Electronic Journal of Qualitative Theory of Differential Equations
2009, No. **62**, 1-12; **http://www.math.u-szeged.hu/ejqtde/**

ON Ψ-BOUNDED SOLUTIONS FOR NON-HOMOGENEOUS MATRIX LYAPUNOV SYSTEMS ON $\mathbb{R}$

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Abstract. In this paper we provide necesssary and sufficient conditions for the existence of at least one Ψ-bounded solution on $\mathbb{R}$ for the system $X' = A(t)X + XB(t) + F(t)$, where $F(t)$ is a Lebesgue Ψ-integrable matrix valued function on $\mathbb{R}$. Further, we prove a result relating to the asymptotic behavior of the Ψ-bounded solutions of this system.

1. Introduction

The importance of matrix Lyapunov systems, which arise in a number of areas of control engineering problems, dynamical systems, and feedback systems are well known. This paper deals with the linear matrix differential system

$$X' = A(t)X + XB(t) + F(t) \tag{1.1}$$

where $A(t)$, $B(t)$ and $F(t)$ are continuous $n \times n$ matrix-valued functions on $\mathbb{R}$. The basic problem under consideration is the determination of necessary and sufficient conditions for the existence of a solution with some specified boundedness condition. A Clasical result of this type, for system of differential equations is given by Coppel [4, Theorem 2, Chapter V].

The problem of Ψ-boundedness of the solutions for systems of ordinary differential equations has been studied in many papers, [1, 2, 3, 5, 9, 10]. Recently [11, 7], extended the concept of Ψ-boundedness of the solutions to Lyapunov matrix differential equations. In [6], the author obtained necessary and sufficient conditions for the non homogenous system $x' = A(t)x + f(t)$, to have at least one Ψ-bounded solution on $\mathbb{R}$ for every Lebesgue Ψ-integrable function f on $\mathbb{R}$.

The aim of present paper is to give a necessary and sufficient condition so that the nonhomogeneous matrix Lyapunov system (1.1) has at least one Ψ-bounded solution on $\mathbb{R}$ for every Lebesgue Ψ-integrable

2000 *Mathematics Subject Classification.* 34D05, 34C11.
Key words and phrases. fundamental matrix; Ψ-bounded; Ψ-integrable.

matrix function F on $\mathbb{R}$. The introduction of the matrix function Ψ permits to obtain a mixed asymptotic behavior of the components of the solutions. Here, Ψ is a continuous matrix-valued function on $\mathbb{R}$. The results of this paper include results of Diamandescu [6], as a particular case when $B(t) = O_n$.

2. Preliminaries

In this section we present some basic definitions, notations and results which are useful for later discussion.

Let $\mathbb{R}^n$ be the Euclidean n-space. For $u = (u_1, u_2, u_3, \ldots, u_n)^T \in \mathbb{R}^n$, let $\|u\| = \max\{|u_1|, |u_2|, |u_3|, \ldots, |u_n|\}$ be the norm of u. Let $\mathbb{R}^{n\times n}$ be the linear space of all $n\times n$ real valued matrices. For a $n\times n$ real matrix $A = [a_{ij}]$, we define the norm $|A| = \sup_{\|u\|\leq 1} \|Au\|$. It is well-known that

$$|A| = \max_{1\leq i\leq n}\{\sum_{j=1}^{n} |a_{ij}|\}.$$

Let $\Psi_k : \mathbb{R} \to \mathbb{R}-\{0\}$ ($\mathbb{R}-\{0\}$ is the set of all nonzero real numbers), $k = 1, 2, \ldots n$, be continuous functions, and let

$$\Psi = \operatorname{diag}[\Psi_1, \Psi_2, \ldots, \Psi_n].$$

Then the matrix $\Psi(t)$ is an invertible square matrix of order n, for all $t \in \mathbb{R}$.

