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Singularly perturbed semilinear Neumann problem with non-normally hyperbolic critical manifold*

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Abstract

In this paper, we investigate the problem of existence and asymptotic behavior of the solutions for the nonlinear boundary value problem

$$\epsilon y'' + ky = f(t,y), \quad t \in \langle a,b \rangle, \quad k > 0, \quad 0 < \epsilon << 1$$

satisfying Neumann boundary conditions and where critical manifold is not normally hyperbolic. Our analysis relies on the method upper and lower solutions.
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1 Introduction

We will consider the singularly perturbed Neumann problem

$$\epsilon y'' + ky = f(t,y), \quad t \in \langle a,b \rangle, \quad k > 0, \quad 0 < \epsilon << 1 \tag{1.1}$$

$$y'(a) = 0, \quad y'(b) = 0. \tag{1.2}$$

The qualitative behavior of the dynamical systems near a normally hyperbolic manifold of critical points is well known (Theorem on persistence of normally hyperbolic manifold, see [2, 3, 5, 9, 12], for reference). However, the framework of the geometric singular perturbation theory is not useful for the non-hyperbolic critical manifolds, i.e. when the characteristic roots of the linearization of (1.1) along a solution u of the reduced problem $ku = f(t,u)$ lie on the imaginary axis.

The main result (Theorem 1) is the existence of a solution $y_\epsilon(t)$ for ϵ belonging to a non-resonant set and an estimate of the difference between the solution $y_\epsilon(t)$ and a solution $u(t)$ of the reduced problem. It is accomplished

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by a construction of a lower and an upper solution for the corresponding boundary value problem.

As usual, we say that $\alpha_\epsilon \in C^2(\langle a,b\rangle)$ is a lower solution for problem (1.1), (1.2) if $\epsilon\alpha''_\epsilon(t) + k\alpha_\epsilon(t) \geq f(t,\alpha_\epsilon(t))$ and $\alpha'_\epsilon(a) \geq 0$, $\alpha'_\epsilon(b) \leq 0$ for every $t \in \langle a,b\rangle$. An upper solution $\beta_\epsilon \in C^2(\langle a,b\rangle)$ satisfies $\epsilon\beta''_\epsilon(t) + k\beta_\epsilon(t) \leq f(t,\beta_\epsilon(t))$ and $\beta'_\epsilon(a) \leq 0$, $\beta'_\epsilon(b) \geq 0$ for every $t \in \langle a,b\rangle$. Then

Lemma 1 ([1, 8])**.** *If $\alpha_\epsilon, \beta_\epsilon$ are lower and upper solutions for (1.1), (1.2) such that $\alpha_\epsilon \leq \beta_\epsilon$, then there exists solution y_ϵ of (1.1), (1.2) with $\alpha_\epsilon \leq y_\epsilon \leq \beta_\epsilon$.*

Denote $\mathcal{D}_\delta(u) = \{(t,y)| \quad a \leq t \leq b, |y-u(t)| < \delta\}$, δ is a positive constant and $u \in C^2$ is a solution of reduced problem $ku = f(t,u)$.
Let

$$v_{1,\epsilon}(t) = |u'(a)| \frac{\cos\left[\sqrt{\frac{m}{\epsilon}}(b-t)\right]}{\sqrt{\frac{m}{\epsilon}}\sin\left[\sqrt{\frac{m}{\epsilon}}(b-a)\right]}$$

and

$$v_{2,\epsilon}(t) = -|u'(b)| \frac{\cos\left[\sqrt{\frac{m}{\epsilon}}(t-a)\right]}{\sqrt{\frac{m}{\epsilon}}\sin\left[\sqrt{\frac{m}{\epsilon}}(b-a)\right]}$$

where $m = k + w$ (for the constant w see Theorem 1 below).

Let

$$J_n(\lambda) = \left\langle m\left(\frac{b-a}{(n+1)\pi-\lambda}\right)^2, m\left(\frac{b-a}{n\pi+\lambda}\right)^2\right\rangle, \quad n = 0,1,2,\ldots,$$

$\lambda > 0$ is an arbitrarily small, but fixed constant and

$$\mathcal{M} = \left\{\bigcup J_n, n = 0,1,2,\ldots\right\}.$$

The function $v_{1,\epsilon}(t)$ satisfies:

1. $\epsilon v''_{1,\epsilon} + m v_{1,\epsilon} = 0$
2. $v'_{1,\epsilon}(a) = |u'(a)|$, $v'_{1,\epsilon}(b) = 0$
3. $v_{1,\epsilon}(t)$ be periodic in the variable t with the period $\frac{2\pi\sqrt{\epsilon}}{\sqrt{m}} \to 0$
4. $v_{1,\epsilon_n}(t)$ converges uniformly to 0 for every sequence $\{\epsilon_n\}_{n=0}^\infty$ such that $\epsilon_n \in J_n$ and $|v_{1,\epsilon_n}(t)| \leq \frac{\sqrt{\epsilon_n}}{\sqrt{m}\sin\lambda}$, $t \in \langle a,b\rangle$.

