Electronic Journal of Qualitative Theory of Differential Equations
2009, No. **43**, 1-14; **http://www.math.u-szeged.hu/ejqtde/**

Positive Solutions for Singular m-Point Boundary Value Problems with Sign Changing Nonlinearities Depending on x' *

Ya Ma, Baoqiang Yan †

Department of Mathematics, Shandong Normal University, Jinan, 250014, P.R. China

Abstract

Using the theory of fixed point theorem in cone, this paper presents the existence of positive solutions for the singular m-point boundary value problem

$$\begin{cases} x''(t) + a(t)f(t, x(t), x'(t)) = 0, 0 < t < 1, \\ x'(0) = 0, \quad x(1) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i), \end{cases}$$

where $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1, \alpha_i \in [0, 1),\ i = 1,\ 2,\ \cdots,\ m-2$, with $0 < \sum_{i=1}^{m-2} \alpha_i < 1$ and f may change sign and may be singular at $x = 0$ and $x' = 0$.

Keywords: m-point boundary value problem; Singularity; Positive solutions; Fixed point theorem
Mathematics subject classification: 34B15, 34B10

1. Introduction

The study of multi-point BVP (boundary value problem) for linear second-order ordinary differential equations was initiated by Il'in and Moiseev [3-4]. Since then, many authors studied more general nonlinear multi-point BVP, for examples [2, 5-8], and references therein. In [7], Gupta, Ntouyas, and Tsamatos considered the existence of a $C^1[0, 1]$ solution for the m-point boundary value problem

$$\begin{cases} x''(t) = f(t, x(t), x'(t)) + e(t), 0 < t < 1, \\ x'(0) = 0, x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i), \end{cases}$$

*The project is supported by the fund of National Natural Science (10871120), the fund of Shandong Education Committee (J07WH08) and the fund of Shandong Natural Science (Y2008A06)
†Corresponding author: yanbqcn@yahoo.com.cn

where $\xi_i \in (0,1)$, $i = 1, 2, \cdots, m-2$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $a_i \in R$, $i = 1, 2, \cdots, m-2$, have the same sign, $\sum_{i=1}^{m-2} a_i \neq 1$, $e \in L^1[0,1]$, $f : [0,1] \times R^2 \to R$ is a function satisfying Carathéodory's conditions and a growth condition of the form $| f(t,u,v) | \leq p_1(t)|u| + q_1(t)|v| + r_1(t)$ with $p_1, q_1, r_1 \in L^1[0,1]$. Recently, using Leray-Schauder continuation theorem, R.Ma and Donal O'Regan proved the existence of positive solutions of $C^1[0,1)$ solutions for the above BVP, where $f : [0,1] \times R^2 \to R$ satisfies the Carathéodory's conditions (see [8]).

Motivated by the works of [7,8], in this paper, we discuss the equation

$$\begin{cases} x''(t) + a(t)f(t, x(t), x'(t)) = 0, 0 < t < 1, \\ x'(0) = 0, \quad x(1) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i), \end{cases} \tag{1.1}$$

where $0 < \xi_i < 1$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $\alpha_i \in [0,1)$ with $0 < \sum_{i=1}^{m-2} \alpha_i < 1$ and f may change sign and may be singular at $x = 0$ and $x' = 0$.

Our main features are as follows. Firstly, the nonlinearity af possesses singularity, that is, $a(t)f(t,x,x')$ may be singular at $t = 0$, $t = 1$, $x = 0$ and $x' = 0$; also the degree of singularity in x and x' may be arbitrary(i. e., if f contains $\frac{1}{x^\alpha}$ and $\frac{1}{(-x')^\gamma}$, α and γ may be big enough). Secondly, f is allowed to change sign. Finally, we discuss the maximal and minimal solutions for equations (1.1). Some ideas come from [11-12].

2. Preliminaries

Now we list the following conditions for convenience .

(H$_1$) $\beta, a, k \in C((0,1), R_+), F \in C(R_+, R_+), G \in C(R_-, R_+), ak \in L[0,1]$;

(H$_2$) F is bounded on any interval $[z, +\infty), z > 0$;

(H$_3$) $\int_{-\infty}^{-1} \frac{1}{G(y)} dy = +\infty$;

and the following conditions are satisfied

(P$_1$) $f \in C((0,1) \times R_+ \times R_-, R)$;

(P$_2$) $0 < \sum_{i=1}^{m-2} \alpha_i < 1, 0 < \xi_i < 1$ and $|f(t,x,y)| \leq k(t)F(x)G(y)$;

(P$_3$) There exists $\delta > 0$ such that $f(t,x,y) \geq \beta(t), y \in (-\delta, 0)$;

where $R_+ = (0, +\infty), R_- = (-\infty, 0), R = (-\infty, +\infty)$.

Lemma 2.1[1] Let E be a Banach space, K a cone of E, and $B_R = \{x \in E : \|x\| < R\}$, where $0 < r < R$. Suppose that F: $K \cap \overline{B_R \backslash B_r} = K_{R,r} \to K$ is a completely continuous operator and the following conditions are satisfied

(1) $\|F(x)\| \geq ||x||$ for any $x \in K$ with $\|x\| = r$.

(2) If $x \neq \lambda F(x)$ for any $x \in K$ with $\|x\| = R$ and $0 < \lambda < 1$.

Then F has a fixed point in $K_{R,r}$.

Let $C[0,1]=\{x:[0,1]\to R|x(t)$ is continuous on $[0,1]\}$ with norm $\|y\|=\max\limits_{t\in[0,1]}|y(t)|$. Then $C[0,1]$ is a Banach space.