Definition 2.1. [8] Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$ then the Kronecker product of A and B written $A \otimes B$ is defined to be the partitioned matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & . & . & . & a_{1n}B \\ a_{21}B & a_{22}B & . & . & . & a_{2n}B \\ . & . & . & . & . & . \\ a_{m1}B & a_{m2}B & . & . & . & a_{mn}B \end{bmatrix}$$

is an $mp \times nq$ matrix and is in $\mathbb{R}^{mp\times nq}$.

Definition 2.2.[8] Let $A = [a_{ij}] \in \mathbb{R}^{m\times n}$, then the vectorization operator
$Vec : \mathbb{R}^{n\times n} \to \mathbb{R}^{n^2}$, defined and denote by

$$\hat{A} = VecA = \begin{bmatrix} A_{.1} \\ A_{.2} \\ . \\ . \\ A_{.n} \end{bmatrix}, \text{ where } A_{.j} = \begin{bmatrix} a_{1j} \\ a_{2j} \\ . \\ . \\ a_{mj} \end{bmatrix} (1 \leq\ j \leq\ n).$$

Lemma 2.1. The vectorization operator $Vec : \mathbb{R}^{n\times n} \to \mathbb{R}^{n^2}$, is a linear and one-to-one operator. In addition, Vec and Vec^{-1} are continuous operators.

Proof. The fact that the vectorization operator is linear and one-to-one is immediate. Now, for$A = [a_{ij}] \in \mathbb{R}^{n\times n}$, we have

$$\|Vec(A)\| = \max_{1\leq i,j\leq n} \{|a_{ij}|\} \leq \max_{1\leq i\leq n} \left\{ \sum_{j=1}^{n} |a_{ij}| \right\} = |A| \, .$$

Thus, the vectorization operator is continuous and $\|Vec\| \leq 1$.

In addition, for $A = I_n$ (identity $n \times n$ matrix) we have $\|Vec(I_n)\| = 1 = |I_n|$ and then,$\|Vec\| = 1$.

Obviously, the inverse of the vectorization operator, $Vec^{-1} : \mathbb{R}^{n^2} \to \mathbb{R}^{n\times n}$, is defined by

$$Vec^{-1}(u) = \begin{bmatrix} u_1 & u_{n+1} & . & . & . & u_{n^2-n+1} \\ u_2 & u_{n+2} & . & . & . & u_{n^2-n+2} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ u_n & u_{2n} & . & . & . & u_{n^2} \end{bmatrix} .$$

Where $u = (u_1, u_2, u_3, ..., u_{n^2})^T \in \mathbb{R}^{n^2}$.

We have$|Vec^{-1}(u)| = \max_{1\leq i\leq n} \left\{ \sum_{j=0}^{n-1} |u_{nj+i}| \right\} \leq n. \max_{1\leq i\leq n} \{|u_i|\} = n. \|u\|$.

Thus, Vec^{-1} is a continuous operator. Also, if we take $u = VecA$ in the above inequality, then the following inequality holds

$$|A| \leq n\|VecA\|,$$

for every $A \in \mathbb{R}$. □

Regarding properties and rules for Kronecker product of matrices we refer to [8].

Now by applying the Vec operator to the nonhomogeneous matrix Lyapunov system (1.1) and using Kronecker product properties, we have

$$\hat{X}'(t) = H(t)\hat{X}(t) + \hat{F}(t), \tag{2.1}$$

where $H(t) = (B^T \otimes I_n) + (I_n \otimes A)$ is a $n^2 \times n^2$ matrix and $\hat{F}(t) = VecF(t)$ is a column matrix of order n^2. System (2.1) is called the Kronecker product system associated with (1.1).
The corresponding homogeneous system of (2.1) is

$$\hat{X}'(t) = H(t)\hat{X}(t). \tag{2.2}$$

Definition 2.3. A function $\gamma : \mathbb{R} \to \mathbb{R}^n$ is said to be Ψ- bounded on $\mathbb{R}$ if $\Psi\gamma$ is bounded on $\mathbb{R}$ $\left(i.e., \sup\limits_{t\in\mathbb{R}} \|\Psi(t)\gamma(t)\| < +\infty\right)$.

