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Positive solutions of boundary value problems for systems of second-order differential equations with integral boundary condition on the half-line

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Abstract In this paper, we study the existence of positive solutions of boundary value problems for systems of second-order differential equations with integral boundary condition on the half-line. By using the fixed-point theorem in cones, we show the existence of at least one positive solution with suitable growth conditions imposed on the nonlinear terms. Moreover, the associated integral kernels for the boundary value problems are given.

MSC 34B10; 34B15; 34B18; 34B40

Keywords Boundary value problems; Positive solutions; Fixed-point theorem in cones; Integral boundary condition

1 Introduction

In this paper we consider the existence of positive solutions for second-order boundary value problems (BVPs) for systems of differential equations with integral boundary condition on the half-line:

$$\begin{gathered} u_1''(t) + f_1(t, u_1(t), u_2(t)) = 0,\ t \in (0, \infty), \\ u_2''(t) + f_2(t, u_1(t), u_2(t)) = 0,\ t \in (0, \infty), \\ u_1(0) = u_2(0) = 0, \\ u_1'(\infty) = \int_0^{+\infty} g_1(s)u_1(s)\mathrm{d}s,\ \ u_2'(\infty) = \int_0^{+\infty} g_2(s)u_2(s)\mathrm{d}s, \end{gathered} \tag{1.1}$$

where $u_i'(\infty) = \lim_{t\to+\infty} u_i'(t),\ i = 1, 2$.

Nonlocal BVPs have been well studied especially on a compact interval. Gupta and co-authors, for example [1], has made an extensive study of multi-point BVPs. The existence of *positive* solutions was studied by Ma in 1999 for a type of three-point boundary value problem [2]. Many authors have studied the existence of positive solutions for multi-point boundary value problems, and obtained many sufficient conditions for the existence of positive solutions. See [3] for a survey of some of the work done and for references to many contributions.

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Boundary value problems with Riemann-Stieltjes integral boundary conditions (BCs) are now being studied since they include BVPs with two-point, multipoint and integral BCs as special cases. See [4]–[16] and the references therein.

In 2000, Ma [4] considered the existence of positive solutions for second order ordinary differential equations

$$u'' + a(t)f(u) = 0$$
$$u(0) = 0, \quad u(1) = \int_\alpha^\beta h(t)u(t)\mathrm{d}t,$$

where $[\alpha, \beta] \subset (0, 1)$ and f is either superlinear or sublinear, by reducing this to a three point BVP. Karakostas and Tsamatos [5]–[8] weakened the restrictions on the function f and considered boundary conditions given by Riemann-Stieltjes integral which improved the result of Ma [4]. This was further improved in [15].

Yang [10]–[12] investigated the boundary value problems of differential equations with some nonlinear BCs

$$-(au')' + bu = g(t)f(t, u), \quad t \in [0, 1]$$
$$\cos\gamma_0 u(0) - \sin\gamma_0 u'(0) = H_1\big(\int_0^1 u(\tau)\mathrm{d}\alpha(\tau)\big),$$
$$\cos\gamma_1 u(1) + \sin\gamma_1 u'(1) = H_2\big(\int_0^1 u(\tau)\mathrm{d}\beta(\tau)\big).$$

Afterward, see [14], he studied the boundary value problem for systems of second-order differential equations with integral boundary condition.

In [15, 16], Webb and Infante used fixed point index theory and gave a general method for solving problems with integral BCs of Riemann-Stieltjes type. In [16] they studied the existence of multiple positive solutions of nonlinear differential equations of the form

$$-u''(t) = g(t)f(t, u(t)), \quad t \in (0, 1)$$

with boundary conditions including the following

$$u(0) = \alpha[u], u(1) = \beta[u],$$
$$u(0) = \alpha[u], u'(1) = \beta[u],$$
$$u(0) = \alpha[u], u'(1) + \beta[u] = 0,$$
$$u'(0) = \alpha[u], u(1) = \beta[u],$$
$$u'(0) + \alpha[u] = 0, u(1) = \beta[u],$$

where α, β are linear functionals on $C[0, 1]$ given by

$$\alpha[u] = \int_0^1 u(s)\mathrm{d}A(s), \beta[u] = \int_0^1 u(s)\mathrm{d}B(s),$$

with A, B functions of bounded variation. By giving a general approach to cover all of these boundary conditions (and others) in a unified way, this work includes much previous work as special cases and improves the previous results.

