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Three point boundary value problem for singularly perturbed semilinear differential equations

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Abstract

In this paper, we investigate the problem of existence and asymptotic behavior of solutions for the nonlinear boundary value problem

$$\epsilon y'' + ky = f(t, y), \quad t \in \langle a, b\rangle, \quad k < 0, \quad 0 < \epsilon << 1$$

satisfying three point boundary conditions. Our analysis relies on the method of lower and upper solutions.

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1 Introduction

We will consider the three point problem

$$\epsilon y'' + ky = f(t, y), \quad t \in \langle a, b\rangle, \quad k < 0, \quad 0 < \epsilon << 1 \tag{1.1}$$

$$y'(a) = 0, \quad y(b) - y(c) = 0, \quad a < c < b. \tag{1.2}$$

This is a singularly perturbed problem because the order of differential equation drops when ϵ becomes zero. The situation in the present case is complicated by the fact that there is an inner point in the boundary conditions, in contrast to the "standard" boundary conditions as the Dirichlet problem, Neumann problem, Robin problem, periodic boundary value problem ([2, 3]), for example.

We apply the method of upper and lower solutions and some estimates to prove the existence of a solution for problem (1.1), (1.2) which converges uniformly to the solution of reduced problem (i.e. if we let $\epsilon \to 0^+$ in (1.1)) on every compact set of interval $\langle a, b)$ for $\epsilon \to 0^+$.
As usual, we say that $\alpha_\epsilon \in C^2(\langle a, b\rangle)$ is a lower solution for problem (1.1), (1.2) if $\epsilon\alpha''_\epsilon(t) + k\alpha_\epsilon(t) \geq f(t, \alpha_\epsilon(t))$ and $\alpha'_\epsilon(0) = 0$, $\alpha_\epsilon(b) - \alpha_\epsilon(c) \leq 0$ for every $t \in \langle a, b\rangle$. An upper solution $\beta_\epsilon \in C^2(\langle a, b\rangle)$ satisfies $\epsilon\beta''_\epsilon(t) + k\beta_\epsilon(t) \leq f(t, \beta_\epsilon(t))$ and $\beta'_\epsilon(0) = 0$, $\beta_\epsilon(b) - \beta_\epsilon(c) \geq 0$ for every $t \in \langle a, b\rangle$.

Lemma 1 (cf. [1]). *If $\alpha_\epsilon, \beta_\epsilon$ are lower and upper solutions for (1.1), (1.2) such that $\alpha_\epsilon \leq \beta_\epsilon$, then there exists solution y_ϵ of (1.1), (1.2) with $\alpha_\epsilon \leq y_\epsilon \leq \beta_\epsilon$.*

Denote $D(u) = \{(t,y)| \quad a \le t \le b, |y - u(t)| < d(t)\}$, where $d(t)$ is the positive continuous function on $\langle a,b\rangle$ such that

$$d(t) = \begin{cases} \delta & \text{for } a \le t \le b - \delta \\ |u(b) - u(c)| + \delta & \text{for } b - \frac{\delta}{2} \le t \le b \end{cases}$$

δ is a small positive constant and $u \in C^2$ is a solution of reduced problem $ku = f(t,u)$.

2 Existence and asymptotic behavior of solutions

Theorem 1. *Let $f \in C^1(D(u))$ satisfies the condition*

$$\left|\frac{\partial f(t,y)}{\partial y}\right| \le w < -k \quad \text{for every} \quad (t,y) \in D(u). \quad \text{(hyperbolicity condition)}$$

Then there exists ϵ_0 such that for every $\epsilon \in (0,\epsilon_0\rangle$ the problem (1.1), (1.2) has a unique solution satisfying the inequality

$$\hat{v}_\epsilon(t) - C\epsilon \le y_\epsilon(t) - (u(t) + v_\epsilon(t)) \le -\hat{v}_\epsilon(t) + \tilde{C}\sqrt{\epsilon} \quad \text{for} \quad u'(a) \le 0 \qquad (2.1)$$

and

$$\hat{v}_\epsilon(t) - \tilde{C}\sqrt{\epsilon} \le y_\epsilon(t) - (u(t) + v_\epsilon(t)) \le -\hat{v}_\epsilon(t) + C\epsilon \quad \text{for} \quad u'(a) \ge 0 \qquad (2.2)$$