Extend this definition for matrix functions.

Definition 2.4. A matrix function $F : \mathbb{R} \to \mathbb{R}^{n\times n}$ is said to be Ψ bounded on $\mathbb{R}$ if the matrix function ΨF is bounded on $\mathbb{R}$ $\left(i.e., \sup\limits_{t\in\mathbb{R}} |\Psi(t)F(t)| < \infty\right)$.

Definition 2.5. A function $\gamma : \mathbb{R} \to \mathbb{R}^n$ is said to be Lebesgue Ψ - integrable on $\mathbb{R}$ if γ is measurable and $\Psi\gamma$ is Lebesgue integrable on $\mathbb{R}$ $\left(i.e., \int\limits_{-\infty}^{\infty} \|\Psi(t)\gamma(t)\|dt < \infty\right)$.

Extend this definition for matrix functions.

Definition 2.6. A matrix function $F : \mathbb{R} \to \mathbb{R}^{n\times n}$ is said to be Lebesgue Ψ integrable on $\mathbb{R}$ if F is measurable and ΨF is Lebesgue integrable on $\mathbb{R}$
$\left(i.e., \int\limits_{-\infty}^{\infty} |\Psi(t)F(t)|dt < \infty\right)$.

Now we shall assume that A and B are continuous $n \times n$ matrices on $\mathbb{R}$ and F is a Lebesgue Ψ-integrable matrix function on $\mathbb{R}$.

By a solution of (1.1), we mean an absolutely continuous matrix function $W(t)$ satisfying the equation (1.1) for all most all $t \in \mathbb{R}$.

The following lemmas play a vital role in the proof of main result.

Lemma 2.2. The matrix function $F : \mathbb{R} \to \mathbb{R}^{n\times n}$ is Lebesgue Ψ-integrable on $\mathbb{R}$ if and only if the vector function $VecF(t)$ is Lebesgue $I_n \otimes \Psi$ - integrable on $\mathbb{R}$.

Proof. From the proof of Lemma 2.1, it follows that

$$\frac{1}{n}|A| \leq \|VecA\|_{\mathbb{R}^{n^2}} \leq |A|,$$

for every $A \in \mathbb{R}^{n\times n}$.

Put $A = \Psi(t)F(t)$ in the above inequality, we have

$$\frac{1}{n}|\Psi(t)F(t)| \leq \|(I_n \otimes \Psi(t)).VecF(t)\|_{\mathbb{R}^{n^2}} \leq |\Psi(t)F(t)|, \tag{2.3}$$

$t \in \mathbb{R}$, for all matrix functions $F(t)$. Lemma follows from (2.3). $\square$

Lemma 2.3. The matrix function $F(t)$ is Ψ - bounded on $\mathbb{R}$ if and only if the vector function $VecF(t)$ is $I_n \otimes \Psi$ - bounded on $\mathbb{R}$.

Proof. The proof easily follows from the inequality (2.3). $\square$

Lemma 2.4. Let $Y(t)$ and $Z(t)$ be the fundamental matrices for the systems

$$X'(t) = A(t)X(t), \tag{2.4}$$

and

$$X'(t) = B^T(t)X(t), \ \ t \in \mathbb{R} \tag{2.5}$$

respectively. Then the matrix $Z(t) \otimes Y(t)$ is a fundamental matrix of (2.2).

Proof. Consider

$$\begin{aligned}(Z(t) \otimes Y(t))' &= (Z'(t) \otimes Y(t)) + (Z(t) \otimes Y'(t)) \\ &= (B^T(t)Z(t) \otimes Y(t)) + (Z(t) \otimes A(t)Y(t)) \\ &= (B^T(t) \otimes I_n)(Z(t) \otimes Y(t)) + (I_n \otimes A(t))(Z(t) \otimes Y(t)) \\ &= [B^T(t) \otimes I_n + I_n \otimes A(t)](Z(t) \otimes Y(t)) \\ &= H(t)(Z(t) \otimes Y(t)),\end{aligned}$$

for all $t \in \mathbb{R}$.

