC}}, \qquad u, v \in B_r(0,Y),
$$

it follows that Γ_1 is a contraction on $B_r(0,Y)$.

These remarks prove that Γ is a condensing operator from $B_r(0,Y)$ into $B_r(0,Y)$. Now, the existence of a mild solution is a consequence of [17, Theorem 4.3.2]. The proof is complete. ∎

Theorem 3.2 *Assume that conditions* $(\mathbf{H_1})-(\mathbf{H_4}),(\mathbf{H_6})$ *and* $(\mathbf{H_8})$ *are satisfied. Further, assume that* $\rho(t,\psi)\leq t$ *for every* $(t,\psi)\in I\times\mathcal{B}$ *and that* $G: I\times\mathcal{B}\to X$ *is completely continuous. If* $\mu=\left[1-c_1K_a(1+\int_0^a\gamma(s)ds)-\widetilde{M}K_a\sum_{i=1}^n c_i^1\right]>0$ *and*

$$\frac{\widetilde{M}K_a}{\mu}\int_0^a m(s)ds<\int_D^\infty\frac{ds}{W(s)},$$

where $D=(M_a+J_0^\varphi+\widetilde{M}HK_a)\|\varphi\|_{\mathcal{B}}+\frac{K_aC}{\mu}$ *and* $C=\widetilde{M}\|G(0,\varphi)\|+\widetilde{M}\sum_{i=1}^n c_i^2+(\widetilde{M}\sum_{i=1}^n c_i^1+c_1)(M_a+J_0^\varphi+\widetilde{M}HK_a)\|\varphi\|_{\mathcal{B}}+c_1(M_a+J_0^\varphi+\widetilde{M}HK_a)\|\varphi\|_{\mathcal{B}}\int_0^a\gamma(s)ds+c_2(1+\int_0^a\gamma(s)ds)$, *then there exists a mild solution of (1.1)-(1.3).*

Proof: On the space $\mathcal{BPC}=\{u:(-\infty,a]\to X, u_0=0, u|_I\in\mathcal{PC}\}$ endowed with the norm $\|\cdot\|_{\mathcal{PC}}$, we define the operator $\Gamma:\mathcal{BPC}\to\mathcal{BPC}$ by $(\Gamma u)_0=0$ and

$$\begin{aligned}\Gamma x(t) &= T(t)G(0,\varphi)-G(t,\bar{x}_t)-\int_0^t AT(t-s)G(s,\bar{x}_s)ds\\ &\quad+\int_0^t T(t-s)F(s,\bar{x}_{\rho(s,\bar{x}_s)})ds+\sum_{0<t_i<t}T(t-t_i)I_i(\bar{x}_{t_i}),\qquad t\in I,\end{aligned}$$

where $\bar{x}=x+y$ on $(-\infty,a]$ and $y(\cdot)$ is the function introduced in $(\mathbf{H_8})$. In order to use Theorem 2.1, we establish a priori estimates for the solutions of the integral equation $z=\lambda\Gamma z,\lambda\in(0,1)$. By using Lemma 3.1, the notation $\alpha^\lambda(s)=\sup_{\theta\in[0,s]}\|x^\lambda(\theta)\|$, and the fact that $\rho(s,\overline{(x^\lambda)}_s)\leq s$, for each $s\in I$, we find that

$$\begin{aligned}\|x^\lambda(t)\| &\leq \|T(t)G(0,\varphi)\|+c_1\|\overline{(x^\lambda)}_t\|_{\mathcal{B}}+c_2+\int_0^t\gamma(t-s)(c_1\|\overline{(x^\lambda)}_s\|_{\mathcal{B}}+c_2)ds\\ &\quad+\widetilde{M}\int_0^t m(s)W(\|\overline{(x^\lambda)}_s\|)ds+\widetilde{M}\sum_{0<t_i\leq t}c_i^1[\|\overline{(x^\lambda)}_{t_i}\|_{\mathcal{B}}]+\widetilde{M}\sum_{i=1}^n c_i^2\\ &\leq \widetilde{M}\|G(0,\varphi)\|+c_1\|\varphi\|_{\mathcal{B}}[M_a+J_0^\varphi+K_a\widetilde{M}H+(M_a+J_0^\varphi+K_a\widetilde{M}H)\int_0^a\gamma(s)ds]\\ &\quad+c_2(1+\int_0^a\gamma(s)ds)+c_1K_a\alpha^\lambda(t)(1+\int_0^a\gamma(s)ds)\\ &\quad+\widetilde{M}\int_0^t m(s)W((M_a+J_0^\varphi+K_a\widetilde{M}H)\|\varphi\|_{\mathcal{B}}+K_a\alpha^\lambda(s))ds+\widetilde{M}\sum_{i=1}^n c_i^2\\ &\quad+\widetilde{M}\sum_{0<t_i\leq t}c_i^1((M_a+J_0^\varphi+K_a\widetilde{M}H)\|\varphi\|_{\mathcal{B}}+K_a\alpha^\lambda(t)).\end{aligned}$$

