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Lower Bounds and Positivity Conditions for Green's Functions to Second Order Differential-Delay Equations

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Abstract

We consider the Cauchy problem on the positive half-line for the differential-delay equation

$$\ddot{u}(t) + 2c_0(t)\dot{u}(t) + c_1(t)\dot{u}(t-h) + d_0(t)u(t) + d_1(t)u(t-h) + d_2(t)u(t-2h) = 0$$

where $c_k(t), d_j(t) \quad (t \geq 0;\ k = 0,1;\ j = 0,1,2)$ are continuous functions. Conditions providing the positivity of the Green function and a lower bound for that function are derived. Our results are new even in the case of ordinary differential equations. Applications of the obtained results to equations with nonlinear causal mappings are also discussed. Equations with causal mappings include ordinary differential and integro-differential equations. In addition, we establish positivity conditions for solutions of functional differential equations with variable and distributed delays.

Key words: differential-delay equation, Green's function, positivity, causal mapping
AMS (MOS) subject classification: 34K11, 34K06, 34K99

1 Introduction and statement of the main result

The theory of nonoscillations of solutions to functional differential equations was extensively developed. Mainly the first order equations were considered, see the well-known publications [1, 2, 12, 21, 23, 28], and references therein. Higher order equations were investigated in the very interesting papers [3, 4, 5, 22, 25].

It should be noted that the existence of nonoscillating solutions does not provide the positivity of the Green functions. Obtaining the positivity conditions for the Green functions requires additional efforts, but the positivity is very important for various applications. For instance, the positivity of Green's function provides the existence of non-negative solutions to equations with various nonlinearities, in particular, with nonlinear causal mappings, cf. [17, 18]. Moreover, positive Green's functions play an essential role in the stability theory, see [15, 16].

In this paper we consider the Cauchy problem for a nonautonomous second order differential-delay equation and derive conditions that provide the positivity of the Green function. In addition, a lower bound for that function is established. Applications to equations with nonlinear causal mappings are also discussed. It should be noted that equations with causal mappings include such traditional equations as ordinary differential and integro-differential equations, cf. [7, 27]. Moreover, we establish positivity conditions for solutions of functional differential equations with variable and distributed delays.

Denote $R_+ = [0, \infty)$, $C(\omega)$ is the space of continuous scalar-valued functions defined on a real segment ω with the sup-norm. For $t > 0$, let us consider the equation

$$\ddot{u}(t) + 2c_0(t)\dot{u}(t) + c_1(t)\dot{u}(t-h) + d_0(t)u(t) + d_1(t)u(t-h) + d_2(t)u(t-2h) = 0 \tag{1.1}$$

where $\dot{u}(t) = du(t)/dt, \ddot{u}(t) = d^2u(t)/dt^2$, $c_k(.), d_j(.) \in C(R_+)$ $(k = 0, 1;\ j = 0, 1, 2)$ and $\dot{c}_0(.) \in C(R_+)$.

The Green function $G(t, s)$ to equation (1.1) is a function defined for $t \ge s - 2h$ $(s \ge 0)$, having the continuous first and second derivatives in t for $t > s$, satisfying that equation for all $t > s \ge 0$ and the conditions

$$G(t,s) = 0 \ \ (s - 2h \le t \le s), \ \frac{\partial G(t,s)}{\partial t} = 0 \ \ (s - 2h \le t < s), \ \ \lim_{t \downarrow s} \frac{\partial G(t,s)}{\partial t} = 1. \tag{1.2}$$

Furthermore, put $c_0(t) \equiv c_0(0)$ for $-2h \le t \le 0$ and

$$a_1(t) = c_1(t) e^{\int_{t-h}^t c_0(s)ds}, \ a_2(t) = d_2(t) e^{\int_{t-2h}^t c_0(s)ds} \ \ (t \ge 0). \tag{1.3}$$

We need the following preliminary result.