On the other hand, the matrix $Z(t) \otimes Y(t)$ is a nonsingular matrix for all $t \in \mathbb{R}$ (because $X(t)$ and $Y(t)$ are nonsingular matrices for all $t \in \mathbb{R}$). $\square$

Let the matrix space $\mathbb{R}^{n\times n}$ be represented as a direct sum of three subspaces X_-, X_0, X_+ such that a solution $W(t)$ of (1.1) is Ψ-bounded on $\mathbb{R}$ if and only if $W(0) \in X_0$ and Ψ-bounded on $\mathbb{R}$ if and only if $W(0) \in X_- \oplus X_0$. Also, let $\mathbb{P}_-, \mathbb{P}_0, \mathbb{P}_+$ denote the corresponding projection of $\mathbb{R}^{n\times n}$ onto X_-, X_0, X_+ respectively.

Then the vector space $\mathbb{R}^{n^2}$ represents a direct sum of three sub spaces S_-, S_0, S_+ such that a solution $\hat{W}(t) = VecW(t)$ of (2.1) is $I_n \otimes \Psi$-bounded on $\mathbb{R}^{n^2}$ if and only if $\hat{W}(0) \in S_0$ and $I_n \otimes \Psi$-bounded on $\mathbb{R}$ if and only if $\hat{W}(0) \in S_- \oplus S_0$ and also Q_-, Q_0, Q_+ denote the corresponding projection of $\mathbb{R}^{n^2}$ onto S_-, S_0, S_+ respectively.

Theorem 2.1. *Let $A(t)$, $B(t)$ and $F(t)$ be continuous matrix functions on $\mathbb{R}$. If $Y(t)$ and $Z(t)$ are the fundamental matrices for the systems* (2.4) *and* (2.5), *then*

$$\hat{X}(t) = \int_{-\infty}^{t} (Z(t) \otimes Y(t))\mathbb{P}_-(Z^{-1}(s) \otimes Y^{-1}(s))\hat{F}(s)ds$$
$$+ \int_{0}^{t} (Z(t) \otimes Y(t))\mathbb{P}_0(Z^{-1}(s) \otimes Y^{-1}(s))\hat{F}(s)ds$$
$$- \int_{t}^{\infty} (Z(t) \otimes Y(t))\mathbb{P}_+(Z^{-1}(s) \otimes Y^{-1}(s))\hat{F}(s)ds \tag{2.6}$$

is a solution of (2.1) *on* $\mathbb{R}$.

Proof. It is easily seen that $\hat{X}$ is the solution of (2.1) on $\mathbb{R}$. □

The following theorems are useful in the proofs of our main results.

Theorem 2.2. [6] *Let A be a continuous $n \times n$ real matrix on $\mathbb{R}$ and let Y be the fundamental matrix of the homogeneous system $x' = A(t)x$ with $Y(0) = I_n$. Then the nonhomogeneous system*

$$x' = A(t)x + f(t) \tag{2.7}$$

has at least one Ψ-bounded solution on $\mathbb{R}$ for every Lebesgue Ψ-integrable function $f : \mathbb{R} \to \mathbb{R}^n$ on $\mathbb{R}$ if and only if there exists a positive constant K such that