Consequently,

$$\alpha^\lambda(t)\leq\frac{C}{\mu}+\frac{\widetilde{M}}{\mu}\int_0^t m(s)W((M_a+J_0^\varphi+K_a\widetilde{M}H)\|\varphi\|_{\mathcal{B}}+K_a\alpha^\lambda(s))ds,$$

where

$$\begin{aligned}C &= \widetilde{M}\|G(0,\varphi)\|+\widetilde{M}\sum_{i=1}^n c_i^2+(\widetilde{M}\sum_{i=1}^n c_i^1+c_1)(M_a+J_0^\varphi+\widetilde{M}HK_a)\|\varphi\|_{\mathcal{B}}\\ &\quad+c_1(M_a+J_0^\varphi+\widetilde{M}HK_a)\|\varphi\|_{\mathcal{B}}\int_0^a\gamma(s)ds+c_2(1+\int_0^a\gamma(s)ds).\end{aligned}$$

If $\zeta^\lambda(t) = (M_a + J_0^\varphi + \widetilde{M}HK_a)\|\varphi\|_{\mathcal{B}} + K_a\alpha^\lambda(t)$,

$$\begin{aligned}\zeta^\lambda(t) &\le (M_a + J_0^\varphi + \widetilde{M}HK_a)\|\varphi\|_{\mathcal{B}} + K_a\Big[\frac{C}{\mu} + \frac{\widetilde{M}}{\mu}\int_0^t m(s)W(\zeta^\lambda(s))ds\Big] \\ &\le (M_a + J_0^\varphi + \widetilde{M}HK_a)\|\varphi\|_{\mathcal{B}} + \frac{K_aC}{\mu} + \frac{K_a\widetilde{M}}{\mu}\int_0^t m(s)W(\zeta^\lambda(s))ds\end{aligned}$$

Denoting by $\beta_\lambda(t)$ the right-hand side of the last inequality, it follows that,

$$\beta'_\lambda(t) \le \frac{K_a\widetilde{M}}{\mu}m(t)W(\beta_\lambda(t))$$

and hence

$$\int_{\beta_\lambda(0)=D}^{\beta_\lambda(t)} \frac{ds}{W(s)} \le \frac{\widetilde{M}K_a}{\mu}\int_0^a m(s)ds < \int_D^\infty \frac{ds}{W(s)}, \quad t \in I$$

where $D = (M_a + J_0^\varphi + \widetilde{M}HK_a)\|\varphi\|_{\mathcal{B}} + \frac{K_aC}{\mu}$, which implies that the set of functions $\{\beta_\lambda(\cdot) : \lambda \in (0,1)\}$ is bounded in $C(I;R)$. Thus, $\{x^\lambda(\cdot) : \lambda \in (0,1)\}$ is bounded on $\mathcal{BPC}$.

To prove that Γ is completely continuous, we introduce the decomposition $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3$ where $(\Gamma_i x)_0 = 0, i = 1,2,3.$ and

$$\begin{aligned}\Gamma_1 x(t) &= T(t)G(0,\varphi) - G(t,\bar{x}_t) + \int_0^t T(t-s)F(s,\bar{x}_{\rho(s,\bar{x}_s)})ds, \quad t \in I, \\ \Gamma_2 x(t) &= -\int_0^t AT(t-s)G(s,\bar{x}_s)ds, \quad t \in I, \\ \Gamma_3 x(t) &= \sum_{0<t_i<t} T(t-t_i)I_i(\bar{x}_{t_i}), \quad t \in I.\end{aligned}$$