Lemma 1.1 *Let the conditions*

$$-\dot{c}_0(t) + c_0^2(t) + d_0(t) \le 0 \ \text{and} \ -c_1(t)c_0(t-h) + d_1(t) \le 0 \ (t \ge 0) \tag{1.4}$$

hold and the Green function $G_0(t, s)$ to the equation

$$\ddot{u}(t) + a_1(t)\dot{u}(t-h) + a_2(t)u(t-2h) = 0 \ \ (t > 0) \tag{1.5}$$

be nonnegative. Then the Green function $G(t, s)$ to equation (1.1) is also nonnegative. Moreover,

$$G(t,s) \ge G_0(t,s) \ \ (t > s \ge 0). \tag{1.6}$$

Proof: Substitute the equality

$$u(t) = w(t)e^{-\int_0^t c_0(s)ds}$$

into (1.1). Then, taking into account that

$$\frac{d}{dt}\int_0^{t-h} c_0(s)ds = \frac{d}{dt}\int_h^t c_0(s_1 - h)ds_1 = c_0(t-h),$$

we have

$$e^{-\int_0^t c_0(s)ds}[\ddot{w}(t) - 2c_0(t)\dot{w}(t) + w(t)(-\dot{c}_0(t) + c_0^2(t) + d_0(t)) +$$
$$2(c_0(t)\dot{w}(t) - c_0^2(t)w(t))] + c_1(t)e^{-\int_0^{t-h} c_0(s)ds}[-c_0(t-h)w(t-h) + \dot{w}(t-h)] +$$

$$d_1(t)e^{-\int_0^{t-h} c_0(s)ds}w(t-h) + d_2(t)e^{-\int_0^{t-2h} c_0(s)ds}w(t-2h) = 0.$$

Or

$$\ddot{w}(t) + a_1(t)\dot{w}(t-h) + m_0(t)w(t) + m_1(t)w(t-h) + a_2(t)w(t-2h) = 0, \tag{1.7}$$

where

$$m_0(t) := -\dot{c}_0(t) + c_0^2(t) + d_0(t);\ m_1(t) := e^{\int_{t-h}^{t} c_0(s)ds}[-c_1(t)c_0(t-h) + d_1(t)].$$

According to (1.4), $m_0(t) \leq 0, m_1(t) \leq 0$. Hence, by the comparison principle, cf. [8, Chapter 2, Section 2], it easily follows that if the Green function to equation (1.5) is nonnegative, then the Green function to equation (1.7) is also nonnegative and (1.6) holds. But (1.7) and (1.1) are equivalent. This proves the lemma. □

2 The main result

In the sequel it is assumed that $c_1(t),\ c_2(t)$ *are positive for all* $t \geq 0$. Therefore the functions $a_1(t),\ a_2(t)$ defined by (1.3) are positive. Put

$$b_1 := \sup_{t\geq 0} a_1(t) \text{ and } b_2 := \sup_{t\geq 0} a_2(t) + \frac{b_1}{2}(b_1 - \inf_{t\geq 0}\ a_1(t)), \tag{2.1}$$

and consider the equation

$$\ddot{u}(t) + b_1\dot{u}(t-h) + b_2u(t-2h) = 0. \tag{2.2}$$

Denote by $H(t)$ the Green function to equation (2.2). By the Laplace transform we easily have

$$H(t) = \frac{1}{2\pi i}\int_{\nu-i\infty}^{\nu+i\infty} \frac{e^{\lambda t}d\lambda}{\lambda^2 + b_1\lambda e^{-\lambda h} + b_2 e^{-2\lambda h}}$$

where ν is a real constant. It is assumed that

$$\inf_{t\geq 0}\ a_1(t) > \frac{b_1}{2} + \frac{2}{b_1}\sup_{t\geq 0} a_2(t). \tag{2.3}$$

Then

$$b_2 < \sup_{t\geq 0} a_2(t) + \frac{b_1}{2}(b_1 - \frac{b_1}{2} - \frac{2}{b_1}\sup_{t\geq 0} a_2(t)) = \frac{b_1^2}{4}.$$

That is,

$$b_1^2 > 4b_2.$$

So the polynomial $z^2 + b_1z + b_2$ has the real roots

$$r_1 := -b_1/2 - \sqrt{b_1^2/4 - b_2},\ r_2 := -b_1/2 + \sqrt{b_1^2/4 - b_2}.$$

Now we are in a position to formulate our main result.