Lemma 2.2 Let (H_1)-(P_3) hold. For each given natural number $n>0$, there exists $y_n\in C[0,1]$ with $y_n(t)\le -\frac{1}{n}$ such that

$$y_n(t)=-\frac{1}{n}-\int_0^t a(s)f(s,(Ay_n)(s)+\frac{1}{n},y_n(s))ds, \qquad t\in[0,1], \tag{2.1}$$

where

$$(Ay)(t) = \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\int_0^1 -y(\tau)d\tau-\frac{\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}-y(\tau)d\tau}{1-\sum_{i=1}^{m-2}\alpha_i}-\int_0^t -y(\tau)d\tau, \qquad t\in[0,1].$$

Proof. For $y\in P=\{y\in C[0,1]:y(t)\le 0,t\in[0,1]\}$, define a operator as follows

$$(T_ny)(t)=-\frac{1}{n}+\min\{0,-\int_0^t a(s)f(s,(Ay)(s)+\frac{1}{n},\min\{y(s),-\frac{1}{n}\})ds\}, \qquad t\in[0,1], \tag{2.2}$$

where $n>0$ is a natural number. For $y\in P$, we have

$$\begin{aligned}
&(Ay)(t)=\frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\int_0^1 -y(\tau)d\tau-\frac{\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}-y(\tau)d\tau}{1-\sum_{i=1}^{m-2}\alpha_i}-\int_0^t -y(\tau)d\tau\\
&\ge \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\int_0^1 -y(\tau)d\tau-\frac{\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}-y(\tau)d\tau}{1-\sum_{i=1}^{m-2}\alpha_i}-\int_0^1 -y(\tau)d\tau\\
&\ge \frac{\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i}\int_0^1 -y(\tau)d\tau-\frac{\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_{m-2}}-y(\tau)d\tau}{1-\sum_{i=1}^{m-2}\alpha_i}\\
&\ge \frac{\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i}\int_{\xi_{m-2}}^1 -y(\tau)d\tau\\
&\ge 0, \quad t\in[0,1].
\end{aligned}$$

Let

$$c(y(t))=-\int_0^t a(s)f(s,(Ay)(s)+\frac{1}{n},\min\{y(s),-\frac{1}{n}\})ds, \qquad t\in[0,1],$$
$$c(y_k(t))=-\int_0^t a(s)f(s,(Ay_k)(s)+\frac{1}{n},\min\{y_k(s),-\frac{1}{n}\})ds, \qquad t\in[0,1].$$

By the equality $\min\{c,0\}=\frac{c-|c|}{2}$, it is easy to know

$$(T_ny)(t)=-\frac{1}{n}+\frac{c(y(t))-|c(y(t)|}{2}, \qquad t\in[0,1].$$

Let $y_k,y\in P$ with $\lim_{k\to+\infty}\|y_k-y\|=0$. Then, there exists a constant $h>0$, such that $\|y_k\|\le h$ and $\|y\|\le h$. Thus, $|\min\{y_k(s),-\frac{1}{n}\}-\min\{y(s),-\frac{1}{n}\}|\to 0$, uniformly for $s\in[0,1]$ as $k\to+\infty$. Therefore, $|(Ay_k)(s)+\frac{1}{n}-((Ay)(s)+\frac{1}{n})|\to 0$ for all $s\in[0,1]$ as $k\to+\infty$. (P_1) implies that $\{a(s)f(s,(Ay_k)(s)+\frac{1}{n},\min\{y_k(s),-\frac{1}{n}\})\}\to\{a(s)f(s,(Ay)(s)+\frac{1}{n},\min\{y(s),-\frac{1}{n}\})\}$, for $s\in(0,1)$ as $k\to+\infty$. By the Lebesgue dominated convergence theorem (the dominating function $a(s)k(s)F[\frac{1}{n},+\infty)G[-h-\frac{1}{n},-\frac{1}{n}])$, we have $\|cy_k - cy\| \to 0$, which yields that

$$\begin{aligned}\|T_n y_k - T_n y\| &= \|\frac{c(y_k)-c(y)-|c(y_k)|+|c(y)|}{2}\| \\ &\le \|\frac{c(y_k)-c(y)+|c(y_k)-c(y)|}{2}\| \\ &\le \|c(y_k)-c(y)\| \to 0, \text{ as } k\to+\infty.\end{aligned}$$

Consequently, T_n is a continuous operator.

Let C be a bounded set in P, i.e., there exists $h_1>0$ such that $\|y\|\le h_1$, for any $y\in C$. For any $t_1,t_2\in[0,1], t_1<t_2, y\in C$,

$$\begin{aligned}&|(T_n y)(t_2)-(T_n y)(t_1)| \\ &= |\frac{-\int_{t_1}^{t_2} a(s)f(s,(Ay)(s)+\frac{1}{n},\min\{y(s),-\frac{1}{n}\}ds}{2} \\ &+\frac{|\int_0^{t_2} a(s)f(s,(Ay)(s)+\frac{1}{n},\min\{y(s),-\frac{1}{n})ds|-|\int_0^{t_1} a(s)f(s,(Ay)(s)+\frac{1}{n},\min\{y(s),-\frac{1}{n})ds|}{2}| \\ &\le |\frac{-\int_{t_1}^{t_2} a(s)f(s,(Ay)(s)+\frac{1}{n},\min\{y(s),-\frac{1}{n}\}ds}{2}| \\ &+\frac{|\int_{t_1}^{t_2} a(s)f(s,(Ay)(s),\min\{y(s),-\frac{1}{n}\}ds|}{2} \\ &\le |\int_{t_1}^{t_2} a(s)k(s)ds|\sup F[\frac{1}{n},+\infty)\sup G[-h_1-\frac{1}{n},-\frac{1}{n}].\end{aligned}$$

According to the absolute continuity of the Lebesgue integral, for any $\epsilon>0$, there exists $\delta>0$ such that $|\int_{t_1}^{t_2} a(s)k(s)ds|<\epsilon, |t_2-t_1|<\delta$. Therefore, $\{T_n y, y\in C\}$ is equicontinuous.

$$\begin{aligned}|(T_n y)(t)| &= |-\frac{1}{n}+\min\{0,-\int_0^t a(s)f(s,(Ay)(s)+\frac{1}{n},\min\{y(s),-\frac{1}{n}\})ds\}| \\ &\le |\frac{1}{n}|+|\int_0^t a(s)f(s,(Ay)(s)+\frac{1}{n},\min\{y(s),-\frac{1}{n}\})ds| \\ &\le 1+\int_0^t a(s)|f(s,(Ay)(s)+\frac{1}{n},\min\{y(s),-\frac{1}{n}\}|)ds \\ &\le 1+\int_0^1 a(s)k(s)ds\sup F[\frac{1}{n},+\infty)G[-h-\frac{1}{n},\frac{1}{n}], \quad t\in[0,1].\end{aligned}$$

Therefore $\{T_n y, y\in C\}$ is bounded.

Hence T_n is a completely continuous operator.