In particular, the work of Webb and Infante does not require the functionals $\alpha[u], \beta[u]$ to be positive for all positive u so some sign changing measures are allowed.

Feng [17] studied the existence of positive solutions for a class of boundary-value problem with integral boundary conditions in Banach spaces.

On the other hand, boundary value problems on infinite intervals occur in mathematical modeling of various applied problems, see the references of [18]. Several authors have studied problems on infinite intervals. See [18]–[27] and their references. In [21]–[25], they studied a similar problem

$$\begin{gathered}(p(t)x'(t))' + \lambda q(t)f(t, x(t), x'(t)) = 0, \quad t \in [0, +\infty)\\ \alpha_1 x(0) - \beta_1 x'(0) - \gamma_1 \lim_{t\to 0} p(t)x'(t) = \delta_1,\\ \alpha_2 \lim_{t\to+\infty} x(t) + \beta_2 \lim_{t\to+\infty} p(t)x'(t) = \delta_2.\end{gathered}$$

In [26], by applying fixed-point theorems, Tian, Ge, and Shan [26] considered the existence of positive solutions for the three-point boundary value problem on the half-line. Thereafter, Tian and Ge [27] studied the existence of positive solutions for the multi-point boundary value problem on the half-line.

Motivated by the papers mentioned above, in this paper we investigate the existence of positive solutions of boundary value problems for systems of second-order differential equations (1.1) with integral boundary condition on the half-line.

2 Preliminaries

Throughout the paper, we denote $\mathbb{R}^+ = [0, +\infty)$.

Definition We say $f_i : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is an L^1-Carathéodory function if

(1) $f_i(\cdot, x, y)$ is measurable for any $(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+$;

(2) $f_i(t, \cdot, \cdot)$ is continuous for almost every $t \in \mathbb{R}^+$;

(3) for each $r_1, r_2 > 0$, there exists $\phi_{r_1,r_2} \in L^\infty[0, +\infty)$ such that

$$0 \le f_i\big(t, (1+t)x, (1+t)y\big) \le \phi_{r_1,r_2}(t),$$

for all $x \in [0, r_1]$, $y \in [0, r_2]$ and almost every $t \in [0, +\infty)$.

We assume that the following conditions hold.

(H1) $g_i \in L^1[0, +\infty)$ is nonnegative and $1 - \int_0^{+\infty} sg_i(s)\mathrm{d}s > 0, \quad i = 1, 2;$

(H2) f_i is an L^1-Carathéodory function, $i = 1, 2$.

For convenience, we denote

$$c_i = \frac{1}{1 - \int_0^{+\infty} sg_i(s)\mathrm{d}s},\ i = 1, 2.$$

Let

$$C(\mathbb{R}^+) = \{x : \mathbb{R}^+ \to \mathbb{R} : x \text{ is continuous and } \sup_{t\in\mathbb{R}^+} \frac{|x(t)|}{1+t} < +\infty\}.$$

Define $\|x\|_1 = \sup_{t\in\mathbb{R}^+} \frac{|x(t)|}{1+t}$. Then $(C(\mathbb{R}^+), \|\cdot\|_1)$ is a Banach space.

Let

$$X = \{(u_1, u_2) \in C(\mathbb{R}^+) \times C(\mathbb{R}^+) : \sup_{t \in \mathbb{R}^+} \frac{|u_i|}{1+t} < +\infty,\ i = 1, 2\}$$

with the norm $\|(u_1, u_2)\| = \|u_1\|_1 + \|u_2\|_1$. It is easy to prove that $(X, \|\cdot\|)$ is a Banach space.

In this paper, the main tool we use is the following Krasnosel'skii's fixed point theorem in cones, see for example [28].