$$\begin{gathered}
|\Psi(t)Y(t)P_-Y^{-1}(s)\Psi^{-1}(s)| \le K \quad \textit{for } t > 0,\ s \le 0 \\
|\Psi(t)Y(t)(P_0 + P_-)Y^{-1}(s)\Psi^{-1}(s)| \le K \quad \textit{for } t > 0,\ s > 0, s < t \\
|\Psi(t)Y(t)P_+Y^{-1}(s)\Psi^{-1}(s)| \le K \quad \textit{for } t > 0, s > 0,\ s \ge t \\
|\Psi(t)Y(t)P_-Y^{-1}(s)\Psi^{-1}(s)| \le K \quad \textit{for } t \le 0,\ s < t \\
|\Psi(t)Y(t)(P_0 + P_+)Y^{-1}(s)\Psi^{-1}(s)| \le K \quad \textit{for } t \le 0,\ s \ge t,\ s < 0 \\
|\Psi(t)Y(t)P_+Y^{-1}(s)\Psi^{-1}(s)| \le K \quad \textit{for } t \le 0,\ s \ge t,\ s \ge 0.
\end{gathered} \tag{2.8}$$

Theorem 2.3. [6] *Suppose that:*
(1) the fundamental matrix $Y(t)$ of $x' = A(t)x$ satisfies:

(a) *condition* (2.8) *is satisfied for some $K > 0$;*

(b) *the following conditions are satisfied:*

(i) $\lim_{t\to\pm\infty} |\Psi(t)Y(t)P_0| = 0$;

(ii) $\lim_{t\to-\infty} |\Psi(t)Y(t)P_+| = 0$;

(iii) $\lim_{t\to+\infty} |\Psi(t)Y(t)P_-| = 0$;

(2) the function $f : \mathbb{R} \to \mathbb{R}^n$ is Lebesgue Ψ-integrable on $\mathbb{R}$.

Then, every Ψ-bounded solution x of (2.7) *is such that*

$$\lim_{t\to\pm\infty} \|\Psi(t)x(t)\| = 0.$$

3. Main result

The main theorms of this paper are proved in this section.

Theorem 3.1. *If A and B are continuous $n \times n$ real matrices on $\mathbb{R}$, then* (1.1) *has at least one Ψ-bounded solution on $\mathbb{R}$ for every Lebesgue Ψ-integrable matrix function $F : \mathbb{R} \to \mathbb{R}^{n\times n}$ on $\mathbb{R}$ if and only if there exists a positive constant K such that*

(3.1)

$$|(Z(t) \otimes \Psi(t)Y(t))Q_-(Z^{-1}(s) \otimes Y^{-1}(s)\Psi^{-1}(s))| \le K, \quad \text{for } t > 0,\ s \le 0$$

$$|(Z(t) \otimes \Psi(t)Y(t))(Q_0 + Q_-)(Z^{-1}(s) \otimes Y^{-1}(s)\Psi^{-1}(s))| \le K, \quad \text{for } t > 0,\ s > 0, s < t$$

$$|(Z(t) \otimes \Psi(t)Y(t))Q_+(Z^{-1}(s) \otimes Y^{-1}(s)\Psi^{-1}(s))| \le K, \quad \text{for } t > 0, s > 0,\ s \ge t$$

$$|(Z(t) \otimes \Psi(t)Y(t))Q_-(Z^{-1}(s) \otimes Y^{-1}(s)\Psi^{-1}(s))| \le K, \quad \text{for } t \le 0,\ s < t$$

$$|(Z(t) \otimes \Psi(t)Y(t))(Q_0 + Q_+)(Z^{-1}(s) \otimes Y^{-1}(s)\Psi^{-1}(s))| \le K, \quad \text{for } t \le 0,\ s \ge t,\ s < 0$$

$$|(Z(t) \otimes \Psi(t)Y(t))Q_+(Z^{-1}(s) \otimes Y^{-1}(s)\Psi^{-1}(s))| \le K, \quad \text{for } t \le 0,\ s \ge t,\ s \ge 0.$$

Proof. Suppose that the equation (1.1) has atleast one Ψ - bounded solution on $\mathbb{R}$ for every Lebesgue Ψ - integrable matrix function $F : \mathbb{R} \to \mathbb{R}^{n\times n}$.