By (H$_3$), choose a sufficiently large $R_n>1$ to fit $\int_{-R_n}^{-1}\frac{dy}{G(y)} > \int_0^1 a(s)k(s)ds\sup F[\frac{1}{n},+\infty)$. For $n>\frac{1}{\delta}$, we prove that

$$y(t)\ne\lambda(T_n y)(t)=\frac{-\lambda}{n}+\lambda\min\{0,-\int_0^t a(s)f(s,(Ay)(s)+\frac{1}{n},\min\{y(s),-\frac{1}{n}\})ds\}, \quad t\in[0,1], \tag{2.3}$$

for any $y \in P$ with $||y|| = R_n$ and $0 < \lambda < 1$.

In fact, if there exists $y \in P$ with $\|y\| = R_n$ and $0 < \lambda < 1$ such that

$$y(t) = \frac{-\lambda}{n} + \lambda \min\{0, -\int_0^t a(s)f(s,(Ay)(s) + \frac{1}{n}, \min\{y(s), -\frac{1}{n}\})ds\},\ t \in [0,1]. \quad (2.4)$$

$y(0) = \dfrac{-\lambda}{n}$. Since $n > \dfrac{1}{\delta}$, we have $-\delta < y(0) < 0$, which implies there exists $\delta_0 > 0$ such that $y(t) > -\delta, t \in (0, \delta_0)$. ($P_3$) implies

$$\int_0^t a(s)f(s,(Ay)(s) + \frac{1}{n}, \min\{y(s), -\frac{1}{n}\})ds > 0, \quad t \in [0,1].$$

Let $t^* = \sup\{s \in [0,1] | \int_0^t a(\tau)f(\tau,(Ay)(\tau) + \frac{1}{n}, \min\{y(\tau), -\frac{1}{n}\})d\tau > 0, 0 \le t \le s\}$.

We show that $t^* = 1$. If $t^* < 1$, we have

$$\begin{cases} \int_0^t a(s)f(s,(Ay)(s) + \frac{1}{n}, \min\{y(s), -\frac{1}{n}\})ds > 0,\ t \in (0,t^*), \\ \int_0^t a(s)f(s,(Ay)(s) + \frac{1}{n}, \min\{y(s), -\frac{1}{n}\})ds = 0,\ t = t^*, \end{cases}$$

$$y(t) = \frac{-\lambda}{n} - \lambda \int_0^t a(s)f(s,(Ay)(s) + \frac{1}{n}, \min\{y(s), -\frac{1}{n}\})ds,\ t \in (0,t^*], \quad (2.5)$$

$$y(t^*) = \frac{-\lambda}{n} > -\delta. \quad (2.6)$$

(2.6) and (P_3) imply there exists $r > 0$ such that $f(t,x,y) \ge \beta(t), t \in (t^* - r, t^*)$. So

$$\begin{aligned} y(t^*) &= \frac{-\lambda}{n} - \lambda \int_0^{t^*} a(s)f(s,(Ay)(s) + \frac{1}{n}, \min\{y(s), -\frac{1}{n}\})ds \\ &\le \frac{-\lambda}{n} - \lambda \int_0^{t^*-r} a(s)f(s,(Ay)(s) + \frac{1}{n}, \min\{y(s), -\frac{1}{n}\})ds - \lambda \int_{t^*-r}^{t^*} a(s)\beta(s)ds, \end{aligned}$$

$$\int_0^{t^*-r} a(s)f(s,(Ay)(s) + \frac{1}{n}, \min\{y(s), -\frac{1}{n}\})ds + \int_{t^*-r}^{t^*} a(s)\beta(s)ds < 0,$$

which is a contradiction. Then, $t^* = 1$. Hence,

$$y(t) = \frac{-\lambda}{n} - \lambda \int_0^t a(s)f(s,(Ay)(s) + \frac{1}{n}, \min\{y(s), -\frac{1}{n}\})ds,\ t \in [0,1]. \quad (2.7)$$

Since $\|y\| = R_n > 1$ and $y \in P$, there exists a $t_0 \in (0,1)$ with $y(t_0) = -R_n < -1$ and a $t_1 \in (0,1)$ such that $y(t) < -1 < -\dfrac{1}{n}, t \in (t_0, t_1]$, which together with (2.7) implies that

$$y(t) = \frac{-\lambda}{n} - \lambda \int_0^t a(s)f(s,(Ay)(s) + \frac{1}{n}, y(s))ds,\ t \in (t_0, t_1]. \quad (2.8)$$

Differentiating (2.8) and using (H_2), we obtain

$$-y'(t) = \lambda a(t)f(t,(Ay)(t) + \frac{1}{n}, y(t)) \le a(t)F((Ay)(t) + \frac{1}{n})G(y(t)),\ t \in (t_0, t_1].$$

And then

$$\frac{-y'(t)}{G(y(t))} \leq a(t)k(t)\sup F[(Ay)(t)+\frac{1}{n},+\infty) \leq a(t)k(t)\sup F[\frac{1}{n},+\infty),\ t\in(t_0,t_1). \quad (2.9)$$

Integrating for (2.9) from t_0 to t_1, we have

$$\int_{y(t_0)}^{y(t_1)} \frac{dy}{G(y)} \leq \int_{t_0}^{t_1} a(s)k(s)ds \sup F[\frac{1}{n},+\infty),\ t\in(t_0,t_1). \quad (2.10)$$

Then

$$\int_{-R_n}^{-1} \frac{dy}{G(y)} \leq \int_{-R_n}^{y(t_1)} \frac{dy}{G(y)} \leq \int_{t_0}^{t_1} a(s)k(s)ds \sup F[\frac{1}{n},+\infty) \leq \int_0^1 a(s)k(s)ds \sup F[\frac{1}{n},+\infty),$$

which contradicts

$$\int_{-R_n}^{-1} \frac{dy}{G(y)} > \int_0^1 a(s)k(s)ds \sup F[\frac{1}{n},+\infty).$$

Hence(2.3) holds. Then put $r=\frac{1}{n}$, Lemma 2.1 leads to the desired result. This completes the proof.