From the proof of [11, Theorem 3.1] and our assumptions on G we infer that Γ_1 is completely continuous and easily we can prove that Γ_2 is continuous. It remains to show that Γ_2 is compact and that Γ_3 is completely continuous. Now, by using the proof of [14, Theorem 3.2] together with the Arzela-Ascoli theorem we conclude that Γ_2 is completely continuous. Next, by using Lemma 2.1, the continuiuty of Γ_3 can be proven using phase space axioms. On the other hand for $r > 0, t \in [t_i, t_{i+1}] \cap (0,a]$, $i \ge 1$, and $u \in B_r = B_r(0,\mathcal{BPC})$, we find that

$$\widetilde{\Gamma_3 u}(t) \in \begin{cases} \sum_{j=1}^{i} T(t-t_j)I_j(B_{r^*}(0,X)), & t \in (t_i, t_{i+1}), \\ \sum_{j=0}^{i} T(t_{i+1}-t_j)I_j(B_{r^*}(0,X)), & t = t_{i+1}, \\ \sum_{j=1}^{i-1} T(t_i-t_j)I_j(B_{r^*}(0,X)) + I_i(B_{r^*}(0,X)), & t = t_i, \end{cases}$$

where $r^* = (M_a + H\widetilde{M})\|\varphi\|_{\mathcal{B}} + K_a r$, which proves that $[\widetilde{\Gamma_3(B_r)}]_i(t)]$ is relatively compact in X, for every $t \in [t_i, t_{i+1}]$, since the maps I_j are completely continuous. Moreover, using the compactness of the operators I_i and the strong continuity of $(T(t))_{t\geq 0}$, we can prove that $[\widetilde{\Gamma_3(B_r)}]_i(t)]$ is equicontinuous at t, for every $t \in [t_i, t_{i+1}]$. Now, from Lemma 2.1, we conclude that Γ_3 is completely continuous.

These remarks, in conjunction with Theorem 2.1 show that Γ has a fixed point $x \in \mathcal{BPC}$. Clearly, the function $u = x + y$ is a mild solution of (1.1)-(1.3). The proof is now complete. ■

4 Example

In this section, we consider an applications of our abstract results. At first we introduce the required technical framework. In the rest of this secion, $X = L^2([0,\pi])$ and A be the operator $Au = u''$ with domain $D(A) = \{u \in X : u'' \in X, u(0) = u(\pi) = 0\}$. It is well known that A is the infinitesimal generator of an analytic semigroup on X. Furthermore, A has a discrete spectrum with eigen values of the form $-n^2, n \in N$, whose corresponding (normalized) eigen functions are given by $z_n(\zeta) = \sqrt{\frac{2}{\pi}}\sin(n\zeta)$. In addition, the following properties hold.

(a) $\{z_n : n \in N\}$ is an orthonormal basis of X;

(b) For $u \in X, T(t)u = \sum_{n=1}^{\infty} e^{-n^2 t} < u, z_n > z_n$ and $Au = -\sum_{n=1}^{\infty} n^2 < u, z_n > z_n$, for $u \in D(A)$;

(c) It is possible to define the fractional power $(-A)^\alpha, \alpha \in (0,1)$, as a closed linear operator over its domain $D((-A)^\alpha)$. More precisely, the operator $(-A)^\alpha : D((-A)^\alpha) \subseteq X \to X$ is given by $(-A)^\alpha u = \sum_{n=1}^{\infty} n^{2\alpha} < u, z_n > z_n$, for all $u \in D(-A)^\alpha$, where $D(-A)^\alpha = \{u \in X : \sum_{n=1}^{\infty} n^{2\alpha} < u, z_n > z_n \in X\}$;

(d) If X_α is the space $D(-A)^\alpha$ endowed with the graph norm $\|\cdot\|_\alpha$, then X_α is a Banach space. Moreover, for $0 < \beta \leq \alpha \leq 1, X_\alpha \subset X_\beta$; the inclusion $X_\alpha \to X_\beta$ is completely continuous and there are constans $C_\alpha > 0$ such that $\|T(t)\|_{\mathcal{L}(X_\alpha;X)} \leq \frac{C_\alpha}{t^\alpha}$ for $t \geq 0$.