The function $v_{2,\epsilon}(t)$ satisfies:

1. $\epsilon v''_{2,\epsilon} + m v_{2,\epsilon} = 0$
2. $v'_{2,\epsilon}(a) = 0$, $v'_{2,\epsilon}(b) = |u'(b)|$
3. $v_{2,\epsilon}(t)$ be periodic in the variable t with the period $\frac{2\pi\sqrt{\epsilon}}{\sqrt{m}} \to 0$
4. $v_{2,\epsilon_n}(t)$ converges uniformly to 0 for every sequence $\{\epsilon_n\}_{n=0}^\infty$ such that $\epsilon_n \in J_n$ and $|v_{2,\epsilon_n}(t)| \leq \frac{\sqrt{\epsilon_n}}{\sqrt{m}\sin\lambda}$, $t \in \langle a,b\rangle$.

Denote $\omega_{0,\epsilon}(t) = v_{2,\epsilon}(t) - v_{1,\epsilon}(t)$.
Let $\omega_{1,\epsilon,i}(t)$ be a solution of the linear problem

$$\epsilon y'' + my = \pm\epsilon u''(t), \quad i = \alpha_\epsilon, \beta_\epsilon$$

with the Neumann boundary condition (1.2), where the sign + and − is considered for $i = \alpha_\epsilon$ and $i = \beta_\epsilon$, respectively. These solutions may be computed exactly

$$\omega_{1,\epsilon,i}(t) = \frac{\cos\left[\sqrt{\frac{m}{\epsilon}}(t-a)\right] \int_a^b \cos\left[\sqrt{\frac{m}{\epsilon}}(b-s)\right](\pm u''(s))\,\mathrm{d}s}{\sqrt{\frac{m}{\epsilon}}\sin\left[\sqrt{\frac{m}{\epsilon}}(b-a)\right]}$$
$$+ \int_a^t \frac{\sin\left[\sqrt{\frac{m}{\epsilon}}(t-s)\right](\pm u''(s))\,\mathrm{d}s}{\sqrt{\frac{m}{\epsilon}}}\mathrm{d}s = \mathcal{O}(\epsilon), \epsilon \in \mathcal{M}.$$

Obviously, $\omega_{1,\epsilon,\alpha_\epsilon}(t) = -\omega_{1,\epsilon,\beta_\epsilon}(t)$ on $\langle a, b\rangle$.

Let $r_{\epsilon,i}(t)$ is a continuous solution of the Fredholm equation of the first kind

$$\Gamma(\epsilon)\int_a^b K_\epsilon(t,s) r_{\epsilon,i}(s)\mathrm{d}s + \Omega_{\epsilon,i}(t) = z_{\epsilon,i}(t), \quad z_{\epsilon,i}(t) \geq 0 \quad i = \alpha_\epsilon, \beta_\epsilon \tag{1.3}$$

where $\Gamma(\epsilon) = \frac{1}{\sqrt{\frac{m}{\epsilon}}\sin\left[\sqrt{\frac{m}{\epsilon}}(b-a)\right]} \cdot \frac{1}{\epsilon}$, $\Gamma^{-1}(\epsilon) = \mathcal{O}(\sqrt{\epsilon})$, $\epsilon \in \mathcal{M}$,

$$\Omega_{\epsilon,i}(t) = \omega_{0,\epsilon}(t) + \omega_{1,\epsilon,i}(t)$$

and the kernel

$$K_\epsilon(t,s) = \begin{cases} K_{1,\epsilon}(t,s), & a \leq s \leq t \leq b \\ K_{2,\epsilon}(t,s), & a \leq t \leq s \leq b, \end{cases}$$

$$K_{1,\epsilon}(t,s) = \cos\left[\sqrt{\frac{m}{\epsilon}}(t-a)\right]\cos\left[\sqrt{\frac{m}{\epsilon}}(b-s)\right] +$$
$$\sin\left[\sqrt{\frac{m}{\epsilon}}(b-a)\right]\sin\left[\sqrt{\frac{m}{\epsilon}}(t-s)\right]$$
$$K_{2,\epsilon}(t,s) = \cos\left[\sqrt{\frac{m}{\epsilon}}(t-a)\right]\cos\left[\sqrt{\frac{m}{\epsilon}}(b-s)\right]$$

for $\epsilon \in \mathcal{M}$ and a modulation function $z_{\epsilon,i}(t)$ is an appropriate continuous nonnegative function such that $r_{\epsilon,i}(t) \leq 0$.

