Electronic Journal of Qualitative Theory of Differential Equations
2010, No. **14**, 1-8; **http://www.math.u-szeged.hu/ejqtde/**

Existence of solutions for a class of fourth-order m-point boundary value problems*

Jian-Ping Sun, Qiu-Yan Ren

Department of Applied Mathematics, Lanzhou University of Technology,
Lanzhou, Gansu, 730050, People's Republic of China

E-mail: jpsun@lut.cn (Sun)

Abstract

Some existence criteria are established for a class of fourth-order m-point boundary value problem by using the upper and lower solution method and the Leray-Schauder continuation principle.

Keywords: Fourth-order m-point boundary value problem; Upper and lower solution method; Leray-Schauder continuation principle; Nagumo condition
2000 AMS Subject Classification: 34B10, 34B15

1 Introduction

Boundary value problems (BVPs for short) of fourth-order differential equations have been used to describe a large number of physical, biological and chemical phenomena. For example, the deformations of an elastic beam in the equilibrium state can be described as some fourth-order BVP. Recently, fourth-order BVPs have received much attention. For instance, [3, 5, 6, 7] discussed some fourth-order two-point BVPs, while [1, 2, 4, 9] studied some fourth-order three-point or four-point BVPs. It is worth mentioning that Ma, Zhang and Fu [7] employed the upper and lower solution method to prove the existence of solutions for the BVP

$$\begin{cases} u^{(4)}(t) = f\left(t, u\left(t\right), u''\left(t\right)\right),\ t \in (0,1), \\ u\left(0\right) = u'\left(1\right) = u''\left(0\right) = u'''\left(1\right) = 0, \end{cases}$$

and Bai [3] considered the existence of a solution for the BVP

$$\begin{cases} u^{(4)}(t) = f\left(t, u\left(t\right), u'\left(t\right), u''\left(t\right), u'''\left(t\right)\right),\ t \in (0,1), \\ u\left(0\right) = u'\left(1\right) = u''\left(0\right) = u'''\left(1\right) = 0 \end{cases}$$

by using the upper and lower solution method and Schauder′s fixed point theorem.

Although there are many works on fourth-order two-point, three-point or four-point BVPs, a little work has been done for more general fourth-order m-point BVPs [8]. Motivated greatly by the

*Supported by the National Natural Science Foundation of China (10801068).

above-mentioned excellent works, in this paper, we will investigate the following fourth-order m-point BVP

$$\begin{cases} u^{(4)}(t)+f(t,u\left(t\right),u'\left(t\right),u''\left(t\right),u'''\left(t\right))=0,\ t\in\left[0,1\right], \\ u\left(0\right)=\sum\limits_{i=1}^{m-2}a_iu\left(\eta_i\right),\ u'\left(1\right)=0, \\ u''\left(0\right)=\sum\limits_{i=1}^{m-2}b_iu''\left(\eta_i\right),\ u'''\left(1\right)=0. \end{cases} \tag{1.1}$$

Throughout this paper, we always assume that $0<\eta_1<\eta_2<\cdots<\eta_{m-2}<1$, a_i and b_i $(i=1,2,\cdots,m-2)$ are nonnegative constants and $f:[0,1]\times R^4\to R$ is continuous. Some existence criteria are established for the BVP (1.1) by using the upper and lower solution method and the Leray-Schauder continuation principle.

2 Preliminaries

Let $E=C\left[0,1\right]$ be equipped with the norm $\|v\|_\infty=\max\limits_{t\in[0,1]}|v\left(t\right)|$ and

$$K=\left\{v\in E|\,v\left(t\right)\geq 0\text{ for }t\in\left[0,1\right]\right\}.$$

Then K is a cone in E and (E,K) is an ordered Banach space. For Banach space $X=C^1\left[0,1\right]$, we use the norm $\|v\|=\max\left\{\|v\|_\infty,\|v'\|_\infty\right\}$.