Theorem 2.1 *Let E be a Banach space and let $K \subset E$ be a cone in E. Suppose that Ω_1, Ω_2 are bounded open subsets of K with $0 \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$, and $A : K \to K$ is a completely continuous operator such that either*

$$\|Aw\| \leq \|w\|, \quad w \in \partial\Omega_1, \quad \|Aw\| \geq \|w\|, \quad w \in \partial\Omega_2,$$

or

$$\|Aw\| \geq \|w\|, \quad w \in \partial\Omega_1, \quad \|Aw\| \leq \|w\|, \quad w \in \partial\Omega_2.$$

Then A has a fixed point in $\overline{\Omega}_2 \setminus \Omega_1$.

Lemma 2.2 *Assume that $(H1)$ holds. Then for any $y_i \in L^1[0, +\infty)$ and $y_i \geq 0$, the boundary value problem*

$$u_i''(t) + y_i(t) = 0 \tag{2.1}$$

$$u_i(0) = 0, u_i'(\infty) = \int_0^{+\infty} g_i(s) u_i(s) \mathrm{d}s, \tag{2.2}$$

has a unique solution $u_i \in C(\mathbb{R}^+)$, and

$$u_i(t) = \int_0^{+\infty} H_i(t, s) y_i(s) \mathrm{d}s,\ i = 1, 2, \tag{2.3}$$

where

$$H_i(t, s) = G(t, s) + tc_i \int_0^{+\infty} g_i(r) G(s, r) \mathrm{d}r,\ i = 1, 2,$$

and

$$G(t, s) = \min\{s, t\}$$

Proof. In fact, for any $y_i \in L^1[0, +\infty)$ and $y_i \geq 0$, by the methods of [15] and [16], it is easy to show that the unique solution of (2.1) and (2.2) is given by

$$u_i(t) = \int_0^{+\infty} H_i(t, s) y_i(s) \mathrm{d}s,\ i = 1, 2,$$

where $H_i(t, s)$ can be written

$$H_i(t, s) = G(t, s) + tc_i \mathcal{G}_i(s),\ i = 1, 2,$$

with $\mathcal{G}_i(s) = \int_0^{+\infty} g_i(r) G(s, r) dr$.

Lemma 2.3 *Assume that $(H1)$ holds. Let $\delta \in (0,1)$, then for all $t \in [\delta, \frac{1}{\delta}], \tau, s \in \mathbb{R}^+$, we have*

$$H_i(\tau, s) \geq 0, H_i(t, s) \geq \frac{\delta}{1+\tau} H_i(\tau, s). \tag{2.4}$$

Proof. It is clear that $H_i(\tau, s) \geq 0$.

For $t \in [\delta, \frac{1}{\delta}], \tau, s \in \mathbb{R}^+$, noting that $G(t, s) = \min\{t, s\}$, we have

$$\frac{\delta}{1+\tau} G(\tau, s) \leq \frac{\delta}{1+\tau} \tau < \delta \leq t.$$

and

$$\frac{\delta}{1+\tau} G(\tau, s) \leq \frac{\delta}{1+\tau} s < s.$$

This proves $\frac{\delta}{1+\tau} G(\tau, s) \leq \min\{t, s\} = G(t, s)$. Therefore, it follows that

$$H_i(t, s) \geq \frac{\delta}{1+\tau} H_i(\tau, s), \quad for \quad s, \tau \in \mathbb{R}^+,\ t \in [\delta, \frac{1}{\delta}].$$

The lemma is proved.

Lemma 2.4 *Assume that $(H1)$ holds. If $y_i \in L^1[0, +\infty), y_i \geq 0$, then the unique solution $u_i(t)$ of the boundary value problem (2.1) and (2.2) satisfies $u_i(t) \geq 0$ and $\min_{\delta \leq t \leq \frac{1}{\delta}} u_i(t) \geq \delta \|u_i\|_1, i = 1, 2$.*

Proof. It is clear that $u_i(t) \geq 0$, for all $t \in \mathbb{R}^+, i = 1, 2$.