Theorem 2.1 *Let the conditions (1.4), (2.3) and*

$$-ehr_k < 1 \quad (k = 1, 2) \tag{2.4}$$

hold. Then the Green function $G(t, s)$ to equation (1.1) is nonnegative and

$$G(t, s) \geq H(t - s) \quad (t > s \geq 0). \tag{2.5}$$

This theorem is proved below. To the best of our knowledge, Theorem 2.1 is new even in the case of ordinary differential equations, cf. [9, 10, 24].

To prove Theorem 2.1 we need the following result.

Lemma 2.2 *Let conditions (2.3) and (2.4) hold. Then the Green function $G_0(t, s)$ to equation (1.5) is nonnegative and*

$$G_0(t, s) \geq H(t - s) \quad (t > s \geq 0). \tag{2.6}$$

Proof: With a real number b, let us consider the equation

$$\dot{u}(t) - bu(t - h) = 0 \quad (h = const > 0;\ \dot{u} = du/dt,\ t \geq 0). \tag{2.7}$$

The Green function $H_1(b, t)$ to this equation is a solution of (2.7) with the initial conditions

$$u(0) = 1;\ u(t) = 0\ (t < 0). \tag{2.8}$$

We need the following result: let the condition

$$-ehb < 1 \tag{2.9}$$

hold. Then the Green function to equation (1.1) is positive. For the proof see for instance [16]. Clearly, for any $b \geq 0$, condition (2.9) holds.

According to this result, under condition (2.4) with $b = r_k$ we have $H_1(r_k, t) \geq 0$ and thanks to the convolution property,

$$H(t) = \frac{1}{2\pi i} \int_{\nu - i\infty}^{\nu + i\infty} \frac{e^{\lambda t} d\lambda}{(\lambda - r_1 e^{-\lambda h})(\lambda - r_2 e^{-\lambda h})} = \int_0^t H_1(r_1, t - s) H_1(r_2, s) ds \geq 0.$$

Hence,

$$\dot{H}(t) = \frac{1}{2\pi i} \int_{\nu - i\infty}^{\nu + i\infty} \frac{\lambda e^{\lambda t} d\lambda}{(\lambda - r_1 e^{-\lambda h})(\lambda - r_2 e^{-\lambda h})}. \tag{2.10}$$

But

$$\frac{2\lambda}{(\lambda - e^{-\lambda} r_1)(\lambda - r_2 e^{-\lambda})} =$$

$$\frac{1}{\lambda - r_1 e^{-\lambda}} + \frac{1}{\lambda - r_2 e^{-\lambda}} + \frac{(r_1 + r_2) e^{-\lambda}}{(\lambda - r_k e^{-\lambda})(\lambda - r_k e^{-\lambda})}$$

Since $r_1 + r_2 = -b_1$, and $H(t) \geq 0$, we have

$$\dot{H}(t) = \frac{1}{2}[H_1(r_1, t) + H_1(r_2, t) - b_1 H(t - h)] \geq -\frac{b_1}{2} H(t - h).$$

Consequently

$$\dot{H}(t-h) \geq -\frac{b_1}{2} H(t-2h). \tag{2.11}$$

Now consider the equation

$$Eu(t) := \ddot{u}(t) + a_1(t)\dot{u}(t-h) + a_2(t)u(t-2h) = f(t) \ \ (t > 0;\ f \in C(R_+), f(t) \geq 0) \tag{2.12}$$

with the zero initial condition

$$u(t) = 0 \ \ (-2h \leq t \leq 0). \tag{2.13}$$

The derivatives at zero are understood as the right derivatives. Simultaneously, with the same zero initial condition, consider the equation

$$E_0 v(t) := \ddot{v}(t) + b_1 \dot{v}(t-h) + b_2 v(t-2h) = f(t) \ \ (t > 0). \tag{2.14}$$