Let $\hat{F} : \mathbb{R} \to \mathbb{R}^{n^2}$ be a Lebesgue $I_n \otimes \Psi$ - integrable function on $\mathbb{R}$. From Lemma 2.2, it follows that the matrix function $F(t) = Vec^{-1}\hat{F}(t)$ is Lebesgue Ψ - integrable matrix function on $\mathbb{R}$. From the hypothesis, the system (1.1) has at least one Ψ - bounded solution $W(t)$ on $\mathbb{R}$. From Lemma 2.3, it follows that the vector valued function $\hat{W}(t) = VecW(t)$ is a $I_n \otimes \Psi$- bounded solution of (2.1) on $\mathbb{R}$.

Thus, system (2.1) has at least one $I_n \otimes \Psi$ - bounded solution on $\mathbb{R}$ for every Lebesgue $I_n \otimes \Psi$ - integrable function $\hat{F}$ on $\mathbb{R}$.

From Theorem 2.2, there is a positive constant K such that the fundamental matrix $S(t) = Z(t) \otimes Y(t)$ of the system (2.2) satisfies the condition

$$\begin{aligned}
&|(I_n \otimes \Psi(t))S(t)Q_- S^{-1}(s)(I_n \otimes \Psi^{-1}(s))| \leq K, \\
&\qquad\qquad \text{for } t > 0,\ s \leq 0 \\
&|(I_n \otimes \Psi(t))S(t)(Q_0 + Q_-)S^{-1}(s)(I_n \otimes \Psi^{-1}(s))| \leq K, \\
&\qquad\qquad \text{for } t > 0,\ s > 0, s < t \\
&|(I_n \otimes \Psi(t))S(t)Q_+ S^{-1}(s)(I_n \otimes \Psi^{-1}(s))| \leq K, \\
&\qquad\qquad \text{for } t > 0, s > 0,\ s \geq t \\
&|(I_n \otimes \Psi(t))S(t)Q_- S^{-1}(s)(I_n \otimes \Psi^{-1}(s))| \leq K, \\
&\qquad\qquad \text{for } t \leq 0,\ s < t \\
&|(I_n \otimes \Psi(t))S(t)(Q_0 + Q_+)S^{-1}(s)(I_n \otimes \Psi^{-1}(s))| \leq K, \\
&\qquad\qquad \text{for } t \leq 0,\ s \geq t,\ s < 0 \\
&|(I_n \otimes \Psi(t))S(t)Q_+ S^{-1}(s)(I_n \otimes \Psi^{-1}(s))| \leq K, \\
&\qquad\qquad \text{for } t \leq 0,\ s \geq t,\ s \geq 0.
\end{aligned}$$

Putting $S(t) = Z(t) \otimes Y(t)$ and using Kronecker product properties, (3.1) holds.

Conversly suppose that (3.1) holds for some $K \geq 0$.

Let $F : \mathbb{R} \to \mathbb{R}^{n \times n}$ be a lebesgue Ψ - integrable matrix function on $\mathbb{R}$. From Lemma 2.2, it follows that the vector valued function $\hat{F}(t) = VecF(t)$ is a Lebesgue $I_n \otimes \Psi$ - integrable function on $\mathbb{R}$.

From Theorem 2.2, it follows the differential system (2.1) has at least one $I_n \otimes \Psi$ - bounded solution on $\mathbb{R}$. Let w(t) be this solution.

From Lemma 2.3, it follows that the matrix function
$W(t) = Vec^{-1}w(t)$ is a Ψ - bounded solution of the equation (1.1) on $\mathbb{R}$ (because $F(t) = Vec^{-1}\hat{F}(t)$).

Thus the matrix Lyapunov system (1.1) has at least one Ψ - bounded solution on $\mathbb{R}$ for every Lebesgue Ψ - integrable matrix function F on $\mathbb{R}$. $\square$

In a particular case, we have the following result.