From (2.3) and (2.4), for any $t \in [\delta, \frac{1}{\delta}], s, \tau \in \mathbb{R}^+, i = 1, 2$, we have

$$\begin{aligned} u_i(t) &= \int_0^{+\infty} H_i(t, s) y_i(s) \mathrm{d}s \\ &\geq \delta \int_0^{+\infty} \frac{1}{1+\tau} H_i(\tau, s) y_i(s) \mathrm{d}s \\ &= \delta \frac{1}{1+\tau} u_i(\tau). \end{aligned}$$

Hence,

$$\min_{\delta \leq t \leq \frac{1}{\delta}} u_i(t) \geq \delta \|u_i\|_1,\ i = 1, 2.$$

The lemma is proved.

Let

$$K = \{(u_1, u_2) \in X : u_i \geq 0, \min_{\delta \leq t \leq \frac{1}{\delta}} u_i(t) \geq \delta \|u_i\|_1,\ i = 1, 2\}.$$

Then $K \subset X$ is a cone in X.

By (H2), for $(u_1, u_2) \in K$, let $\dfrac{u_i(s)}{1+s} \leq r_i$ for all $s \geq 0$, then we have

$$\begin{aligned} \int_0^{+\infty} f_i(s, u_1(s), u_2(s)) \mathrm{d}s &= \int_0^{+\infty} f_i\big(s, (1+s)\frac{u_1(s)}{1+s}, (1+s)\frac{u_2(s)}{1+s}\big) \mathrm{d}s \\ &\leq \int_0^{+\infty} \phi_{r_1, r_2}(s) \mathrm{d}s < +\infty. \end{aligned}$$

So $\int_0^{+\infty} f_i(s, u_1(s), u_2(s))\mathrm{d}s$ is convergent and $f_i \in L^1[0, +\infty)$. Hence, from Lemma 2.2, we know that the boundary value problem (1.1) is equivalent to

$$u_i(t) = \int_0^{+\infty} H_i(t, s) f_i(s, u_1(s), u_2(s))\mathrm{d}s, \ i = 1, 2.$$

Define $T_i : K \to C(\mathbb{R}^+)$ by

$$T_1(u_1, u_2)(t) = \int_0^{+\infty} H_1(t, s) f_1(s, u_1(s), u_2(s))\mathrm{d}s,$$
$$T_2(u_1, u_2)(t) = \int_0^{+\infty} H_2(t, s) f_2(s, u_1(s), u_2(s))\mathrm{d}s.$$

Let

$$T(u_1, u_2)(t) = (T_1(u_1, u_2)(t), T_2(u_1, u_2)(t)).$$

Lemma 2.5 ([29]) *Let $M \subset C(\mathbb{R}^+)$, then M is a relatively compact if the following conditions hold:*

(a) all functions from M are uniformly bounded in $C(\mathbb{R}^+)$;

(b) the functions from $\{y : y = \frac{x}{1+t}, x \in M\}$ are equicontinuous on any compact interval of $[0, +\infty)$;

(c) the functions from $\{y : y = \frac{x}{1+t}, x \in M\}$ and are equiconvergent at infinity, that is , for any $\epsilon > 0$, there exists a $T = T(\epsilon) > 0$, such that $|y(t) - y(+\infty)| < \epsilon$, for all $t > T$, and $x \in M$.

Lemma 2.6 *Assume that (H1)(H2) hold. Then $T : K \to K$ is completely continuous.*

Proof. (1) $T : K \to K$. For any $(u_1, u_2) \in K$,

$$\begin{aligned}
\frac{T_i(u_1, u_2)(t)}{1+t} &= \int_0^{+\infty} \frac{H_i(t, s)}{1+t} f_i(s, u_1(s), u_2(s))\mathrm{d}s \\
&= \int_0^{+\infty} \frac{G(t, s) + tc_i\mathcal{G}_i(s)}{1+t} f_i(s, u_1(s), u_2(s))\mathrm{d}s \\
&\leq \big(1 + c_i \int_0^{+\infty} rg_i(r)\mathrm{d}r\big)\big(\int_0^{+\infty} \phi_{r_1,r_2}(s)\mathrm{d}s\big) \\
&= c_i \int_0^{+\infty} \phi_{r_1,r_2}(s)\mathrm{d}s < +\infty.
\end{aligned}$$

So, $T(u_1, u_2) \in X$. From Lemma 2.4, we have $T(u_1, u_2) \in K$.