Lemma 2.3[10] Let $\{x_n(t)\}$ be an infinite sequence of bounded variation function on $[a,b]$ and $\{x_n(t_0)\}(t_0\in[a,b])$ and $\{V(x_n)\}$ be bounded($V(x)$ denotes the total variation of x). Then there exists a subsequence $\{x_{n_k}(t)\}$ of $\{x_n(t)\}, i\neq j, n_i\neq n_j$, such that $\{x_{n_k}(t)\}$ converges everywhere to some bounded variation function $x(t)$ on $[a,b]$.

Lemma 2.4[9](Zorn) If X is a partially ordered set in which every chain has an upper bound, then X has a maximal element.

3. Main results

Theorem 3.1 Let (H_1)-(P_3) hold. Then the m-point boundary value problem (1.1) has at least one positive solution.

Proof. Put $M_n=\min\{y_n(t): t\in[0,\xi_{m-2}]\}$, (H_1) implies $\gamma=\sup\{M_n\}<0$. In fact, if $\gamma=0$, there exists $n_k>N>0$ such that $M_{n_k}\to 0$ and $-\delta<y_{n_k}<0$. (H_1) implies

$$\begin{aligned} y_{n_k}(t) &= -\frac{1}{n}-\int_0^t a(s)f(s,(Ay_{n_k})(s)+\frac{1}{n},y_{n_k}(s))ds \\ &< -\frac{1}{n}-\int_0^t a(s)\beta(s)ds \\ &< -\int_0^t a(s)\beta(s)ds,\quad t\in[0,\xi_{m-2}]. \end{aligned}$$

Then $y_{n_k}(\xi_{m-2})<-\int_0^{\xi_{m-2}} a(s)\beta(s)ds$, which contradicts to $M_{n_k}\to 0$.

Set $\tau=\max\{\gamma,-\delta,-\int_0^{\xi_{m-2}} a(s)\beta(s)\}$. In the remainder of the proof, assume $n>-\frac{1}{\tau}$.

1). First, we prove there exists a $t_n\in(0,\xi_{m-2}]$ with $y_n(t_n)=\tau$. In fact, since $y_n(0)=-\frac{1}{n}>\tau$, there exists $\delta_0>0$ such that $y_n(t)>\tau, t\in(0,\delta_0)$. Let $t_n=\sup\{t|s\in$

$[0,t], y_n(s) > \tau\}$ .Then $y_n(t_n) = \tau$. If $t_n > \xi_{m-2}$, we have $y_n(t) > \tau > -\delta, t \in [0, \xi_{m-2}]$. (H_1) shows that

$$\begin{aligned} y_n(t) &= -\frac{1}{n} - \int_0^t a(s) f(s, (Ay_n)(s) + \frac{1}{n}, y_n(s)) ds \\ &\le -\frac{1}{n} - \int_0^t a(s)\beta(s) ds \\ &\le -\int_0^t a(s)\beta(s) ds, \quad t \in [0, \xi_{m-2}]. \end{aligned}$$

Then $\tau < y_n(\xi_{m-2}) \le -\int_0^{\xi_{m-2}} a(s)\beta(s)ds < \tau$, which is a contradiction.

Second, we prove

$$y_n(t) \le \tau, \quad t \in [t_n, 1]. \tag{3.1}$$

In fact, if there exists a $t \in (t_n, 1]$ such that $y_n(t) > \tau$, and we choose $t', t'' \in [t_n, 1], t' < t''$ to fit $y_n(t') = \tau, \tau < y_n(t) < -\frac{1}{n}, t \in (t', t'']$, from (2.1)

$$0 < \int_{t'}^{t''} a(s) f(s, (Ay_n)(s) + \frac{1}{n}, y_n(s)) ds = y_n(t') - y_n(t'') < 0.$$

This contradiction implies that (3.1) holds. Then

$$\begin{cases} y_n(t) \le -\int_0^t a(s)\beta(s)ds, & t \in [0, t_n], \\ y_n(t) \le \tau, & t \in [t_n, 1]. \end{cases}$$

Let $W(t) = \max\{-\int_0^t a(s)\beta(s)ds, \tau\}, t \in (0,1)$. Obviously, $W(t)$ is bounded on $[\frac{1}{3k}, 1 - \frac{1}{3k}]$ and $y_n(t) \le W(t), t \in [0,1]$.

2). $\{y_n(t)\}$ is equicontinuous on $[\frac{1}{3k}, 1 - \frac{1}{3k}]$($k \ge 1$ is a natural number) and uniformly bounded on $[0,1]$.

Notice that

$$\begin{aligned} (Ay_n)(t) + \frac{1}{n} &= \frac{1}{1 - \sum_{i=1}^{m-2}\alpha_i} \int_0^1 -y_n(\tau)d\tau - \frac{\sum_{i=1}^{m-2}\alpha_i \int_0^{\xi_i} -y_n(\tau)d\tau}{1 - \sum_{i=1}^{m-2}\alpha_i} - \int_0^t -y_n(\tau)d\tau + \frac{1}{n} \\ &> \frac{\sum_{i=1}^{m-2}\alpha_i}{1 - \sum_{i=1}^{m-2}\alpha_i} \int_\xi^1 -y_n(\tau)d\tau \ge \frac{\sum_{i=1}^{m-2}\alpha_i}{1 - \sum_{i=1}^{m-2}\alpha_i}(-\tau)(1-\xi) = \Theta, t \in [0,1]. \end{aligned}$$

We know from (2.9)

$$\int_{y_n(t)}^{-\frac{1}{n}} \frac{dy_n}{G(y_n)} \le \int_0^t a(s)k(s)ds \sup F[\Theta, +\infty), t \in [0,1]. \tag{3.2}$$