Then

$$v(t) := E_0^{-1} f(t) = \int_0^t H(t-s) f(s) ds.$$

Put $W = E_0 - E$. With the notation $\Delta_k(t) = b_k - a_k(t)$ $(k = 1, 2)$, according to (2.11) and (2.1) we have

$$WH(t-s) = \Delta_1(t)\dot{H}(t-s-h) + \Delta_2(t)H(t-t-2h) \geq$$

$$H(t-s-2h)(-b_1\Delta_1(t)/2 + \Delta_2(t)) \geq 0. \tag{2.15}$$

Rewrite (2.12) as $E_0 u - Wu = f$. With $u = E_0^{-1} y$ we obtain

$$y - WE_0^{-1} y = f. \tag{2.16}$$

But

$$WE_0^{-1} f(t) = (b_1 - a_1(t))\frac{d}{dt}\int_0^t H(t-s-h)f(s)ds + (b_2 - a_2(t))\int_0^t H(t-s-2h)f(s)ds =$$

$$\int_0^t [\Delta_1(t)\dot{H}(t-s-h) + \Delta_2(t)H(t-s-2h)]f(s)ds,$$

since $H(t) = 0, t \leq 0$. So thanks to (2.15), WE_0^{-1} is a positive Volterra operator. Consequently, for each positive $T < \infty$, the spectral radius of WE_0^{-1} in space $C(0, T)$ is equal to zero. Hence,

$$y = \sum_{k=0}^{\infty} (WE_0^{-1})^k f,$$

and therefore,

$$0 \leq f \leq y = (I - WE_0^{-1})^{-1} f \ \ (f \geq 0).$$

Consequently

$$0 \leq E_0^{-1} f \leq x = E_0^{-1} y \leq E_0^{-1}(I - WE_0^{-1})^{-1} f \ \ (f \geq 0).$$

This proves the lemma. □

The assertion of Theorem 2.1 follows from Lemmas 1.1 and 2.2.

3 Equations with nonlinear causal mappings

For a positive $T < \infty$, let E be a Banach space of functions defined on $[0, T]$ with the unit operator I. For all $\tau \in [0, T)$ and $w \in E$, let the projections P_τ be defined by

$$(P_\tau w)(t) = \begin{cases} w(t) & \text{if } 0 \le t \le \tau, \\ 0 & \text{if } \tau < t \le T \end{cases}.$$

In addition, $P_T = I$. A mapping $F : E \to E$ satisfying the condition

$$P_\tau F P_\tau = P_\tau F \quad (\tau \in [0, T])$$

will be called *a causal mapping (operator)*. This definition is somewhat different from the definition of the causal operator presented in [7]. In the linear case our definition coincides with the one accepted in [13].

Denote by $B(0, T)$ the space of real bounded measurable functions defined on $[0, T]$ with the sup-norm $|.|_{B(0,T)}$. Let us point an example of a causal mapping. To this end consider in $B(0, T)$ the mapping

$$(Fw)(t) = f(t, w(t)) + \int_0^t k(t, s, w(s))ds \quad (w \in B(0, T))$$

with a continuous kernel k, defined on $[0, T]^2 \times \mathbb{R}$ and a continuous function $f : [0, T] \times \mathbb{R} \to \mathbb{R}$. For each $\tau \in [0, T)$, we have

$$(P_\tau F w)(t) = f_\tau(t, w(t)) + \int_0^\tau k_\tau(t, s, w(s))ds$$

where

$$k_\tau(t, s, w(s)) = \begin{cases} k(t, s, w(s)) & \text{if } 0 \le t \le \tau, \\ 0 & \text{if } \tau < t \le T \end{cases}$$

$(0 \le s \le t)$, and

$$f_\tau(t, w(t)) = \begin{cases} f(t, w(t)) & \text{if } 0 \le t \le \tau, \\ 0 & \text{if } \tau < t \le T \end{cases}.$$

Thus the considered mapping is causal. Note that, the integral operator

$$\int_0^c k(t, s, w(s))ds$$

with a fixed positive $c \le T$ is not causal.