(2) $T : K \to K$ is continuous.

For any convergent sequence $(u_{1,n}, u_{2,n}) \to (u_1, u_2)$, then $u_{1,n} \to u_1$, and $u_{2,n} \to u_2$, as $n \to +\infty$, and there are constants $r_1, r_2 > 0$, such that

$$\|u_{1,n}\|, \|u_1\| \leq r_1 \ \text{ and } \ \|u_{2,n}\|, \|u_2\| \leq r_2.$$

By (H_2), for *a.e.* $s \in [0, +\infty)$, we have

$$|f_i(s, u_{1,n}(s), u_{2,n}(s)) - f_i(s, u_1(s), u_2(s))| \to 0, \text{ as } n \to +\infty,$$

and

$$|f_i(s, u_{1,n}(s), u_{2,n}(s)) - f_i(s, u_1(s), u_2(s))| \leq 2\phi_{r_1,r_2}(s).$$

From the Lebesgue dominated convergence theorem, we have

$$\begin{aligned}
&\|T_i(u_{1,n}, u_{2,n}) - T_i(u_1, u_2)\|_1 \\
&\leq \sup_{t \in \mathbb{R}^+} \int_0^{+\infty} \frac{H_i(t,s)}{1+t} |f_i(s, u_1(s), u_2(s)) - f_i(s, u_{1,n}(s), u_{2,n}(s))| \mathrm{d}s \to 0.
\end{aligned}$$

So,

$$\|T_i(u_{1,n}, u_{2,n}) - T_i(u_1, u_2)\|_1 \to 0, \ n \to +\infty.$$

Hence, $T : K \to K$ continuous.

(3) $T : K \to K$ is relatively compact.

The proof of compactness follows from the Lemma 2.5 as in [29]

So, $T : K \to K$ is completely continuous. The lemma is proved.

3 The main result

We assume

(H3) There exist nonnegative functions $a_i \in L^1[0, +\infty)$, and continuous functions $h_i \in C[\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+], i = 1, 2$, such that

$$f_i(t, x, y) \leq a_i(t) h_i(x, y), \quad t, x, y \in \mathbb{R}^+,$$

$$\int_0^{+\infty} s a_i(s) \mathrm{d}s < +\infty.$$

Obviously, if (H3) holds then $\int_0^{+\infty} a_i(s)\mathrm{d}s < +\infty$ and if (H1) and (H3) hold then $\int_0^{+\infty} \frac{H_i(t,s)}{1+t}(1+s)a_i(s)\mathrm{d}s < +\infty$ for $t \in \mathbb{R}^+$ and $i = 1, 2$.

Denote

$$h_i^0 = \overline{\lim_{x+y \to 0^+}} \frac{h_i(x,y)}{x+y}, \quad h_i^\infty = \overline{\lim_{x+y \to +\infty}} \frac{h_i(x,y)}{x+y};$$

$$f_i^0 = \underline{\lim_{x+y \to 0^+}} \inf_{\delta \leq t \leq \frac{1}{\delta}} \frac{f_i(t,x,y)}{x+y}, \quad f_i^\infty = \underline{\lim_{x+y \to +\infty}} \inf_{\delta \leq t \leq \frac{1}{\delta}} \frac{f_i(t,x,y)}{x+y};$$

$$N_i = \inf_{\delta \leq t \leq \frac{1}{\delta}} \int_\delta^{\frac{1}{\delta}} \frac{H_i(t,s)}{1+t} \mathrm{d}s, \quad n_i = \sup_{t \in \mathbb{R}^+} \int_0^{+\infty} \frac{H_i(t,s)}{1+t}(1+s)a_i(s)\mathrm{d}s.$$

Theorem 3.1 *Suppose that hypotheses* $(H1) - (H3)$ *hold and satisfy*

(1) there exist positive numbers δ, m_i, M_i, $0 < \delta < 1$, *such that* $m_1 n_1 + m_2 n_2 \leq 1$, $M_1 N_1 \delta + M_2 N_2 \delta \geq 1$;

(2) $0 \leq h_i^0 < m_i$, $M_i < f_i^\infty \leq +\infty$.