Now (H_3) and (3.2) show that $\omega(t) = \inf\{y_n(t)\} > -\infty$ is bounded on $[0,1]$. On the other hand, it follows from (2.1) and (3.1) that

$$|y_n'(t)| \le k(t)a(t) \sup F[\Theta, +\infty) \sup G[\omega_k, \max\{\tau, W(\frac{1}{3k})\}], \ (n \ge k), \tag{3.3}$$

where $\omega_k = \inf\{\omega(t), t \in [\frac{1}{3k}, 1-\frac{1}{3k}]\}$. Thus (3.3) and the absolute continuity of Lebesgue integral show that $\{y_n(t)\}$ is equicontinuous on $[\frac{1}{3k}, 1-\frac{1}{3k}]$. Now the Arzela-Ascoli theorem guarantees that there exists a subsequence of $\{y_n^{(k)}(t)\}$, which converges uniformly on $[\frac{1}{3k}, 1-\frac{1}{3k}]$. When $k=1$, there exists a subsequence $\{y_n^{(1)}(t)\}$ of $\{y_n(t)\}$, which converges uniformly on $[\frac{1}{3}, \frac{2}{3}]$.When $k=2$, there exists a subsequence $\{y_n^{(2)}(t)\}$ of $\{y_n^{(1)}(t)\}$, which converges uniformly on $[\frac{1}{6}, \frac{5}{6}]$. In general, there exists a subsequence $\{y_n^{(k+1)}(t)\}$ of $\{y_n^{(k)}(t)\}$, which converges uniformly on $[\frac{1}{3(k+1)}, 1-\frac{1}{3(k+1)}]$. Then the diagonal sequence $\{y_k^{(k)}(t)\}$ converges pointwise in $(0,1)$ and it is easy to verify that $\{y_k^{(k)}(t)\}$ converges uniformly on any interval $[c,d] \subseteq (0,1)$. Without loss of generality, let $\{y_k^{(k)}(t)\}$ be itself of $\{y_n(t)\}$ in the rest. Put $y(t) = \lim_{n\to\infty} y_n(t), t \in (0,1)$. Then $y(t)$ is continuous on $(0,1)$ and since $y_n(t) \le W(t) < 0$, we have $y(t) \le 0, t \in (0,1)$.

3) Now (3.2) shows

$$\sup\{\max\{-y_n(t), t \in [0,1]\}\} < +\infty.$$

We have

$$\lim_{t\to 0^+} \sup\{\int_0^t -y_n(s)ds\} = 0, \quad \lim_{t\to 1^-} \sup\{\int_t^1 -y_n(s)ds\} = 0, \quad t \in [0,1], \tag{3.4}$$

and

$$\begin{aligned}(Ay_n)(t) &= \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\int_0^1 -y_n(\tau)d\tau - \frac{\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i} -y_n(\tau)d\tau}{1-\sum_{i=1}^{m-2}\alpha_i} - \int_0^t -y_n(\tau)d\tau \\ &< \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\int_0^1 -y_n(\tau)d\tau \\ &< +\infty, \ t \in [0,1].\end{aligned} \tag{3.5}$$

Since (3.4) and (3.5) hold, the Fatou theorem of the Lebesgue integral implies $(Ay)(t) < +\infty$, for any fixed $t \in (0,1)$.

4) $y(t)$ satisfies the following equation

$$y(t) = -\int_0^t a(s)f(s,(Ay)(s),y(s))ds, \quad t \in (0,1). \tag{3.6}$$

Since $y_n(t)$ converges uniformly on $[a,b] \subset (0,1)$, (3.4) implies that $(Ay_n)(s)$ converges to $(Ay)(s)$ for any $s \in (0,1)$. For fixed $t \in (0,1)$ and any d , $0 < d < t$, we have

$$y_n(t) - y_n(d) = -\int_d^t a(s)f(s,(Ay_n)(s) + \frac{1}{n}, y_n(s))ds. \tag{3.7}$$

for all $n > k$. Since $y_n(s) \le \max\{\tau, W(d)\}$, $(Ay_n)(s) + \frac{1}{n} \ge \Theta$, $s \in [d,t]$, $\{(Ay_n)(s)\}$ and $\{y_n(s)\}$ are bounded and equicontinuous on $[d,t]$

$$y(t) - y(d) = -\int_d^t a(s)f(s,(Ay)(s),y(s))ds. \tag{3.8}$$

Putting $t = d$ in (3.2), we have

$$\int_{y_n(d)}^{-\frac{1}{n}} \frac{dy_n}{G(y_n)} \leq \int_0^d a(s)k(s)ds \sup F[\Theta, +\infty). \tag{3.9}$$

Letting $n \to \infty$ and $d \to 0^+$, we obtain

$$y(0^+) = \lim_{d \to 0^+} y(d) = 0.$$

Letting $d \to 0^+$ in (3.8), we have

$$y(t) = -\int_0^t a(s)f(s, (Ay)(s), y(s))ds, \quad t \in (0, 1), \tag{3.10}$$

and

$$(Ay)(1) = \sum_{i=1}^{m-2} \alpha_i (Ay)(\xi_i).$$

Hence $x(t) = (Ay)(t)$ is a positive solution of (1.1). □

Theorem 3.2 Suppose that (H_1)-(P_3) hold. Then the set of positive solutions of (1.1) is compact in $C^1[0, 1]$.

Proof Let $M = \{y \in C[0, 1]$: $(Ay)(t)$ is a positive solution of equation (1.1) $\}$. We show that

(1) M is not empty;

(2) M is relatively compact(bounded, equicontinuous);

(3) M is closed.

Obviously, Theorem 3.1 implies M is not empty.

First, we show that $M \subset C[0, 1]$ is relatively compact. For any $y \in M$, differentiating (3.10) and using (H_2), we obtain

$$\begin{aligned} -y'(t) &= a(t)f(t, (Ay)(t), y(t)) \\ &\leq a(t)|f(t, (Ay)(t), y(t))| \\ &\leq a(t)k(t)F[\Theta, +\infty)G(y(t)), \ t \in (0, 1), \end{aligned}$$

$$\begin{aligned} \frac{-y'(t)}{G(y(t))} &\leq a(t)k(t) \sup F[(Ay)(t), +\infty) \\ &\leq a(t)k(t) \sup F[\Theta, +\infty), t \in [0, 1]. \end{aligned} \tag{3.11}$$

Integrating for (3.11) from 0 to t, we have

$$\int_{y(t)}^0 \frac{dy}{G(y)} \leq \int_0^1 a(s)k(s)ds \sup F[\Theta, +\infty), t \in [0, 1]. \tag{3.12}$$

Now (H_3) and (3.12) show that for any $y \in M$, there exists $K > 0$ such that $|y(t)| < K$,$\forall\ t \in [0, 1]$. Then M is bounded.