Denote by K_+ the cone of nonnegative functions from $B(0, T)$. For a positive number $R \le \infty$, put

$$K_R := \{w \in K_+ : 0 \le w(t) \le R, \ \ 0 \le t \le T\}.$$

Everywhere below F is a continuous causal operator mapping K_R into K_+. Consider the equation

$$x(t) = f(t) + \int_0^t Q(t, t_1)(Fx)(t_1)dt_1 \quad (0 < t \le T < \infty), \tag{3.1}$$

where $Q : [0 \le s \le t \le T] \to [0, \infty)$ is a nonnegative measurable kernel and $f \in K_+$ is given. *A solution of (3.1)* is a function $x \in B(0,T)$ which satisfies (3.1) for all $t \in [0,T]$.

It is assumed that there is a linear positive causal operators A in $B(0,T)$, such that

$$Fv \le Av \quad (v \in K_R). \tag{3.2}$$

That is, the operator V defined on $B(0,T)$ by

$$v \to (Vv)(t) := \int_0^t Q(t,t_1)(Av)(t_1)dt_1$$

is a majorant for the mapping

$$\int_0^t Q(t,t_1)(Fv)(t_1)dt_1 \quad (0 < t \le T).$$

Assume that V is compact and

$$b(s) := \sup_{s \le t \le T} Q(t,s) \in L^1(0,T) \tag{3.3}$$

where $L^1(0,T)$ is the space of scalar integrable functions defined on $[0,T]$. We need the following result proved in [18].

Theorem 3.1 *Let V be compact and the conditions (3.2), (3.3) and*

$$e^{|A|_{B(0,T)} \int_0^T b(s)ds} |f|_{B(0,T)} \le R \tag{3.4}$$

hold. Then (3.1) has a solution $x \in K_R$, satisfying the inequality

$$|x|_{B(0,T)} \le e^{|A|_{B(0,T)} \int_0^T b(s)ds} |f|_{B(0,T)}.$$

Now let us consider the equation

$$(Eu)(t) = (Fu)(t) \tag{3.5}$$

$(0 < t \le T < \infty)$ where E is the linear operator defined by (2.12). Take the initial conditions

$$u(t) = \phi_0(t) \ (-2h \le t \le 0); \dot{u}(t) = \phi_1(t) \ \ (-h \le t \le 0) \tag{3.6}$$

with given continuous functions ϕ_0, ϕ_1. Equation (3.5) is equivalent to the equation

$$x(t) = y(t) + \int_0^t G(t,t_1)(Fx)(t_1)dt_1 \quad (t \in (0,T)), \tag{3.7}$$

where G is the Green function to equation (1.1) and $y(t)$ is a solution of problem (1.1),(3.6). A continuous solution of (3.7) will be called *a mild solution of problem (3.5), (3.6).* Now Theorems 2.1 and 3.1 imply

Corollary 3.2 *Under the hypothesis of Theorem 2.1, let condition (3.2) hold for $R = \infty$. Then equation (3.5) has a nonnegative mild solution.*

4 Functional differential equations with variable and distributed delays

Consider the equation

$$u(t) = f(t) + \int_0^t K(t,s) \int_0^{2h} u(s-\tau)R(s,\tau)dr(\tau)ds \tag{4.1}$$

where $f(t)$ and $R(t,\tau)$ are continuous functions, and $r(\tau)$ $(t \geq 0; 0 \leq \tau \leq 2h)$ is nondecreasing. Take the zero initial conditions

$$u(t) = \dot{u}(t) = 0 \ (t \leq 0). \tag{4.2}$$

Lemma 4.1 *Let $K(t,s)$, $f(t)$ and $R(s,\tau)$ be non-negative functions. Then a solution of the problem (4.1), (4.2) is non-negative. Moreover,*

$$u(t) \geq f(t), t > 0. \tag{4.3}$$

Proof: Put

$$Vu(t) := \int_0^t K(t,s) \int_0^{2h} u(s-\tau)R(s,\tau)dr(\tau)\ ds.$$

We have

$$Vu(t) = \int_0^{2h} \int_0^t K(t,\tau)u(s-\tau)R(s,\tau)ds\ dr(\tau) =$$

$$\int_0^{2h} \int_0^{t-\tau} K(t,s_1+\tau)u(s_1)R(s_1+\tau,\tau)ds_1 dr(\tau).$$