Then, the boundary value problem (1.1) *has at least one positive solution.*

Proof. For $0 \le h_i^0 < m_i$ and by continuity of h_i, there exists a $\delta_1 > 0$ ($\delta_1 < 1$), such that

$$h_i(u_1, u_2) \le m_i(u_1 + u_2), u_1 + u_2 \in (0, \delta_1].$$

Let

$$\Omega_1 = \{(u_1, u_2) \in K : \|(u_1, u_2)\| < \delta_1\}.$$

For any $(u_1, u_2) \in \partial\Omega_1, t \in \mathbb{R}^+$, we have

$$\begin{aligned}
\frac{T_i(u_1, u_2)(t)}{1+t} &= \int_0^{+\infty} \frac{H_i(t,s)}{1+t} f_i(s, u_1(s), u_2(s))\mathrm{d}s \\
&\le \int_0^{+\infty} \frac{H_i(t,s)}{1+t} a_i(s) h_i(u_1(s), u_2(s))\mathrm{d}s \\
&\le m_i(\|u_1\|_1 + \|u_2\|_1) \int_0^{+\infty} \frac{H_i(t,s)}{1+t}(1+s)a_i(s)\mathrm{d}s \\
&\le m_i n_i(\|u_1\|_1 + \|u_2\|_1).
\end{aligned}$$

So,

$$\begin{aligned}
\|T(u_1, u_2)\| &= \|T_1(u_1, u_2)\|_1 + \|T_2(u_1, u_2)\|_1 \\
&\le m_1 n_1(\|u_1\|_1 + \|u_2\|_1) + m_2 n_2(\|u_1\|_1 + \|u_2\|_1) \\
&\le \|u_1\|_1 + \|u_2\|_1 = \|(u_1, u_2)\|.
\end{aligned}$$

Therefore

$$\|T(u_1, u_2)\| \le \|(u_1, u_2)\|,\ (u_1, u_2) \in \partial\Omega_1. \tag{3.1}$$

For $M_i < f_i^\infty \le +\infty$, there exists $\delta_2 > 1$ such that

$$f_i(t, u_1, u_2) \ge M_i(u_1 + u_2),\ t \in [\delta, \frac{1}{\delta}],\ u_1 + u_2 > \delta\delta_2.$$

Let

$$\Omega_2 = \{(u_1, u_2) \in K : \|(u_1, u_2)\| < \delta_2\}.$$

For $(u_1, u_2) \in \overline{\Omega}_2 \subset K$, by Lemma 2.4, we have

$$\min_{\delta \le t \le \frac{1}{\delta}} u_i(t) \ge \delta\|u_i\|_1,\ i = 1, 2.$$

Hence,

$$\min_{\delta \le t \le \frac{1}{\delta}} (u_1(t) + u_2(t)) \ge \delta(\|u_1\|_1 + \|u_2\|_1).$$

So for $(u_1, u_2) \in \partial\Omega_2$ and $t \in \mathbb{R}^+$, we get

$$\begin{aligned}
\frac{T_i(u_1, u_2)(t)}{1+t} &\ge \int_\delta^{\frac{1}{\delta}} \frac{H_i(t,s)}{1+t} f_i(s, u_1(s), u_2(s))\mathrm{d}s \\
&\ge \delta M_i(\|u_1\|_1 + \|u_2\|_1) \int_\delta^{\frac{1}{\delta}} \frac{H_i(t,s)}{1+t}\mathrm{d}s \\
&\ge \delta M_i N_i(\|u_1\|_1 + \|u_2\|_1).
\end{aligned}$$