This is an integral equation of the kernel $K_\epsilon(t,s)$ that is continuous on the square $\langle a,b\rangle \times \langle a,b\rangle$. The problem (1.3) is defined as ill-posed and, in general, may be described numerically with Tikhonov regularization ([6, 7, 10, 11]). By substituing $z_{\epsilon,i}(t) = r_{\epsilon,i}(t) + \tilde{z}_{\epsilon,i}(t)$, $i = \alpha_\epsilon, \beta_\epsilon$ into (1.3) we obtain

$$\Gamma(\epsilon)\int_a^b K_\epsilon(t,s) r_{\epsilon,i}(s)\mathrm{d}s + \tilde{\Omega}_{\epsilon,i}(t) = r_{\epsilon,i}(t), \quad i = \alpha_\epsilon, \beta_\epsilon,$$

i.e. $r_{\epsilon,i}(t)$ is a solution of Fredholm integral equation of second kind

$$\Gamma(\epsilon)\int_a^b K_\epsilon(t,s) y(s)\mathrm{d}s + \tilde{\Omega}_{\epsilon,i}(t) = y(t), \quad i = \alpha_\epsilon, \beta_\epsilon, \tag{1.4}$$

where $\tilde{\Omega}_{\epsilon,i}(t) = \Omega_{\epsilon,i}(t) - \tilde{z}_{\epsilon,i}(t)$ and $\tilde{z}_{\epsilon,i}(t)$ is an appropriate chosen function such that

$$\tilde{z}_{\epsilon,i}(t) \geq -r_{\epsilon,i}(t), \tag{1.5}$$

$$r_{\epsilon,i}(t) \leq 0, \tag{1.6}$$

$t \in \langle a, b\rangle$, $i = \alpha_\epsilon, \beta_\epsilon$.

The kernel K_ϵ is semiseparable ([4]), therefore the equation (1.4) can be rewritten as

$$y(t) = \sum_{k=1}^{3} A_{k,\epsilon,a}(t)\int_a^t B_{k,\epsilon,a}(s)y(s)\mathrm{d}s + A_{1,\epsilon,b}(t)\int_t^b B_{1,\epsilon,b}(s)y(s)\mathrm{d}s + \tilde{\Omega}_{\epsilon,i}(t)$$

where

$$\begin{aligned}
A_{1,\epsilon,a}(t) &= \Gamma(\epsilon)\cos\left[\sqrt{\frac{m}{\epsilon}}(t-a)\right]\\
A_{2,\epsilon,a}(t) &= \Gamma(\epsilon)\sin\left[\sqrt{\frac{m}{\epsilon}}(b-a)\right]\sin\left[\sqrt{\frac{m}{\epsilon}}t\right]\\
A_{3,\epsilon,a}(t) &= -\Gamma(\epsilon)\sin\left[\sqrt{\frac{m}{\epsilon}}(b-a)\right]\cos\left[\sqrt{\frac{m}{\epsilon}}t\right]\\
A_{1,\epsilon,b}(t) &= \Gamma(\epsilon)\cos\left[\sqrt{\frac{m}{\epsilon}}(t-a)\right]\\
B_{1,\epsilon,a}(s) &= \cos\left[\sqrt{\frac{m}{\epsilon}}(b-s)\right]\\
B_{2,\epsilon,a}(s) &= \cos\left[\sqrt{\frac{m}{\epsilon}}s\right]\\
B_{3,\epsilon,a}(s) &= \sin\left[\sqrt{\frac{m}{\epsilon}}s\right]\\
B_{1,\epsilon,b}(s) &= \cos\left[\sqrt{\frac{m}{\epsilon}}(b-s)\right]
\end{aligned}$$

or

$$y(t) = \sum_{k=1}^{3} A_{k,\epsilon,a}(t)X_{k,\epsilon,a,i}(t) + A_{1,\epsilon,b}(t)X_{1,\epsilon,b,i}(t) + \tilde{\Omega}_{\epsilon,i}(t), \quad i = \alpha_\epsilon, \beta_\epsilon \tag{1.7}$$

where

$$X_{k,\epsilon,a,i}(t) = \int\limits_a^t B_{k,\epsilon,a}(s)y(s)\mathrm{d}s, \quad X_{1,\epsilon,b,i}(t) = \int\limits_t^b B_{1,\epsilon,b}(s)y(s)\mathrm{d}s, \quad k = 1,2,3.$$

Multiply both sides of the integral equation (1.7) by $B_{j,\epsilon,a}(t)$ and integrate from a to t and by $B_{1,\epsilon,b}(t)$ and integrate from t to b, respectively. We obtain

$$X_{j,\epsilon,a,i} = \sum_{k=1}^{3} \int\limits_a^t A_{k,\epsilon,a}B_{j,\epsilon,a}X_{k,\epsilon,a,i}\mathrm{d}t + \int\limits_a^t A_{1,\epsilon,b}B_{1,\epsilon,a}X_{1,\epsilon,b,i}\mathrm{d}t + \int\limits_a^t B_{j,\epsilon,a}\tilde{\Omega}_{\epsilon,i}\mathrm{d}t$$

$$X_{1,\epsilon,b,i} = \sum_{k=1}^{3} \int\limits_t^b A_{k,\epsilon,a}B_{1,\epsilon,b}X_{k,\epsilon,a,i}\mathrm{d}t + \int\limits_t^b A_{1,\epsilon,b}B_{1,\epsilon,b}X_{1,\epsilon,b,i}\mathrm{d}t + \int\limits_t^b B_{1,\epsilon,b}\tilde{\Omega}_{\epsilon,i}\mathrm{d}t$$

$j = 1,2,3, i = \alpha_\epsilon, \beta_\epsilon$.