Consider the differential system

$$\frac{d}{dt}\Big[u(t,\zeta) + \int_{-\infty}^{t}\int_{0}^{\pi} b(t-s,\eta,\zeta)u(s,\eta)d\eta ds\Big] = \frac{\partial^2}{\partial\zeta^2}u(t,\zeta)$$
$$+ \int_{-\infty}^{t} a(s-t)u(s-\rho_1(t)\rho_2(\|u(t)\|),\zeta)ds, \quad t \in I,\ \zeta \in [0,\pi] \tag{4.1}$$
$$u(t,0) = u(t,\pi) = 0, \quad t \in I \tag{4.2}$$
$$u(\tau,\zeta) = \varphi(\tau,\zeta), \quad \tau \leq 0, \quad 0 \leq \zeta \leq \pi \tag{4.3}$$
$$\Delta u(t_j,\zeta) = \int_{-\infty}^{t_j} \gamma_j(s-t_j)u(s,\zeta)ds, \quad j = 1,2,\ldots,n. \tag{4.4}$$

where $\varphi \in \mathcal{B} = \mathcal{PC}_0 \times L^2(g,X)$ and $0 < t_1 < t_2 < \cdots < t_n < a$ are prefixed.

To treat this system, we will assume that $g(\cdot)$ satisfies the conditions $(g-5)-(g-7)$ in [15]. We know from Theorem 1.37 and 7.1.1 in [15] that $C_b((-\infty,0];X)$ is continuously included in $\mathcal{B}$. Additionally we assume that the functions $\rho_i : [0,\infty) \to [0,\infty),\ i = 1,2.\ \ a : R \to R$ are continuous; $L_F = \left(\int_{-\infty}^{0} \frac{(a^2(s))}{g(s)} ds\right)^{\frac{1}{2}} < \infty$ and that the following conditinos holds.

(a) The funtions $\gamma_i : R \to R, i = 1,2,\ldots,n$, are continuous, bounded and for every $i = 1,2,\ldots,n,\ L_i = \left(\int_{-\infty}^{0} \frac{(\gamma_i(s))^2}{g(s)} ds\right)^{\frac{1}{2}} < \infty$.

(b) The functions $b(s,\eta,\zeta), \frac{\partial b(s,\eta,\zeta)}{\partial \zeta}$ are measurable, $b(s,\eta,\pi) = b(s,\eta,0) = 0$ and $L_g = \max\left\{\left(\int_0^{\pi}\int_{-\infty}^{0}\int_0^{\pi} \frac{1}{g(s)}\left(\frac{\partial^i b(s,\eta,\zeta)}{\partial \zeta^i}\right)^2 d\eta ds d\zeta\right)^{\frac{1}{2}} : i = 0,1\right\} < \infty.$

Under these conditions, we can define the operators, $\rho, G, F : I \times \mathcal{B} \to X$ and $I_i : \mathcal{B} \to X$ by

$$\begin{aligned}
\rho(t,\psi) &= \rho_1(t)\rho_2(\|\psi(0)\|),\\
G(\psi)(\zeta) &= \int_{-\infty}^{0}\int_0^{\pi} b(s,\nu,\zeta)\psi(s,\nu)d\nu ds,\\
F(\psi)(\zeta) &= \int_{-\infty}^{0} a(s)\psi(s,\zeta)ds\\
I_i(\psi)(\zeta) &= \int_{-\infty}^{\infty} \gamma_i(s)\psi(s,\zeta)ds, \quad i = 1,2,\ldots,n,
\end{aligned}$$

which permit to transform system (4.1)-(4.4) into the system (1.1)-(1.3). Moreover, the maps, $G, F, I_i, i = 1,2,\ldots,n$ are bounded linear operators with $\|G\|_{\mathcal{L}(X)} \le L_G$ and $\|F\|_{\mathcal{L}(X)} \le L_F$ and $\|I_i\|_{\mathcal{L}(X)} \le L_i$, for every $j = 1,2,\ldots,n$.

Moreover, a straightforward estimation invloving (a) enables us to prove that G is $D(-A)^{\frac{1}{2}}$-valued with $\|(-A)^{\frac{1}{2}}G\| \le L_G$, which implies that G is completely continuous from $I \times \mathcal{B}$ into X since the inclusion $i : X_{\frac{1}{2}} \to X$ is completely continuous. Thus, the assumptions $(\mathbf{H_1}), (\mathbf{H_4})$ and $(\mathbf{H_5})$ are hold with $Y = X_{\frac{1}{2}}$.

From the Theorem 3.1 and Remark 3.3, we deduce the following propositions immediately.

Proposition 4.1 *Assume that condition* $(\mathbf{H_2})$ *holds and that the functions* ρ_1, ρ_2 *are bounded. If*

$$K_a\left(L_G + 2C_1\sqrt{a} + aL_F + \sum_{i=1}^{n} L_i\right) < 1,$$

there exists a mild solution of (4.1)-(4.4).