Lemma 2.1 *Let* $\sum\limits_{i=1}^{m-2}a_i\neq 1$. *Then for any* $h\in E$, *the second-order* m*-point BVP*

$$\begin{cases} -u''(t)=h\left(t\right),\ t\in\left[0,1\right], \\ u\left(0\right)=\sum\limits_{i=1}^{m-2}a_iu\left(\eta_i\right),\ u'\left(1\right)=0 \end{cases} \tag{2.1}$$

has a unique solution

$$u\left(t\right)=\int_0^1 G_1\left(t,s\right)h\left(s\right)ds,$$

where

$$G_1\left(t,s\right)=K\left(t,s\right)+\frac{1}{1-\sum\limits_{i=1}^{m-2}a_i}\sum_{i=1}^{m-2}a_iK\left(\eta_i,s\right),$$

here

$$K\left(t,s\right)=\begin{cases} s,\ 0\leq s\leq t\leq 1, \\ t,\ 0\leq t\leq s\leq 1 \end{cases}$$

is Green's function of the second-order two-point BVP

$$\begin{cases} -u''(t)=0,\ t\in\left[0,1\right], \\ u\left(0\right)=u'\left(1\right)=0. \end{cases}$$

Proof. If u is a solution of the BVP (2.1), then we may suppose that

$$u(t) = \int_0^1 K(t,s)h(s)ds + At + B.$$

By the boundary conditions in (2.1), we know that

$$A = 0 \text{ and } B = \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \sum_{i=1}^{m-2} a_i \int_0^1 K(\eta_i, s)h(s)\,ds.$$

Therefore, the unique solution of the BVP (2.1)

$$\begin{aligned} u(t) &= \int_0^1 K(t,s)h(s)ds + \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \sum_{i=1}^{m-2} a_i \int_0^1 K(\eta_i, s)h(s)\,ds \\ &= \int_0^1 G_1(t,s)\,h(s)\,ds. \end{aligned}$$

□

In the remainder of this paper, we always assume that $\sum_{i=1}^{m-2} a_i < 1$ and $\sum_{i=1}^{m-2} b_i < 1$, which imply that $G_1(t,s)$ and $G_2(t,s)$ are nonnegative on $[0,1] \times [0,1]$, where

$$G_2(t,s) = K(t,s) + \frac{1}{1 - \sum_{i=1}^{m-2} b_i} \sum_{i=1}^{m-2} b_i K(\eta_i, s).$$

Now, we define operators A and $B : E \to E$ as follows:

$$(Av)(t) = -\int_0^1 G_1(t,s)\,v(s)\,ds, \ t \in [0,1] \tag{2.2}$$

and

$$(Bv)(t) = -\int_t^1 v(s)\,ds, \ t \in [0,1]. \tag{2.3}$$

Remark 2.1 *A and B are decreasing operators on E.*

Lemma 2.2 *If the following BVP*

$$\begin{cases} v''(t) + f(t, (Av)(t), (Bv)(t), v(t), v'(t)) = 0, \ t \in [0,1], \\ v(0) = \sum_{i=1}^{m-2} b_i v(\eta_i), \ v'(1) = 0 \end{cases} \tag{2.4}$$

has a solution, then does the BVP (1.1).

Proof. Suppose that v is a solution of the BVP (2.4). Then it is easy to prove that $u = Av$ is a solution of the BVP (1.1). □

Definition 2.1 *If* $\alpha \in C^2[0,1]$ *satisfies*

$$\begin{cases} \alpha''(t) + f(t, (A\alpha)(t), (B\alpha)(t), \alpha(t), \alpha'(t)) \geq 0, \ t \in [0,1], \\ \alpha(0) \leq \sum\limits_{i=1}^{m-2} b_i \alpha(\eta_i), \ \alpha'(1) \leq 0, \end{cases} \tag{2.5}$$

then α *is called a lower solution of the BVP (2.4).*

Definition 2.2 *If* $\beta \in C^2[0,1]$ *satisfies*

$$\begin{cases} \beta''(t) + f(t, (A\beta)(t), (B\beta)(t), \beta(t), \beta'(t)) \leq 0, \ t \in [0,1], \\ \beta(0) \geq \sum\limits_{i=1}^{m-2} b_i \beta(\eta_i), \ \beta'(1) \geq 0, \end{cases} \tag{2.6}$$

then β *is called an upper solution of the BVP (2.4).*

Remark 2.2 *If the inequality in Definition (2.1)*

$$\alpha''(t) + f(t, (A\alpha)(t), (B\alpha)(t), \alpha(t), \alpha'(t)) \geq 0, \ t \in [0,1]$$

is replaced by

$$\alpha''(t) + f(t, (A\alpha)(t), (B\alpha)(t), \alpha(t), \alpha'(t)) > 0, \ t \in [0,1],$$

then α *is called a strict lower solution of the BVP (2.4). Similarly, we can also give the definition of a strict upper solution for the BVP (2.4).*