Theorem 3.2. *If the homogeneous system ($F = O$ in (1.1)) has no nontrivial Ψ-bounded solution on $\mathbb{R}$, then the system (1.1) has a unique Ψ-bounded solution on $\mathbb{R}$ for every Lebesgue Ψ-integrable matrix function $F : \mathbb{R} \to \mathbb{R}^{n\times n}$ on $\mathbb{R}$ if and only if there exists a positive constant K such that*

(3.2)
$$|(Z(t) \otimes \Psi(t)Y(t))Q_-(Z^{-1}(s) \otimes Y^{-1}(s)\Psi^{-1}(s))| \leq K \quad \text{for } s < t$$
$$|(Z(t) \otimes \Psi(t)Y(t))Q_+(Z^{-1}(s) \otimes Y^{-1}(s)\Psi^{-1}(s))| \leq K \quad \text{for } t \leq s$$

Proof. In this case, $Q_0 = O$. The proof is simple by putting $Q_0 = O$ in Theorem 3.1. □

Next, we prove a theorem in which we will see that the asymptotic behavior of solutions to (1.1) is determined completely by the asymptotic behavior of the fundamental matrices $Y(t)$ and $Z(t)$ of (2.4) and (2.5) respectively.

Theorem 3.3. *Suppose that:*
(1) the fundamental matrices $Y(t)$ and $Z(t)$ of (2.4) and (2.5) satisfies:
  (a) *condition (3.1) is satisfied for some $K > 0$;*
  (b) *the following conditions are satisfied:*
    (i) $\lim\limits_{t\to\pm\infty} ||(Z(t) \otimes \Psi(t)Y(t))Q_0|| = 0$;
    (ii) $\lim\limits_{t\to-\infty} ||(Z(t) \otimes \Psi(t)Y(t))Q_+|| = 0$;
    (iii) $\lim\limits_{t\to+\infty} ||(Z(t) \otimes \Psi(t)Y(t))Q_-|| = 0$;

(2) the matrix function $F : \mathbb{R} \to \mathbb{R}^{n\times n}$ is Lebesgue Ψ-integrable on $\mathbb{R}$.
Then, every Ψ-bounded solution X of (1.1) is such that

$$\lim_{t\to\pm\infty} |\Psi(t)X(t)| = 0.$$

Proof. Let $X(t)$ be a Ψ - bounded solution of (1.1). From Lemma 2.3, it follows that the function $\hat{X}(t) = VecX(t)$ is a $I_n \otimes \Psi$- bounded solution on $\mathbb{R}$ of the differential system (2.1). Also from Lemma 2.2, the function $\hat{F}(t)$ is Lebesgue $I_n \otimes \Psi$ - integrable on $\mathbb{R}$. From the Theorem 2.2, it follows that

$$\lim_{t\to\pm\infty} \left\| (I_n \otimes \Psi(t))\, \hat{X}(t) \right\| = 0.$$

Now, from the inequality (2.3) we have

$$|\Psi(t)X(t)| \leq n \left\| (I_n \otimes \Psi(t))\, \hat{X}(t) \right\|, t \in \mathbb{R}$$

and, then
$$\lim_{t\to\pm\infty} |\Psi(t)X(t)| = 0.$$
$\square$

The next result follows from Theorems 3.2 and 3.3.

Corollary 3.4. *Suppose that*

(1) *the homogeneous system ($F = O$ in* (1.1)*) has no nontrivial Ψ-bounded solution on $\mathbb{R}$;*
(2) *the fundamental matrices $Y(t)$ and $Z(t)$ of* (2.4) *and* (2.5) *satisfies:*
(i) the condition (3.2) *for some $K > 0$.*
(ii) $\lim_{t\to-\infty} ||(Z(t)\otimes\Psi(t)Y(t))Q_+|| = 0$*;*
(iii) $\lim_{t\to+\infty} ||(Z(t)\otimes\Psi(t)Y(t))Q_-|| = 0$*;*
(3) *the matrix function $F : \mathbb{R} \to \mathbb{R}^{n\times n}$ is Lebesgue Ψ-integrable on $\mathbb{R}$.*

Then (1.1) *has a unique solution X on $\mathbb{R}$ such that*
$$\lim_{t\to\pm\infty} \|\Psi(t)X(t)\| = 0.$$

Note that Theorem 3.3 is no longer true if we require that the function F be Ψ-bounded on $\mathbb{R}$ (more, even $\lim\limits_{t\to\pm\infty} |\Psi(t)F(t)| = 0$), instead of the condition (3) in the above Theorem. This is shown in the following example.