V is a nonnegative operator in $C(R_+)$. Moreover, for any finite T,

$$|Vu(t)| \leq M_T \int_0^t |u(s_1)|ds_1 \ \ (0 \leq t \leq T),$$

where

$$M_T := \sup_{0 \leq t, s_1 \leq T} \int_0^{2h} K(t,s_1+\tau)R(s_1+\tau,\tau)dr(\tau).$$

Hence it easily follows that V is a Volterra operator and its spectral radius is equal to zero. By the above mentioned comparison principle, inequality (4.3) holds, as claimed. $\square$

Now let us consider the equation

$$\ddot{u}(t) + 2c_0(t)\dot{u}(t) + c_1(t)\dot{u}(t-h) + d_0(t)u(t) + d_1(t)u(t-h) + d_2(t)u(t-2h) =$$

$$f_0(t) + \int_0^{2h} u(t-\tau)R(t,\tau)dr(\tau) \ \ (f_0 \in C(R_+);\ t > 0). \tag{4.4}$$

This equation under condition (4.2) is equivalent to (4.1) with

$$K(t,s) = G(t,s),\ f(t) = \int_0^t G(t,s)f_0(s)ds. \tag{4.5}$$

Now the previous lemma and Theorem 2.1 imply

Corollary 4.2 *Under the hypothesis of Theorem 2.1, let f_0 and $R(t,\tau)$ be non-negative. Then a solution $u(t)$ of problem (4.4), (4.2) is non-negative and (4.3) holds with f defined by (4.5).*

Furthermore, consider the equation

$$u(t) = f(t) + \int_0^t K(t,s)m(s)u(s-v(s))ds \tag{4.6}$$

where $m(t) \in C(R_+)$; $v(t)$ is non-negative, continuously differentiable, $v(t) \le 2h$, and

$$0 \le \beta := \inf_{t\ge 0} \dot v(t) < 1. \tag{4.7}$$

Lemma 4.3 *Let $K(t,s)$, $m(t)$ and $f(t)$ be non-negative functions and condition (4.7) holds. Then a solution of problem (4.6), (4.2) is non-negative. Moreover, inequality (4.3) holds.*

Proof: Put

$$V_0 u(t) := \int_0^t K(t,s)u(s-v(s))ds.$$

Let $w(t) \ge 0$. Then

$$V_0 w(t) \ge \frac{1}{1-\beta}\int_0^t K(t,s)w(s-v(s))(1-\dot v(s))ds = \frac{1}{1-\beta}\int_0^{t-v(t)} K(t,\psi(s_1))w(s_1)ds_1$$

where $\psi(s)$ is inverse to the function $s \to s - v(s)$. Clearly, V_0 is a nonnegative operator in $C(R_+)$. Moreover, for any finite T,

$$|V_0 u(t)| \le N_T \int_0^t |u(s_1)|ds_1 \ \ (0 \le t \le T),$$

where

$$N_T := \frac{1}{1-\beta}\sup_{0\le t,s_1\le T} K(t,\psi(s_1)).$$

Hence it easily follows that V_0 is a Volterra operator and its spectral radius is equal to zero. By the above mentioned comparison principle, inequality (4.3) holds, as claimed. □

Now let us consider the equation

$$\ddot u(t) + 2c_0(t)\dot u(t) + c_1(t)\dot u(t-h) + d_0(t)u(t) + d_1(t)u(t-h) + d_2(t)u(t-2h) =$$

$$f_0(t) + m(t)u(t-v(t)) \ \ (m(t) \in C(R_+), t \ge 0). \tag{4.8}$$

This equation under condition (4.2) is equivalent to (4.6) with (4.5) taken into account. Now the previous lemma and Theorem 2.1 imply

Corollary 4.4 *Under the hypothesis of Theorem 2.1, let $f_0(t)$ and $m(t)$ be non-negative. Then a solution $u(t)$ of problem (4.8), (4.2) is non-negative and (4.3) holds with f defined by (4.5).*

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