Proposition 4.2 *Assume that* $\varphi \in C_b((-\infty,0);X)$. *If*

$$K_a\left(L_G + 2C_1\sqrt{a} + aL_F + \sum_{i=1}^{n} L_i\right) < 1,$$

there exists a mild solution of (4.1)-(4.4).

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References

[1] Aiello, Walter; Freedman, H.I.; Wu, J., Analysis of a model representing stage-structured population growth with state-dependent time delay, *SIAM J. Appl. Math.,* Vol. 52(3)(1992), 855–869.

[2] A. Anguraj, M. Mallika Arjunan and E. Hernández, Existence results for an impulsive neutral functional differential equation with state-dependent delay, *Applicable Analysis*, Vol. 86(7)(2007), 861-872.

[3] D. D. Bainov and P. S. Simeonov, *Systems with Impulse Effect*, Ellis Horwood Ltd., Chichister, 1989.

[4] Bartha, Maria, Periodic solutions for differential equations with state-dependent delay and positive feedback, *Nonlinear Analysis TMA.,* Vol. 53(6)(2003), 839–857.

[5] Driver, R. D., A neutral system with state-dependent delay. *J. Differential Equations,* Vol. 54(1)(1984), 73–86.

[6] Granas, A. and Dugundji, J., *Fixed Point Theory.* Springer-Verlag, New York, 2003.

[7] J. K. Hale and J. Kato, Phase space for retarded equations with infinite delay, *Funckcial. Ekvac.,* Vol. 21(1)(1978), 11–41.

[8] Hartung, Ferenc; Turi, Janos, Identification of parameters in delay equations with state-dependent delays, *Nonlinear Analysis TMA.,* Vol. 29(11)(1997), 1303–1318.

[9] Hartung, Ferenc; Herdman, Terry L; Turi, Janos, Parameter identification in classes of neutral differential equations with state-dependent delays, *Nonlinear Analysis TMA.,* Vol. 39(3)(2000), 305–325.

[10] Hartung, Ferenc, Parameter estimation by quasilinearization in functional differential equations with state-dependent delays: a numerical study, Proceedings of the Third World Congress of Nonlinear Analysis, Part 7 (Catania, 2000). *Nonlinear Analysis TMA.,* Vol. 47(7)(2001), 4557–4566.

[11] Hernández, E; M. Pierri and G. Goncalves, Existence results for an impulsive abstract partial differential equation with state-dependent delay, *Comput. Appl. Math.,* 52(2006), 411–420.

[12] Hernández, E; Mark A. Mckibben, On state- dependent delay partial neutral functional-differential equations, *Appl. Math. Comput.,* Vol. 186(1)(2007), 294–301

[13] Hernández, E; Prokopczyk, A; Ladeira, Luiz, A note on partial functional differential equations with state-dependent delay, *Nonlinear Analysis: Real World Applications,* 7(2006), 510–519.

[14] Hernández, E; Mark A. Mckibben and Hernan R. Henriquez, Existence results for partial neutral functional differential equations with state-dependent delay, *Mathematical and Computer Modelling,* 49(2009), 1260–1267.

[15] Hino, Yoshiyuki; Murakami, Satoru; Naito, Toshiki, *Functional-differential equations with infinite delay,* Lecture Notes in Mathematics, 1473. Springer-Verlag, Berlin, 1991.

[16] V. Lakshmikantham, D. D. Bainov, and P. S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, Singapore, 1989.

[17] Martin, R. H., *Nonlinear Operators and Differential Equations in Banach Spaces,* Robert E. Krieger Publ. Co., Florida, 1987.

[18] Pazy, A., *Semigroups of linear operators and applications to partial differential equations,* Applied Mathematical Sciences, 44. Springer-Verlag, New York-Berlin, 1983.

[19] A. M. Samoilenko and N. A. Perestyuk, *Impulsive Differential Equations*, World Scientific, Singapore, 1995.

[20] Yang, Zhihui; Cao, Jinde, Existence of periodic solutions in neutral state-dependent delays equations and models, *J. Comput. Appl. Math.,* Vol. 174(1)(2005), 179–199.

[21] Alexander V. Rezounenko, Partial differential equations with discrete and distributed state-dependent delays, *J. Math. Anal. Appl.,* Vol. 326(2)(2007), 1031–1045.

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