Differentiating these equations and taking into consideration the definition of $X_{j,\epsilon,a}$, $X_{1,\epsilon,b}$ we obtain the boundary value problem for the system of linear differential equations

$$X'_{j,\epsilon,a,i} = \sum_{k=1}^{3} A_{k,\epsilon,a}B_{j,\epsilon,a}X_{k,\epsilon,a,i} + A_{1,\epsilon,b}B_{1,\epsilon,a}X_{k,\epsilon,b,i} + B_{j,\epsilon,a}\tilde{\Omega}_{\epsilon,i} \qquad (1.8)$$

$$X'_{1,\epsilon,b,i} = -\sum_{k=1}^{3} A_{k,\epsilon,a}B_{1,\epsilon,b}X_{k,\epsilon,a,i} - A_{1,\epsilon,b}B_{1,\epsilon,b}X_{1,\epsilon,b,i} - B_{1,\epsilon,b}\tilde{\Omega}_{\epsilon,i} \qquad (1.9)$$

$$X_{j,\epsilon,a,i}(a) = 0, \quad X_{1,\epsilon,b,i}(b) = 0 \qquad (1.10)$$

$j = 1,2,3, i = \alpha_\epsilon, \beta_\epsilon$ or in the block matrix notation

$$X' = \begin{pmatrix} P_{1,\epsilon}(t) & P_{3,\epsilon}(t) \\ P_{2,\epsilon}(t) & P_{4,\epsilon}(t) \end{pmatrix} X + D_{\epsilon,i}(t)$$

where

$$X = (X_{1,\epsilon,a,i}(t), X_{2,\epsilon,a,i}(t), X_{3,\epsilon,a,i}(t), X_{1,\epsilon,b,i}(t))^T,$$

$$P_{1,\epsilon}(t) = \begin{pmatrix} A_{1,\epsilon,a}(t)B_{1,\epsilon,a}(t) & A_{2,\epsilon,a}(t)B_{1,\epsilon,a}(t) & A_{3,\epsilon,a}(t)B_{1,\epsilon,a}(t) \\ A_{1,\epsilon,a}(t)B_{2,\epsilon,a}(t) & A_{2,\epsilon,a}(t)B_{2,\epsilon,a}(t) & A_{3,\epsilon,a}(t)B_{2,\epsilon,a}(t) \\ A_{1,\epsilon,a}(t)B_{3,\epsilon,a}(t) & A_{2,\epsilon,a}(t)B_{3,\epsilon,a}(t) & A_{3,\epsilon,a}(t)B_{3,\epsilon,a}(t) \end{pmatrix},$$

$$P_{2,\epsilon}(t) = -\begin{pmatrix} A_{1,\epsilon,a}(t)B_{1,\epsilon,b}(t) & A_{2,\epsilon,a}(t)B_{1,\epsilon,b}(t) & A_{3,\epsilon,a}(t)B_{1,\epsilon,b}(t) \end{pmatrix},$$

$$P_{3,\epsilon}(t) = \begin{pmatrix} A_{1,\epsilon,b}(t)B_{1,\epsilon,a}(t) \\ A_{1,\epsilon,b}(t)B_{2,\epsilon,a}(t) \\ A_{1,\epsilon,b}(t)B_{3,\epsilon,a}(t) \end{pmatrix}, P_{4,\epsilon}(t) = -\left(A_{1,\epsilon,b}(t)B_{1,\epsilon,b}(t)\right)$$

and

$$D_{\epsilon,i}(t) = \tilde{\Omega}_{\epsilon,i}(t)\left(B_{1,\epsilon,a}(t), B_{2,\epsilon,a}(t), B_{3,\epsilon,a}(t), -B_{1,\epsilon,b}(t)\right)^T,$$

$i = \alpha_\epsilon, \beta_\epsilon$. Thus,

$$\begin{aligned} r_{\epsilon,i}(t) &= r_{\epsilon,i}\left(\tilde{z}_{\epsilon,i}(t)\right) \\ &= \sum_{k=1}^{3} A_{k,\epsilon,a}(t) X_{k,\epsilon,a,i}(t) + A_{1,\epsilon,b}(t) X_{1,\epsilon,b,i}(t) + \tilde{\Omega}_{\epsilon,i}(t) \end{aligned} \tag{1.11}$$

where X is a solution of the linear boundary value problem (1.8), (1.9), (1.10).