Definition 2.3 *Assume that* $f \in C\left([0,1] \times R^4, R\right)$, α, $\beta \in E$ *and* $\alpha(t) \leq \beta(t)$ *for* $t \in [0,1]$. *We say that* f *satisfies Nagumo condition with respect to* α *and* β *provided that there exists a function* $h \in C([0,+\infty), (0,+\infty))$ *such that*

$$|f(t, x_1, x_2, x_3, x_4)| \leq h(|x_4|),$$

for all $(t, x_1, x_2, x_3, x_4) \in [0,1] \times [(A\beta)(t), (A\alpha)(t)] \times [(B\beta)(t), (B\alpha)(t)] \times [\alpha(t), \beta(t)] \times R$, *and*

$$\int_\lambda^{+\infty} \frac{s}{h(s)} ds > \max_{t\in[0,1]} \beta(t) - \min_{t\in[0,1]} \alpha(t), \tag{2.7}$$

where $\lambda = \max\{|\beta(1) - \alpha(0)|, |\beta(0) - \alpha(1)|\}$.

Lemma 2.3 *Assume that* α *and* β *are, respectively, the lower and the upper solution of the BVP (2.4) with* $\alpha(t) \leq \beta(t)$ *for* $t \in [0,1]$, *and* f *satisfies the Nagumo condition with respect to* α *and* β. *Then there exists* $N > 0$ *(depending only on* α *and* β*) such that any solution* ω *of the BVP (2.4) lying in* $[\alpha, \beta]$ *satisfies*

$$\left|\omega'(t)\right| \leq N, \ t \in [0,1].$$

Proof. It follows from the definition of λ and the mean-value theorem that there exists $t_0 \in (0,1)$ such that

$$\left|\omega'(t_0)\right| = |\omega(1) - \omega(0)| \leq \lambda. \tag{2.8}$$

By (2.7), we know that there exists $N > \lambda$ such that

$$\int_\lambda^{N} \frac{s}{h(s)} ds > \max_{t\in[0,1]} \beta(t) - \min_{t\in[0,1]} \alpha(t). \tag{2.9}$$

Now, we will prove that $|\omega'(t)| \le N$ for any $t \in [0,1]$. Suppose on the contrary that there exists $t_1 \in [0,1]$ such that

$$\left|\omega'(t_1)\right| > N. \tag{2.10}$$

In view of (2.8) and (2.10), we know that there exist $t_2, t_3 \in (0,1)$ with $t_2 < t_3$ such that one of the following cases holds:

Case 1. $\lambda < \omega'(t) < N$ for $t \in (t_2, t_3)$, $\omega'(t_2) = \lambda$ and $\omega'(t_3) = N$;

Case 2. $\lambda < \omega'(t) < N$ for $t \in (t_2, t_3)$, $\omega'(t_2) = N$ and $\omega'(t_3) = \lambda$;

Case 3. $-N < \omega'(t) < -\lambda$ for $t \in (t_2, t_3)$, $\omega'(t_2) = -N$ and $\omega'(t_3) = -\lambda$;

Case 4. $-N < \omega'(t) < -\lambda$ for $t \in (t_2, t_3)$, $\omega'(t_2) = -\lambda$ and $\omega'(t_3) = -N$.

Since the others is similar, we only consider Case 1. By the Nagumo condition, we have

$$\begin{aligned}\left|\omega''(t)\right| \cdot \omega'(t) &= \left|f(t, (A\omega)(t), (B\omega)(t), \omega(t), \omega'(t))\right| \cdot \omega'(t) \\ &\le h\left(\left|\omega'(t)\right|\right) \cdot \omega'(t), \ t \in [t_2, t_3].\end{aligned}$$

So,

$$\frac{|\omega''(t)| \cdot \omega'(t)}{h(\omega'(t))} \le \omega'(t), \ t \in [t_2, t_3],$$

and so,

$$\left|\int_{t_2}^{t_3} \frac{\omega''(t) \cdot \omega'(t)}{h(\omega'(t))} dt\right| \le \int_{t_2}^{t_3} \left|\frac{\omega''(t) \cdot \omega'(t)}{h(\omega'(t))}\right| dt \le \int_{t_2}^{t_3} \omega'(t)\, dt,$$

which implies that

$$\int_{\lambda}^{N} \frac{s}{h(s)} ds \le \omega(t_3) - \omega(t_2) \le \max_{t\in[0,1]} \beta(t) - \min_{t\in[0,1]} \alpha(t),$$