Example. Consider (1.1) with $A(t) = I_2$, $B(t) = -I_2$ and
$F(t) = \begin{bmatrix} \sqrt{1+|t|} & \frac{1}{1+|t|} \\ 1 & 1 \end{bmatrix}$.

Then, $Y(t) = \begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix}$, $Z(t) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-t} \end{bmatrix}$ are the fundamental matrices for (2.4) and (2.5) respectively. Consider
$$\Psi(t) = \begin{bmatrix} \frac{1}{1+|t|} & 0 \\ 0 & \frac{1}{(1+|t|)^2} \end{bmatrix}.$$

Therefore, $Q_- = I_2$, $Q_+ = O_2$ and $Q_0 = O_2$. The conditions (3.1) are satisfied with $K = 1$. In addition, the hypothesis (1b) of Theorem 3.3 is satisfied. Because
$$\Psi(t)F(t) = \begin{bmatrix} \frac{1}{\sqrt{1+|t|}} & \frac{1}{(1+|t|)^2} \\ \frac{1}{(1+|t|)^2} & \frac{1}{(1+|t|)^2} \end{bmatrix},$$

the matrix function F is not Lebesgue Ψ-integrable on $\mathbb{R}$, but it is Ψ-bounded on $\mathbb{R}$, with $\lim\limits_{t\to\pm\infty} |\Psi(t)F(t)| = 0$. The solutions of the system (1.1) are

$$X(t) = \begin{bmatrix} p(t)+c_1 & q(t)+c_2 \\ t+c_3 & t+c_4 \end{bmatrix},$$

where

$$p(t) = \begin{cases} -\frac{2}{3}(1-t)^{3/2}, & t<0 \\ \frac{2}{3}(1+t)^{3/2}, & t\geq 0 \end{cases}$$

and

$$q(t) = \begin{cases} -\ln(1-t), & t<0 \\ \ln(1+t), & t\geq 0\,. \end{cases}$$

It is easily seen that $\lim\limits_{t\to\pm\infty} \|\Psi(t)X(t)\| = +\infty$, for all $c_1, c_2, c_3, c_4 \in \mathbb{R}$. It follows that the solutions of the system (1.1) are Ψ-unbounded on $\mathbb{R}$.

Remark. If in the above example, $F(t) = \begin{bmatrix} 0 & \frac{1}{1+|t|} \\ 1 & 1 \end{bmatrix}$, then $\int_{-\infty}^{+\infty} \|\Psi(t)F(t)\| dt = 2$. On the other hand, the solutions of (1.1) are

$$X(t) = \begin{bmatrix} c_1 & q(t)+c_2 \\ t+c_3 & t+c_4 \end{bmatrix},$$

where

$$q(t) = \begin{cases} -\ln(1-t), & t<0 \\ \ln(1+t), & t\geq 0\,. \end{cases}$$

We observe that the asymptotic properties of the components of the solutions are not the same. The first row first column element is bounded and the remaining elements are unbounded on $\mathbb{R}$. However, all solutions of (1.1) are Ψ-bounded on $\mathbb{R}$ and $\lim\limits_{t\to\pm\infty} \|\Psi(t)X(t)\| = 0$. This shows that the asymptotic properties of the components of the solutions are the same, via the matrix function Ψ. This is obtained by using a matrix function Ψ rather than a scalar function.

Acknowledgement. The authors would like to thank the anonymous referee for his valuable suggestions which helped to improve the quality of the presentation.

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(Received June 18, 2009)

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