The conditions (1.5), (1.6) we may write in the form

$$-\tilde{z}_{\epsilon,i}(t) \leq r_{\epsilon,i}(t) \leq 0, \quad i = \alpha_\epsilon, \beta_\epsilon \tag{1.12}$$

or

$$0 \leq \sum_{k=1}^{3} A_{k,\epsilon,a}(t) X_{k,\epsilon,a,i}(t) + A_{1,\epsilon,b}(t) X_{1,\epsilon,b,i}(t) + \Omega_{\epsilon,i}(t) \leq \tilde{z}_{\epsilon,i}(t). \tag{1.13}$$

Remark 1. The matrix

$$\begin{pmatrix} P_{1,\epsilon}(t) & P_{3,\epsilon}(t) \\ P_{2,\epsilon}(t) & P_{4,\epsilon}(t) \end{pmatrix}$$

of the system is periodic with period p tendings to 0 for $\epsilon \to 0^+$, $\epsilon \in \mathcal{M}$ and using the Floquet theory, then the solution of the linear homogeneous system

$$X' = \begin{pmatrix} P_{1,\epsilon}(t) & P_{3,\epsilon}(t) \\ P_{2,\epsilon}(t) & P_{4,\epsilon}(t) \end{pmatrix} X$$

can be written as $X_{hom,\epsilon}(t) = p_\epsilon(t) e^{\Theta_\epsilon t}$ where $p_\epsilon(t)$ is a periodic function and a matrix Θ_ϵ is time independent. This fact is instructive for the numerical description and the computer simulation of the system (1.8), (1.9).

Remark 2. The condition (1.13) is the fundamental assumption for existence of the barrier functions α_ϵ, β_ϵ for proving Theorem 1.

Now let $v_{c,\epsilon,i}(t)$ be a solution of Neumann boundary value problem (1.2) for Diff. Eq.

$$\epsilon y'' + my = r_{\epsilon,i}(t), \quad i = \alpha_\epsilon, \beta_\epsilon \tag{1.14}$$

i.e.

$$\begin{aligned} v_{c,\epsilon,i}(t) &= \frac{\cos\left[\sqrt{\frac{m}{\epsilon}}(t-a)\right] \int\limits_a^b \cos\left[\sqrt{\frac{m}{\epsilon}}(b-s)\right] \frac{r_{\epsilon,i}(s)}{\epsilon} \mathrm{d}s}{\sqrt{\frac{m}{\epsilon}} \sin\left[\sqrt{\frac{m}{\epsilon}}(b-a)\right]} \\ &+ \int_a^t \frac{\sin\left[\sqrt{\frac{m}{\epsilon}}(t-s)\right] \frac{r_{\epsilon,i}(s)}{\epsilon} \mathrm{d}s}{\sqrt{\frac{m}{\epsilon}}} \mathrm{d}s = \mathcal{O}\left(r_{\epsilon,i}(t)\right), \epsilon \in \mathcal{M}. \end{aligned}$$

As follows from (1.3), the functions $v_{c,\epsilon,i}(t)$ must appear in the region as illustrated in Figure 1.1.

Now we may state the main result of this article.

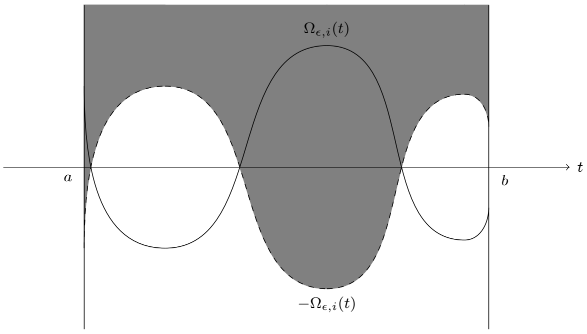

Figure 1.1: The region for $v_{c,\epsilon,i}(t)$

2 Main result

Theorem 1.

($\mathcal{A}1$) *Let* $\tilde{z}_{\epsilon,i}(t)$, $\epsilon \in (0,\epsilon_0] \cap \mathcal{M}$, $i = \alpha_\epsilon, \beta_\epsilon$ *be the continuous functions such that (1.13) holds.*

($\mathcal{A}2$) *Let* $f \in C^1(\mathcal{D}_\delta(u))$ *satisfies the condition*

$$\left|\frac{\partial f(t,y)}{\partial y}\right| \le w < k \quad \text{for every} \quad (t,y) \in \mathcal{D}_\delta(u)$$

(nonhyperbolicity condition)
where

$$\delta \ge \max\left\{\omega_{0,\epsilon}(t) + \omega_{1,\epsilon,i}(t) + v_{c,\epsilon,i}(t) : i = \alpha_\epsilon, \beta_\epsilon; t \in \langle a,b\rangle; \epsilon \in (0,\epsilon_0] \cap \mathcal{M}\right\}.$$

Then the problem (1.1), (1.2) has for $\epsilon \in (0,\epsilon_0] \cap \mathcal{M}$ *a solution satisfying the inequality*

$$-\omega_{0,\epsilon}(t) - \omega_{1,\epsilon,\alpha_\epsilon}(t) - v_{c,\epsilon,\alpha_\epsilon}(t) \le y_\epsilon(t) - u(t) \le \omega_{0,\epsilon}(t) + \omega_{1,\epsilon,\beta_\epsilon}(t) + v_{c,\epsilon,\beta_\epsilon}(t)$$

on $\langle a,b\rangle$.