For any $y \in M$, we obtain from (3.11)

$$\begin{aligned} -y'(t) &= a(t)f(t, (Ay)(t), y(t)) \\ &\leq a(t)|f(t, (Ay)(t), y(t))| \\ &\leq a(t)k(t)F[\Theta, +\infty)G(y(t)), \ t \in (0, 1), \end{aligned}$$

and

$$\begin{aligned} y'(t) &= -a(t)f(t,(Ay)(t),y(t)) \\ &\le a(t)|f(t,(Ay)(t),y(t))| \\ &\le a(t)k(t)F[\Theta,+\infty)G(y(t)),\ t\in(0,1), \end{aligned}$$

which yields

$$\frac{-y'(t)}{G(y(t))+1} \le a(t)k(t)\sup F[\Theta,+\infty),\ t\in(0,1), \tag{3.13}$$

and

$$\frac{y'(t)}{G(y(t))+1} \le a(t)k(t)\sup F[\Theta,+\infty),\ t\in(0,1). \tag{3.14}$$

Notice that the rights are always positive in (3.13) and (3.14). Let $I(y(t)) = \int_0^{y(t)} \frac{dy}{G(y)+1}$. For any $t_1, t_2 \in [0,1]$, integrating for (3.13) and (3.14) from t_1 to t_2 ,we obtain

$$|I(y(t_1)) - I(y(t_2))| \le \int_{t_1}^{t_2} a(t)k(t)F[\Theta,+\infty)dt. \tag{3.15}$$

Since I^{-1} is uniformly continuous on $[I(-K),0]$, for any $\overline{\epsilon} > 0$, there is a $\epsilon' > 0$ such that

$$|I^{-1}(s_1) - I^{-1}(s_2)| < \overline{\epsilon}, \forall |s_1 - s_2| < \epsilon', s_1, s_2 \in [I(-K),0]. \tag{3.16}$$

And (3.15) guarantees that for $\epsilon' > 0$, there is a $\delta' > 0$ such that

$$|I(y(t_1)) - I(y(t_2))| < \epsilon', \forall |t_1 - t_2| < \delta', t_1, t_2 \in [0,1]. \tag{3.17}$$

Now (3.16) and (3.17) yield that

$$|y(t_1) - y(t_2)| = |I^{-1}(I(y(t_1)) - I^{-1}(I(y(t_2))| < \overline{\epsilon},\ t_1, t_2 \in [0,1], \tag{3.18}$$

which means that M is equicontinuous. So M is relatively compact.

Second, we show that M is closed. Suppose that $\{y_n\} \subseteq M$ and $\lim_{n\to+\infty} \max_{t\in[0,1]} |y_n(t) - y_0(t)| = 0$. Obviously $y_0 \in C[0,1]$ and $\lim_{n\to+\infty} (Ay_n)(t) = (Ay_0)(t)$, $t \in [0,1]$. Moreover,

$$\begin{aligned} (Ay_n)(t) &= \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\int_0^1 -y_n(\tau)d\tau - \frac{\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i} -y_n(\tau)d\tau}{1-\sum_{i=1}^{m-2}\alpha_i} - \int_0^t -y_n(\tau)d\tau \\ &< \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\int_0^1 -y_n(\tau)d\tau \\ &< \frac{K}{1-\sum_{i=1}^{m-2}\alpha_i}, t\in[0,1]. \end{aligned} \tag{3.19}$$

For $y_n \in M$, from (3.10) we obtain

$$y_n(t) = -\int_0^t a(s)f(s,(Ay_n)(s),y_n(s))ds, \quad t\in(0,1). \tag{3.20}$$

For fixed $t \in (0,1)$, there exists $0 < d < t$ such that

$$y_n(t) - y_n(d) = -\int_d^t a(s)f(s,(Ay_n)(s),y_n(s))ds. \tag{3.21}$$

Since $y_n(s) \leq \max\{\tau, W(d)\}, (Ay_n)(s) \geq \Theta, s \in [d,t]$, the Lebesgue Dominated Convergence Theorem yields that

$$y_0(t) - y_0(d) = -\int_d^t a(s)f(s,(Ay_0)(s),y_0(s))ds, \;\; t \in (0,1). \tag{3.22}$$

From (3.10), we have

$$\begin{aligned} -y_n'(t) &= a(t)f(t,(Ay_n)(s),y_n(s)) \\ &\leq a(t)k(t)F[\Theta,+\infty)G(y_n(t)), \quad t \in (0,1), \end{aligned}$$

which yields

$$\frac{-y_n'(t)}{G(y_n(t))} \leq a(t)k(t)\sup F[\Theta,+\infty), \;\; t \in (0,1).$$

Integrating from 0 to d

$$\int_{y_n(d)}^0 \frac{dy_n}{G(y_n)} \leq \int_0^d a(s)k(s)ds \sup F[\Theta,+\infty). \tag{3.23}$$

Letting $n \to \infty$ and $d \to 0^+$, we obtain

$$y_0(0^+) = \lim_{d\to 0^+} y_0(d) = 0.$$

Letting $d \to 0^+$ in (3.22), we have

$$y_0(t) = -\int_0^t a(s)f(s,(Ay_0)(s),y_0(s))ds, \quad t \in (0,1), \tag{3.24}$$

and

$$(Ay_0)(1) = \sum_{i=1}^{m-2} \alpha_i (Ay_0)(\xi_i).$$

Then $x_0(t) = (Ay_0)(t)$ is a positive solution of (1.1). So $y_0 \in M$ and M is a closed set.

Hence $\{Ay, y \subseteq M\} \in C^1[0,1]$ is compact.

Theorem 3.3 Suppose (H_1)-(P_3) hold. Then (1.1) has a minimal positive solution and a maximal positive solution in $C^1[0,1]$.

Proof. Let $\Omega = \{x(t) : x(t)$ is a $C^1[0,1]$ positive solution of (1.1)$\}$. Theorem 3.1 implies that is nonempty. Define a partially ordered $\leq$ in $\Omega : x \leq y$ iff $x(t) \leq y(t)$ for any $t \in [0,1]$. We prove only that any chain in $<\Omega, \leq>$ has a lower bound in Ω. The rest is obtained from Zorn's lemma. Let $\{x_\alpha(t)\}$ be a chain in $<\Omega, \leq>$. Since $C[0,1]$ is a separable Banach space, there exists countable set at most $\{x_n(t)\}$, which is dense in $\{x_\alpha(t)\}$. Without loss of generality, we may assume that $\{x_n(t)\} \subseteq \{x_\alpha(t)\}$. Put $z_n(t) = \min\{x_1(t), x_2(t), \cdots, x_n(t)\}$. Since $\{x_\alpha(t)\}$ is a chain, $z_n(t) \in \Omega$ for any n (in fact, $z_n(t)$ equals one of $x_n(t)$) and $z_{n+1}(t) \leq z_n(t)$ for any n. Put $z(t) = \lim_{m\to+\infty} z_n(t)$. We prove that $z(t) \in \Omega$.