We have

$$\begin{aligned}\|T(u_1,u_2)\| &= \|T_1(u_1,u_2)\|_1 + \|T_2(u_1,u_2)\|_1\\ &\geq \delta M_1N_1(\|u_1\|_1+\|u_2\|_1) + \delta M_2N_2(\|u_1\|_1+\|u_2\|_1)\\ &\geq (\|u_1\|_1+\|u_2\|_1) \geq \|(u_1,u_2)\|.\end{aligned}$$

Therefore

$$\|T(u_1,u_2)\| \geq \|(u_1,u_2)\|,\ (u_1,u_2)\in\partial\Omega_2. \tag{3.2}$$

From (3.1) and (3.2) and Theorem 2.1, we know that T has at least one positive solution in $K\cap(\overline{\Omega}_2\setminus\Omega_1)$. Therefore the boundary value problem (1.1) has at least one positive solution.

The theorem is proved.

Theorem 3.2 *Suppose that hypotheses* $(H1)-(H3)$ *hold and satisfy*

(1) there exist positive numbers δ, p_i, P_i, $0<\delta<1$, *such that* $0\leq h_i^\infty < p_i$, $P_i < f_i^0 \leq +\infty$;

(2) $p_1n_1+p_2n_2\leq 1$, $P_1N_1\delta + P_2N_2\delta \geq 1$.

Then, the boundary value problem (1.1) *has at least one positive solution.*

Proof. For $P_i < f_i^0 \leq +\infty$, there exists a $\delta_3>0$ $(\delta_3<1)$, such that

$$f_i(t,u_1,u_2)\geq P_i(u_1+u_2), t\in[\delta,\frac{1}{\delta}], 0<u_1+u_2<\delta_3.$$

Let

$$\Omega_3=\{(u_1,u_2)\in K:\|(u_1,u_2)\|<\delta_3\}.$$

For $(u_1,u_2)\in\overline{\Omega}_3\subset K$, from Lemma 2.4 we have

$$\min_{\delta\leq t\leq\frac{1}{\delta}} u_i(t)\geq\delta\|u_i\|_1, i=1,2.$$

Hence,

$$\min_{\delta\leq t\leq\frac{1}{\delta}}(u_1(t)+u_2(t))\geq\delta(\|u_1\|_1+\|u_2\|_1).$$

For $(u_1,u_2)\in\partial\Omega_3$ and $t\in\mathbb{R}^+$,

$$\begin{aligned}\frac{T_i(u_1,u_2)(t)}{1+t} &\geq \int_\delta^{\frac{1}{\delta}}\frac{H_i(t,s)}{1+t}f_i(s,u_1(s),u_2(s))\mathrm{d}s\\ &\geq \delta P_i(\|u_1\|_1+\|u_2\|_1)\int_\delta^{\frac{1}{\delta}}\frac{H_i(t,s)}{1+t}\mathrm{d}s\\ &\geq \delta P_iN_i(\|u_1\|_1+\|u_2\|_1).\end{aligned}$$

We can get

$$\begin{aligned}\|T(u_1,u_2)\| &= \|T_1(u_1,u_2)\|_1 + \|T_2(u_1,u_2)\|_1\\ &\geq \delta P_1N_1(\|u_1\|_1+\|u_2\|_1) + \delta P_2N_2(\|u_1\|_1+\|u_2\|_1)\\ &\geq (\|u_1\|_1+\|u_2\|_1) \geq \|(u_1,u_2)\|.\end{aligned}$$

Therefore

$$\|T(u_1,u_2)\| \geq \|(u_1,u_2)\|, (u_1,u_2) \in \partial\Omega_3. \tag{3.3}$$

For $0 \leq h_i^\infty < p_i$, there exists a $R_0 > 0$, such that

$$h_i(u_1,u_2) \leq p_i(\|u_1\|_1 + \|u_2\|_1), \;\; \|u_1\|_1 + \|u_2\|_1 > R_0.$$

Denote $q_i = \max_{0\leq\|u_1\|_1+\|u_2\|_1\leq R_0} h_i(u_1,u_2)$, then we have

$$h_i(u_1,u_2) \leq q_i + p_i(\|u_1\|_1 + \|u_2\|_1), \;\; (u_1,u_2) \in X.$$

There exists a δ_4 ($\delta_4 > \max\{1, \delta_3, (q_1n_1 + q_2n_2)(1 - p_1n_1 - p_2n_2)^{-1}\}$).