Proof. We define the lower solutions by

$$\alpha_\epsilon(t) = u(t) - (\omega_{0,\epsilon}(t) + \omega_{1,\epsilon,\alpha_\epsilon}(t) + v_{c,\epsilon,\alpha_\epsilon}(t))$$

and the upper solutions by

$$\beta_\epsilon(t) = u(t) + (\omega_{0,\epsilon}(t) + \omega_{1,\epsilon,\beta_\epsilon}(t) + v_{c,\epsilon,\beta_\epsilon}(t)).$$

After simple algebraic manipulation we obtain

$$\omega_{0,\epsilon}(t) + \omega_{1,\epsilon,i}(t) + v_{c,\epsilon,i}(t) = z_{\epsilon,i}(t) \geq 0, \quad i = \alpha_\epsilon, \beta_\epsilon$$

on $\langle a, b \rangle$. The functions $\alpha_\epsilon, \beta_\epsilon$ satisfy the boundary conditions prescribed for the lower and upper solutions of (1.1), (1.2) and $\alpha_\epsilon(t) \leq \beta_\epsilon(t)$ on $\langle a, b \rangle$.
Now we show that

$$\epsilon \alpha''_\epsilon(t) + k\alpha_\epsilon(t) \geq f(t, \alpha_\epsilon(t)) \tag{2.1}$$

and

$$\epsilon \beta''_\epsilon(t) + k\beta_\epsilon(t) \leq f(t, \beta_\epsilon(t)). \tag{2.2}$$

Denote $h(t, y) = f(t, y) - ky$. From the assumption $(\mathcal{A}2)$ on the function $f(t, y)$ we have

$$-m \leq \frac{\partial h(t, y)}{\partial y} \leq 2w - m < 0$$

in $\mathcal{D}_\delta(u)$. By the Taylor theorem we obtain

$$\begin{aligned}
\epsilon \alpha''_\epsilon(t) - h(t, \alpha_\epsilon(t)) &= \epsilon \alpha''_\epsilon(t) - [h(t, \alpha_\epsilon(t)) - h(t, u(t))] \\
&= \epsilon u''(t) - \epsilon \omega''_{0,\epsilon}(t) - \epsilon \omega''_{1,\epsilon,\alpha_\epsilon}(t) - \epsilon v''_{c,\epsilon,\alpha_\epsilon}(t) \\
&- \frac{\partial h\left(t, \theta_\epsilon(t)\right)}{\partial y} \left(-\omega_{0,\epsilon}(t) - \omega_{1,\epsilon,\alpha_\epsilon}(t) - v_{c,\epsilon,\alpha_\epsilon}(t)\right) \\
&\geq \epsilon u''(t) - \epsilon \omega''_{0,\epsilon}(t) - \epsilon \omega''_{1,\epsilon,\alpha_\epsilon}(t) - \epsilon v''_{c,\epsilon,\alpha_\epsilon}(t) \\
&+ (-m) \left(\omega_{0,\epsilon}(t) + \omega_{1,\epsilon,\alpha_\epsilon}(t) + v_{c,\epsilon,\alpha_\epsilon}(t)\right) \\
&= -\epsilon v''_{c,\epsilon,\alpha_\epsilon}(t) - m v_{c,\epsilon,\alpha_\epsilon}(t) = -r_{\epsilon,\alpha_\epsilon}(t).
\end{aligned}$$

From the condition (1.6) is $-r_{\epsilon,\alpha_\epsilon}(t) \geq 0$ therefore $\epsilon \alpha''_\epsilon(t) - h(t, \alpha_\epsilon(t)) \geq 0$ on $\langle a, b \rangle$.
The inequality for $\beta_\epsilon(t)$:

$$\begin{aligned}
h(t, \beta_\epsilon(t)) - \epsilon \beta''_\epsilon(t) &= \frac{\partial h\left(t, \tilde{\theta}_\epsilon(t)\right)}{\partial y} \left(\omega_{0,\epsilon}(t) + \omega_{1,\epsilon,\beta_\epsilon}(t) + v_{c,\epsilon,\beta_\epsilon}(t)\right) \\
&- \epsilon u''(t) - \epsilon \omega''_{0,\epsilon}(t) - \epsilon \omega''_{1,\epsilon,\beta_\epsilon}(t) - \epsilon v''_{c,\epsilon,\beta_\epsilon}(t) \\
&\geq (-m) \left(\omega_{0,\epsilon}(t) + \omega_{1,\epsilon,\beta_\epsilon}(t) + v_{c,\epsilon,\beta_\epsilon}(t)\right) \\
&- \epsilon u''(t) - \epsilon \omega''_{0,\epsilon}(t) - \epsilon \omega''_{1,\epsilon,\beta_\epsilon}(t) - \epsilon v''_{c,\epsilon,\beta_\epsilon}(t) \\
&= -\epsilon v''_{c,\epsilon,\beta_\epsilon}(t) - m v_{c,\epsilon,\beta_\epsilon}(t) = -r_{\epsilon,\beta_\epsilon}(t) \geq 0
\end{aligned}$$

where $(t, \theta_\epsilon(t))$ is a point between $(t, \alpha_\epsilon(t))$ and $(t, u(t))$, $(t, \theta_\epsilon(t)) \in \mathcal{D}_\delta(u)$. Analogously, $(t, \tilde{\theta}_\epsilon(t))$ is a point between $(t, u(t))$ and $(t, \beta_\epsilon(t))$, $(t, \tilde{\theta}_\epsilon(t)) \in \mathcal{D}_\delta(u)$ for $\epsilon \in (0, \epsilon_0] \cap \mathcal{M}$. The existence of a solution for (1.1), (1.2) satisfying the inequality above follows from Lemma 1.

Remark 3. We note, that if there exists the solution of (1.3) such that $r_{\epsilon,i}(t) = \mathcal{O}(\epsilon^\nu)$, $\nu > 0$ then for every sequence$\{\epsilon_n\}$, $\epsilon_n \in (0, \epsilon_0] \cap \mathcal{M}$, $\epsilon_n \in J_n$ we have

$$|y_{\epsilon_n}(t) - u(t)| \leq \left(\left|u'(a)\right| + \left|u'(b)\right|\right) \mathcal{O}\left(\sqrt{\epsilon_n}\right) + M_{u''} \mathcal{O}\left(\epsilon_n\right) + \mathcal{O}\left(\epsilon_n^\nu\right),$$

$M_{u''} = \max\left\{|u''(t)|, t \in \langle a, b \rangle\right\}$ on $\langle a, b \rangle$.

Remark 4. In the trivial case, when $u(t) = c$ =const is $\omega_{0,\epsilon}(t) = \omega_{1,\epsilon,i}(t) \stackrel{\text{id}}{=} 0$, $r_{\epsilon,i}(t) \stackrel{\text{id}}{=} 0$, $i = \alpha_\epsilon, \beta_\epsilon$ and

$$|y_\epsilon(t) - u(t)| \leq 0$$

i.e. $y_\epsilon(t) = u(t)$ on $\langle a, b\rangle$.

Example 1. Consider nonlinear problem (1.1), (1.2) with $f(t, y) = y^2 + g(t)$, i.e.

$$\epsilon y'' + ky = y^2 + g(t), \quad t \in \langle a, b\rangle, \quad k > 0, \quad 0 < \epsilon << 1$$
$$y'(a) = 0, \quad y'(b) = 0.$$

For $0 \leq g(t) < \frac{k^2}{4}$ on $\langle a, b\rangle$ the solution

$$u(t) = \frac{1}{2}\left(k - \sqrt{k^2 - 4g(t)}\right)$$

of the reduced problem $ku = u^2 + g(t)$ satisfies the assumption $(\mathcal{A}2)$ of Theorem 1. Let $\tilde{z}_{\epsilon,i}(t)$, $\epsilon \in (0, \epsilon_0] \cap \mathcal{M}$, $i = \alpha_\epsilon, \beta_\epsilon$ are the functions satisfying (1.13) (the assumption $(\mathcal{A}1)$).

Thus, according to Theorem 1 above, there is for $\epsilon \in (0, \epsilon_0] \cap \mathcal{M}$ a solution $y_\epsilon(t)$ of the considered boundary value problem satisfying the inequality

$$-\omega_{0,\epsilon}(t) - \omega_{1,\epsilon,\alpha_\epsilon}(t) - v_{c,\epsilon,\alpha_\epsilon}(t) \leq y_\epsilon(t) - u(t) \leq \omega_{0,\epsilon}(t) + \omega_{1,\epsilon,\beta_\epsilon}(t) + v_{c,\epsilon,\beta_\epsilon}(t)$$

on $\langle a, b\rangle$.