By Lemma 2.2, there exists $y_n(t)$ (e.g., $y_n(t)$ may be $z_n'(t)$), which is a solution of

$$(Ty)(t) = -\int_0^t a(s)f(s,(Ay)(s),y(s))ds \qquad t \in [0,1],$$

such that

$$z_n(t) = \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\int_0^1 -y_n(\tau)d\tau - \frac{\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i} -y_n(\tau)d\tau}{1-\sum_{i=1}^{m-2}\alpha_i} - \int_0^t -y_n(\tau)d\tau.$$

(3.2) imply that $\{\|y_n\|\}$ is bounded. From Lemma 2.3, there exists a subsequence $\{y_{n_k}(t)\}$ of $\{y_n(t)\}$, $i \neq j, n_i \neq n_j$, which converges everywhere on $[0, 1]$. Without loss of generality, let $\{y_{n_k}(t)\}$ be itself of $\{y_n(t)\}$. Put $y_0(t) = \lim_{m\to+\infty} y_n(t), t \in [0, 1]$. Use $y_n(t), y_0(t)$, and 0 in place of $y_n(t), y(t)$, and $1/n$ in Theorem 3.1, respectively. A similar argument to show Theorem 3.1 yields that $y_0(t)$ is a solution of

$$y(t) = -\int_0^t a(s)f(s,(Ay)(s), y_n(s))ds, \qquad t \in [0, 1].$$

The boundedness of $\{\|y_n\|\}$ leads to

$$\begin{aligned} z(t) &= \lim_{m\to+\infty} z_n(t) \\ &= \lim_{m\to+\infty}[\frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\int_0^1 -y_n(\tau)d\tau - \frac{\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i} -y_n(\tau)d\tau}{1-\sum_{i=1}^{m-2}\alpha_i} - \int_0^t -y_n(\tau)d\tau] \\ &= \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\int_0^1 -y_0(\tau)d\tau - \frac{\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i} -y_0(\tau)d\tau}{1-\sum_{i=1}^{m-2}\alpha_i} - \int_0^t -y_0(\tau)d\tau. \end{aligned}$$

Hence $z \in \Omega$. By Lemma 2.2, for any $x \in \{x_\alpha\}$, there exists $\{x_{n_k}\} \subseteq \{x_n\}$ such that $\|x_{n_k} - x\| \to 0$. Notice that $x_{n_k}(t) \geq z_{n_k}(t) \geq z(t), t \in [0, 1]$. Letting $k \to +\infty$, we have $x(t) \geq z(t)$, $t \in [0, 1]$; i.e.,$\{x_\alpha\}$ has lower boundedness in Ω. Zorn's lemma shows that (1.1) has a minimal $C^1[0, 1]$ positive solution. By a similar proof, we can get the a maximal $C^1[0, 1]$ positive solution. The proof is complete.

Theorem 3.4 Suppose that (H_1)-(P_3) hold , $f(t, x, z)$ is decreasing in x for all $(t, z) \in [0, 1] \times R_-$, $a(0)f(0, x, z) \neq 0$ and $\lim_{t\to0} f(t, x, y) \neq +\infty$. Then (1.1) has an unique positive solution in $C^1[0, 1]$.

Proof. Assume that x_1 and x_2 are two positive different solutions to (1.1), i.e., there exists $t_0 \in (0, 1]$ such that $x_1(t_0) \neq x_2(t_0)$. Without loss of generality, assume that $x_1(t_0) > x_2(t_0)$. Let $\varphi(t) = x_1(t) - x_2(t)$ for all $t \in [0, 1]$. Obviously, $\varphi \in C[0, 1] \cap C^1(0, 1]$ with $\varphi(t_0) > 0$.

Let $t_* = \inf\{0 < t < t_0 | \varphi(s) > 0 \text{ for all } s \in t \in [t, t_0]\}$ and $t^* = \sup\{t_0 < t < 1 | \varphi(s) > 0 \text{ for all } s \in t \in [t_0, t]\}$. It is easy to see that $\varphi(t) > 0$ for all $t \in (t_*, t^*)$ and φ has maximum in $[t_*, t^*]$. Let t' satisfying that $\varphi(t') = \max_{t\in[t_*,t^*]} \varphi(t)$. There are three cases: (1) $t' \in (t_*, t^*)$; (2) $t' = t^* = 1$;(3) $t' = 0$.

(1) $t' \in (t_*, t^*)$. It is easy to see that $\varphi''(t') \leq 0$ and $\varphi'(t') = 0$. Then $\varphi''(t') = x_1''(t') - x_2''(t')$

$$= -a(t')f(t', x_1(t'), x_1'(t')) + a(t')f(t', x_2(t'), x_2'(t')) > 0,$$

a contradiction.

(2) $t' = t^* = 1$. Since $t' = t^* = 1$, we have $\sum_{i=1}^{m-2} \alpha_i \max\{\varphi(\xi_i)\} > \sum_{i=1}^{m-2} \alpha_i \varphi(\xi_i) = \varphi(1)$, a contradiction to $0 < \sum_{i=1}^{m-2} \alpha_i < 1$.