Let

$$\Omega_4 = \{(u_1,u_2) \in K : \|(u_1,u_2)\| < \delta_4\}.$$

For any $(u_1,u_2) \in \partial\Omega_4$ and $t \in \mathbb{R}^+$, we have

$$\begin{aligned}
\frac{T_i(u_1,u_2)(t)}{1+t} &\leq \int_0^{+\infty} \frac{H_i(t,s)}{1+t} a_i(s) h_i(u_1(s),u_2(s))\mathrm{d}s \\
&\leq \int_0^{+\infty} \frac{H_i(t,s)}{1+t} a_i(s)(q_i + p_i(u_1+u_2))\mathrm{d}s \\
&< p_i(\|u_1\|_1 + \|u_2\|_1) \int_0^{+\infty} \frac{H_i(t,s)}{1+t}(1+s)a_i(s)\mathrm{d}s \\
&\quad + q_i \int_0^{+\infty} \frac{H_i(t,s)}{1+t}(1+s)a_i(s)\mathrm{d}s \\
&\leq p_in_i(\|u_1\|_1 + \|u_2\|_1) + q_in_i.
\end{aligned}$$

So,

$$\begin{aligned}
\|T(u_1,u_2)\| &= \|T_1(u_1,u_2)\|_1 + \|T_2(u_1,u_2)\|_1 \\
&\leq (p_1n_1 + p_2n_2)(\|u_1\|_1 + \|u_2\|_1) + q_1n_1 + q_2n_2 \\
&\leq (\|u_1\|_1 + \|u_2\|_1) = \|(u_1,u_2)\|.
\end{aligned}$$

Therefore

$$\|T(u_1,u_2)\| \leq \|(u_1,u_2)\|, (u_1,u_2) \in \partial\Omega_4. \tag{3.4}$$

From (3.3) and (3.4) and Theorem 2.1, we know T has at least one positive solution at $K \cap (\overline{\Omega}_4 \setminus \Omega_3)$. Therefore the boundary value problem (1.1) has at least one positive solution.

The theorem is proved.

4 Illustration

We give an example to illustrate our results.

Example Consider the following the boundary value problem:

$$\begin{gathered}
u_1''(t) + (u_1+u_2)^2 e^{-t} = 0, \\
u_2''(t) + (u_1+u_2)^{\frac{3}{2}} e^{-t} = 0, \\
u_1(0) = u_2(0) = 0, \\
u_1'(\infty) = \int_0^{+\infty} e^{-2s} u_1(s)\mathrm{d}s, \;\; u_2'(\infty) = \int_0^{+\infty} e^{-2s} u_2(s)\mathrm{d}s.
\end{gathered}$$

In this example,

$$f_1(t,u_1,u_2)=(u_1+u_2)^2e^{-t},\ \ f_2(t,u_1,u_2)=(u_1+u_2)^{\frac{3}{2}}e^{-t},\ \ g_1(s)=g_1(s)=e^{-2s},$$

$$a_1(t)=a_2(t)=e^{-t},\ \ h_1(u_1,u_2)=(u_1+u_2)^2,\ \ h_2(u_1,u_2)=(u_1+u_2)^{\frac{3}{2}}.$$

For the constants r_1, $r_2>0$, we take $\phi_{r_1,r_2}(t)=(1+r_1+r_2)^2(1+t)^2e^{-t}$. Let $\delta=\frac{1}{2}$.

Since $n_i\le\frac{68}{27}$ and $N_i\ge\frac{1}{9}$, we take $m_i=\frac{1}{6}$, $M_i=10$. Then the conditions in Theorem 3.1 are all satisfied. By Theorem 3.1, the boundary value problem mentioned in the example above has at least one positive solution.

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