3 Generalization of the assumption $(\mathcal{A}1)$

The assumption of nonnegativity of $z_{\epsilon,i}(t)$ in (1.3) and the condition (1.12) may be generalized in the following sense.
Denote

$$I_{+,\epsilon,i} = \{t \in \langle a, b\rangle : z_{\epsilon,i}(t) \geq 0\}, \quad i = \alpha_\epsilon, \beta_\epsilon$$

and

$$I_{-,\epsilon,i} = \{t \in \langle a, b\rangle : z_{\epsilon,i}(t) \leq 0\}, \quad i = \alpha_\epsilon, \beta_\epsilon.$$

Let there exist the functions $\tilde{z}_{\epsilon,i}(t)$ such that

$$r_{\epsilon,i}(t) \leq 0 \quad \text{on} \quad I_{+,\epsilon,i}, \quad i = \alpha_\epsilon, \beta_\epsilon \tag{3.1}$$

and

$$r_{\epsilon,i}(t) \leq 2wz_{\epsilon,i}(t) \quad \text{on} \quad I_{-,\epsilon,i}, \quad i = \alpha_\epsilon, \beta_\epsilon \tag{3.2}$$

and

$$v_{c,\epsilon,\alpha_\epsilon}(t) + v_{c,\epsilon,\beta_\epsilon}(t) \geq -2\omega_{0,\epsilon}(t) \quad \text{on} \quad I_{-,\epsilon,\alpha_\epsilon} \cup I_{-,\epsilon,\beta_\epsilon} \tag{3.3}$$

where $r_{\epsilon,i}(t)$ is from (1.11) and $z_{\epsilon,i}(t) = r_{\epsilon,i}(t) + \tilde{z}_{\epsilon,i}(t)$, $i = \alpha_\epsilon, \beta_\epsilon$.
Taking into consideration the fact that

$$\omega_{0,\epsilon}(t) + \omega_{1,\epsilon,i}(t) + v_{c,\epsilon,i}(t) = z_{\epsilon,i}(t) \leq 0 \quad \text{on} \quad I_{-,\epsilon,i}, \quad i = \alpha_\epsilon, \beta_\epsilon, \tag{3.4}$$

for the required inequality (2.1) for $\alpha_\epsilon(t)$ on the interval $I_{-,\epsilon,\alpha_\epsilon}$ (in the case of the inequality for $\beta_\epsilon(t)$ i.e. (2.2) on $I_{-,\epsilon,\beta_\epsilon}$, we proceed analogously) we obtain

$$\begin{aligned}\epsilon\alpha_\epsilon''(t) - h(t,\alpha_\epsilon(t)) &= \epsilon u''(t) - \epsilon\omega_{0,\epsilon}''(t) - \epsilon\omega_{1,\epsilon,\alpha_\epsilon}''(t) - \epsilon v_{c,\epsilon,\alpha_\epsilon}''(t)\\ &- \frac{\partial h\,(t,\theta_\epsilon(t))}{\partial y}\left(-\omega_{0,\epsilon}(t) - \omega_{1,\epsilon,\alpha_\epsilon}(t) - v_{c,\epsilon,\alpha_\epsilon}(t)\right)\\ &\geq \epsilon u''(t) - \epsilon\omega_{0,\epsilon}''(t) - \epsilon\omega_{1,\epsilon,\alpha_\epsilon}''(t) - \epsilon v_{c,\epsilon,\alpha_\epsilon}''(t)\\ &+ (-m+2w)\left(\omega_{0,\epsilon}(t) + \omega_{1,\epsilon,\alpha_\epsilon}(t) + v_{c,\epsilon,\alpha_\epsilon}(t)\right)\\ &= -r_{\epsilon,\alpha_\epsilon}(t) + 2w\left(\omega_{0,\epsilon}(t) + \omega_{1,\epsilon,\alpha_\epsilon}(t) + v_{c,\epsilon,\alpha_\epsilon}(t)\right).\end{aligned}$$

From (3.2) and (3.4), $-r_{\epsilon,\alpha_\epsilon}(t) + 2w\left(\omega_{0,\epsilon}(t) + \omega_{1,\epsilon,\alpha_\epsilon}(t) + v_{c,\epsilon,\alpha_\epsilon}(t)\right) \geq 0$ for $t \in I_{-,\epsilon,\alpha_\epsilon}$. The condition (3.3) guarantees that $\alpha_\epsilon(t) \leq \beta_\epsilon(t)$ on $\langle a,b\rangle$. Hence, Theorem 1 holds.

From (3.2), we get

$$(1-2w)r_{\epsilon,i}(t) \leq 2w\tilde{z}_{\epsilon,i}(t) \leq -2wr_{\epsilon,i}(t) \tag{3.5}$$

and we may generalize the assumption $(\mathcal{A}1)$ as follows.

$(\mathcal{A}1')$ Let $\tilde{z}_{\epsilon,i}(t)$, $i = \alpha_\epsilon, \beta_\epsilon$ be the continuous functions such that

$$\left[(1.12)\right] \vee \left[(3.5) \wedge \left(v_{c,\epsilon,\alpha_\epsilon}(t) + v_{c,\epsilon,\beta_\epsilon}(t) \geq -2\omega_{0,\epsilon}(t)\right)\right]$$

on $\langle a,b\rangle$, $\epsilon \in (0,\epsilon_0] \cap \mathcal{M}$ holds.

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