(3) $t' = 0$. Since $t' = 0$ and x_1 and x_2 are solutions, the proof of lemma 2.2 implies that there exist $x_{n,1}$ and $x_{n,2}$ such that

$$\|x_{n,1} - x_1\| < \frac{\varphi(0)}{2}, \quad \|x_{n,2} - x_2\| < \frac{\varphi(0)}{2}$$

where

$$x_{n,1}(t) = \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i} \int_0^1 -y_{n,1}(\tau) d\tau - \frac{\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} -y_{n,1}(\tau) d\tau}{1 - \sum_{i=1}^{m-2} \alpha_i} - \int_0^t -y_{n,1}(\tau) d\tau, \qquad t \in [0,1],$$

$$x_{n,2}(t) = \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i} \int_0^1 -y_{n,2}(\tau) d\tau - \frac{\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} -y_{n,2}(\tau) d\tau}{1 - \sum_{i=1}^{m-2} \alpha_i} - \int_0^t -y_{n,2}(\tau) d\tau, \qquad t \in [0,1],$$

and

$$y_{n,1}(t) = -\frac{1}{n} - \int_0^t a(s) f(s, x_{n,1}(s) + \frac{1}{n}, y_{n,1}(s)) ds, \quad t \in [0,1],$$

$$y_{n,2}(t) = -\frac{1}{n} - \int_0^t a(s) f(s, x_{n,2}(s) + \frac{1}{n}, y_{n,2}(s)) ds, \quad t \in [0,1],$$

$y_{n,1}(t) \le -\frac{1}{n}$, $y_{n,2}(t) \le -\frac{1}{n}$ for all $t \in [0,1]$.

By a similar proof with above, there exists $t_1 \in (0,1]$ such that $x_{n,1}(t_1) \ne x_{n,2}(t_1)$. Without loss of generality, assume that $x_{n,1}(t_1) > x_{n,2}(t_1)$. Let $\varphi_n(t) = x_{n,1}(t) - x_{n,2}(t)$ for all $t \in [0,1]$. Obviously, $\varphi_n \in C[0,1] \cap C^1(0,1]$ with $\varphi_n(t_1) > 0$. Let $t_* = \inf\{0 < t < t_1 | \varphi_n(s) > 0 \text{ for all } s \in t \in [t, t_1]\}$ and $t^* = \sup\{t_1 < t < 1 | \varphi_n(s) > 0 \text{ for all } s \in t \in [t_1, t]\}$. It is easy to see that $\varphi_n(t) > 0$ for all $t \in (t_{1*}, t^{1*})$ and φ_n has maximum in $[t_{1*}, t^{1*}]$. Let t'' satisfying that $\varphi(t'') = \max_{t \in [t_{1*}, t^{1*}]} \varphi(t)$. There are three cases: 1) $t'' \in (t_{1*}, t^{1*})$; 2) $t'' = t^* = 1$; 3) $t'' = 0$.

The proof of 1) and 2) are similar with (1) and (2).

3) $t'' = 0$. We have $\varphi_n(t) < \varphi_n(0), \ t \in (0,1]$, $\varphi_n'(0) = 0$, $\varphi_n'(t_\xi) < 0$, $t_\xi \in (0,1)$. Then

$$\underline{\lim}_{t_\xi \to 0+} \varphi_n''(t) = \underline{\lim}_{t_\xi \to 0+} \frac{\varphi_n'(t_\xi) - \varphi_n'(0)}{t_\xi - 0} \le 0.$$

On the other hand, since $\varphi_n''(0) = x_{n,1}''(0) - x_{n,2}''(0)$

$$= -a(0) f(0, x_{n,1}(0) + \frac{1}{n}, x_{n,1}'(0)) + a(0) f(0, x_{n,2}(0) + \frac{1}{n}, x_{n,2}'(0)) > 0,$$

a contradiction. Then (1.1) has at most one solution. The proof is complete.

Example 3.1. In (1.1), let $f(t,x,y) = k(t)[1 + x^{-\gamma} + (-y)^{-\sigma} - (-y)\ln(-y)]$, $a(t) = t^{-\frac{1}{3}}$, and

$$k(t) = t^{-\frac{1}{2}}, \quad 0 < t < 1,$$

where $\gamma > 0, \sigma < -2$, and let $F(x) = 1 + x^{-\gamma}, G(y) = 1 + (-y)^{-\sigma} - (-y)\ln(-y)$. Then

$$f(t,x,y) \le k(t)F(x)G(y), \quad \delta = 1, \quad \beta(t) = k(t),$$

and

$$\int_{-\infty}^{-1} \frac{dy}{G(y)} = +\infty.$$

By Theorem 3.1, (1.1) at least has a positive solution and Corollary 3.1 implies the set of solutions is compact.

References

[1] K.Deimling, Nonlinear Functional Analysis, *Springer -Verlag, Berlin,* 1985.

[2] Y.Guo, W.Ge, Positive solutions for three-point boundary value problems with dependence on the first order derivative, *J. Math. Anal. Appl.,* **290**(2004), pp291-301.

[3] V.A.Il'in, E.I.Moiseev, Nonlocal boundary value problem of the second kind for a Sturm-Liouville operator, *Differential Equation* **23 (8)**(1987), pp979-987.

[4] V.A.Il'in, E.I.Moiseev, Nonlocal value problem of the first kind for a Sturm-Liouville operator in its differential and finite difference aspects, *Differential Equations* **23 (7)** (1987), pp803-810.

[5] B.Liu, Positive solutions of a nonlinear three-point boundary value problem, *Applied Mathematics and Computation* **132** (2002), pp11-28.

[6] R.Ma, Positive solutions for a nonlinear three-point boundary value problem, *Electron. J.Differential Equations* **34** (1999), pp1-8.

[7] C.P. Gupta, S.K. Ntouyas, P.Ch. Tsamatos, Solvability of an m-point boundary value problem for second order ordinary differential equations, J. Math. Anal. Appl. **189** (1995) pp575-584.

[8] R.Ma, Donal O'Regan, Solvability of singular second order m-point boundary value problems *Journal of Mathematical Analysis and Applications* **301**(2005), pp124-134.

[9] Mario Petrich and Norman Reilly, Completely regular semigroups, Canadian Mathematical Society series of monographs, *Springer New York* **64** (2002).

[10] D. Xia et al., Foundation of Real Variable Function and Applied Functional Analysis, *Shanghai Science and Technology Press, Shanghai,* (1987) (in Chinese).

[11] G.Yang, Minimal Positive Solutions to Some Singular Second-Order Differential Equations,*Computers J. Math. Anal. Appl.* **266** (2002), pp479-491.

[12] G.Yang, Positive solutions of some second order nonlinear singular differential equations, *Computers Math.Appl.* **45** (2003), pp604-615.

[13] G.Yang, Second order singular boundary value problems with sign-changing nonlinearities on infinity intervals. *Nonlinear Analysis Forum.* **9 (2)** (2004), pp169-174.

(Received September 19, 2008)