on $\langle a,b\rangle$ where

$$\begin{aligned} v_\epsilon(t) &= u'(a) \cdot \frac{e^{\sqrt{\frac{m}{\epsilon}}(t-b)} - e^{\sqrt{\frac{m}{\epsilon}}(b-t)} + e^{\sqrt{\frac{m}{\epsilon}}(c-t)} - e^{\sqrt{\frac{m}{\epsilon}}(t-c)}}{\sqrt{\frac{m}{\epsilon}}\left(e^{\sqrt{\frac{m}{\epsilon}}(a-c)} - e^{\sqrt{\frac{m}{\epsilon}}(a-b)} + e^{\sqrt{\frac{m}{\epsilon}}(c-a)} - e^{\sqrt{\frac{m}{\epsilon}}(b-a)}\right)}, \\ \hat{v}_\epsilon(t) &= |u(b) - u(c)| \cdot \frac{e^{\sqrt{\frac{m}{\epsilon}}(t-a)} + e^{\sqrt{\frac{m}{\epsilon}}(a-t)}}{\left(e^{\sqrt{\frac{m}{\epsilon}}(a-c)} - e^{\sqrt{\frac{m}{\epsilon}}(a-b)} + e^{\sqrt{\frac{m}{\epsilon}}(c-a)} - e^{\sqrt{\frac{m}{\epsilon}}(b-a)}\right)}, \end{aligned}$$

$m = -k - w$ and C, $\tilde{C}$ be the positive constants.

Proof. The functions $v_\epsilon(t)$ and $\hat{v}_\epsilon(t)$ on $\langle a,b\rangle$ satisfy:

1. $\epsilon v''_\epsilon - m v_\epsilon = 0$, $v'_\epsilon(a) = -u'(a)$, $v_\epsilon(b) - v_\epsilon(c) = 0$, $v_\epsilon \le 0$ for $u'(a) \le 0$ and $v_\epsilon \ge 0$ for $u'(a) \ge 0$
2. $\epsilon \hat{v}''_\epsilon - m\hat{v}_\epsilon = 0$, $\hat{v}'_\epsilon(a) = 0$, $\hat{v}_\epsilon(b) - \hat{v}_\epsilon(c) = -|u(b) - u(c)|$, $\hat{v}_\epsilon \le 0$.

For $u'(a) \le 0$ we define the lower solutions by

$$\alpha_\epsilon(t) = u(t) + v_\epsilon(t) + \hat{v}_\epsilon(t) - \Gamma_\epsilon$$

and the upper solutions by

$$\beta_\epsilon(t) = u(t) + v_\epsilon(t) - \hat{v}_\epsilon(t) + \tilde{\Gamma}_\epsilon$$

(for $u'(a) \ge 0$ we proceed analogously).

Here $\Gamma_\epsilon = \frac{\epsilon\Delta}{m}$ and $\tilde{\Gamma}_\epsilon = \frac{\sqrt{\epsilon}\tilde{\Delta}}{m}$ where Δ, $\tilde{\Delta}$ be the constants which shall be defined below. $\alpha \le \beta$ on $\langle a,b\rangle$ and satisfy the boundary conditions prescribed for the lower and upper solutions of (1.1), (1.2).

Now we show that $\epsilon\alpha_\epsilon''(t)+k\alpha_\epsilon(t)\geq f(t,\alpha_\epsilon(t))$ and $\epsilon\beta_\epsilon''(t)+k\beta_\epsilon(t)\leq f(t,\beta_\epsilon(t))$. Denote $h(t,y)=f(t,y)-ky$. By the Taylor theorem we obtain

$$h(t,\alpha_\epsilon(t))=h(t,\alpha_\epsilon(t))-h(t,u(t))=\frac{\partial h(t,\theta_\epsilon(t))}{\partial y}(v_\epsilon(t)+\hat{v}_\epsilon(t)-\Gamma_\epsilon)$$

where $(t,\theta_\epsilon(t))$ is a point between $(t,\alpha_\epsilon(t))$ and $(t,u(t))$, and $(t,\theta_\epsilon(t))\in D(u)$ for sufficiently small ϵ. Hence

$$\epsilon\alpha_\epsilon''(t)-h(t,\alpha_\epsilon(t))\geq\epsilon u''+\epsilon v_\epsilon''(t)+\epsilon\hat{v}_\epsilon''(t)-m(v_\epsilon(t)+\hat{v}_\epsilon(t)-\Gamma_\epsilon)\geq-\epsilon|u''|+\epsilon\Delta.$$

If we choose a constant Δ such that $\Delta\geq|u''(t)|$, $t\in\langle a,b\rangle$ then $\epsilon\alpha_\epsilon''(t)\geq h(t,\alpha_\epsilon(t))$ in $\langle a,b\rangle$.
The inequality for $\beta_\epsilon(t))$:

$$h(t,\beta_\epsilon(t))-\epsilon\beta_\epsilon''(t)=\frac{\partial h(t,\tilde{\theta}_\epsilon(t))}{\partial y}(v_\epsilon(t)-\hat{v}_\epsilon(t)+\tilde{\Gamma}_\epsilon)-\epsilon\beta_\epsilon''(t)=$$

$$\frac{\partial h(t,\tilde{\theta}_\epsilon(t))}{\partial y}(v_\epsilon(t)-\hat{v}_\epsilon(t)+\tilde{\Gamma}_\epsilon)-\epsilon(u''+v_\epsilon''(t)-\hat{v}_\epsilon''(t))\geq$$

$$\frac{\partial h(t,\tilde{\theta}_\epsilon(t))}{\partial y}v_\epsilon(t)+m\tilde{\Gamma}_\epsilon-\epsilon u''-\epsilon v_\epsilon''(t)=$$

$$\left(\frac{\partial h(t,\tilde{\theta}_\epsilon(t))}{\partial y}-m\right)v_\epsilon(t)+m\tilde{\Gamma}_\epsilon-\epsilon u''.$$

Let $L=\max\{|u''(t)|\quad|t\in\langle a,b\rangle\}$ and denote $\tilde{v}_\epsilon=\frac{v_\epsilon}{\sqrt{\epsilon}}$. Then $\epsilon\beta_\epsilon''(t)\leq h(t,\beta_\epsilon(t))$ if

$$m\tilde{\Gamma}_\epsilon-\epsilon L\geq\left(\frac{\partial h(t,\tilde{\theta}_\epsilon(t))}{\partial y}-m\right)|v_\epsilon(t)|$$

i. e.

$$\sqrt{\epsilon}\left(\tilde{\Delta}-\sqrt{\epsilon}L\right)\geq\left(\frac{\partial h(t,\tilde{\theta}_\epsilon(t))}{\partial y}-m\right)\sqrt{\epsilon}|\tilde{v}_\epsilon(t)|$$

$$\tilde{\Delta}\geq\sqrt{\epsilon}L+\left(\frac{\partial h(t,\tilde{\theta}_\epsilon(t))}{\partial y}-m\right)|\tilde{v}_\epsilon(t)|.$$

Thus, from the inequalities $m\leq\frac{\partial h(t,\tilde{\theta}_\epsilon(t))}{\partial y}\leq m+2w$ in $D(u)$ and $v_\epsilon(t)\leq 0$ follows that it is sufficient to choose a constant $\tilde{\Delta}$ such that

$$\tilde{\Delta}\geq\sqrt{\epsilon}L+2w\frac{|u'(a)|}{\sqrt{m}}.$$

The existence of a solution for (1.1), (1.2) satisfying the above inequality follows from Lemma 1.

Remark 1. Theorem 1 implies that $y_\epsilon(t)=u(t)+O(\sqrt{\epsilon})$ on every compact subset of $\langle a,b)$ and $\lim\limits_{\epsilon\to 0^+}y_\epsilon(b)=u(c)$. Boundary layer effect occurs at the point b in the case, when $u(c)\neq u(b)$.

References

[1] J. Mawhin: *Points fixes, points critiques et problemes aux limites*, Semin. Math. Sup. no. 92, Presses Univ. Montreal, 1985.

[2] R. Vrábel': *Asymptotic behavior of T-periodic solutions of singularly perturbed second-order differential equation*, Mathematica Bohemica 121 (1996), 73-76.

[3] R. Vrábel': *Semilinear singular perturbation*, Nonlinear analysis, TMA. - ISSN 0362-546X. - Vol. 25, No. 1 (1995), 17-26.

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