which contradicts with (2.9) and the proof is complete. □

3 Main result

Theorem 3.1 *Assume that α and β are, respectively, the strict lower and the strict upper solution of the BVP (2.4) with $\alpha(t) \le \beta(t)$ for $t \in [0,1]$, and f satisfies the Nagumo condition with respect to α and β. Then the BVP (2.4) has a solution v_0 and*

$$\alpha(t) \le v_0(t) \le \beta(t) \ \text{ for } t \in [0,1].$$

Proof. It follows from Lemma 2.3 that there exists $N > 0$ such that any solution ω of the BVP (2.4) lying in $[\alpha, \beta]$ satisfies

$$\left|\omega'(t)\right| \le N \text{ for } t \in [0,1].$$

We denote $C = \max\left\{N, \max_{t\in[0,1]} |\alpha'(t)|, \max_{t\in[0,1]} |\beta'(t)|\right\}$ and define the auxiliary functions f_1, f_2, f_3 and $F : [0,1] \times R^4 \to R$ as follows:

$$f_1(t, x_1, x_2, x_3, x_4) = \begin{cases} f(t, x_1, x_2, x_3, C), \ x_4 > C, \\ f(t, x_1, x_2, x_3, x_4), \ -C \le x_4 \le C, \\ f(t, x_1, x_2, x_3, -C), \ x_4 < -C; \end{cases}$$

$$f_2(t,x_1,x_2,x_3,x_4)=\begin{cases} f_1(t,(A\alpha)(t),x_2,x_3,x_4), & x_1>(A\alpha)(t),\\ f_1(t,x_1,x_2,x_3,x_4), & (A\beta)(t)\le x_1\le (A\alpha)(t),\\ f_1(t,(A\beta)(t),x_2,x_3,x_4), & x_1<(A\beta)(t);\end{cases}$$

$$f_3(t,x_1,x_2,x_3,x_4)=\begin{cases} f_2(t,x_1,(B\alpha)(t),x_3,x_4), & x_2>(B\alpha)(t),\\ f_2(t,x_1,x_2,x_3,x_4), & (B\beta)(t)\le x_2\le (B\alpha)(t),\\ f_2(t,x_1,(B\beta)(t),x_3,x_4), & x_2<(B\beta)(t)\end{cases}$$

and

$$F(t,x_1,x_2,x_3,x_4)=\begin{cases} f_3(t,x_1,x_2,\beta(t),x_4), & x_3>\beta(t),\\ f_3(t,x_1,x_2,x_3,x_4), & \alpha(t)\le x_3\le \beta(t),\\ f_3(t,x_1,x_2,\alpha(t),x_4), & x_3<\alpha(t).\end{cases}$$

Consider the following auxiliary BVP

$$\begin{cases} v''(t)+F(t,(Av)(t),(Bv)(t),v(t),v'(t))=0, \ t\in[0,1],\\ v(0)=\sum\limits_{i=1}^{m-2} b_i v(\eta_i), \ v'(1)=0.\end{cases} \tag{3.1}$$

If we define an operator $T:X\to X$ by

$$(Tv)(t)=\int_0^1 G_2(t,s)F(s,(Av)(s),(Bv)(s),v(s),v'(s))ds, \ t\in[0,1],$$

then it is obvious that fixed points of T are solutions of the BVP (3.1). Now, we will apply the Leray-Schauder continuation principle to prove that the operator T has a fixed point. Since it is easy to verify that $T:X\to X$ is completely continuous by using the Arzela-Ascoli theorem, we only need to prove that the set of all possible solutions of the homotopy group problem $v=\lambda Tv$ is a priori bounded in X by a constant independent of $\lambda\in(0,1)$. Denote

$$\alpha_m=\min_{t\in[0,1]}\alpha(t), \ \beta_M=\max_{t\in[0,1]}\beta(t),$$
$$(A\beta)_m=\min_{t\in[0,1]}(A\beta)(t), \ (A\alpha)_M=\max_{t\in[0,1]}(A\alpha)(t),$$
$$(B\beta)_m=\min_{t\in[0,1]}(B\beta)(t), \ (B\alpha)_M=\max_{t\in[0,1]}(B\alpha)(t),$$
$$L=\sup\{|f(t,x_1,x_2,x_3,x_4)|:(t,x_1,x_2,x_3,x_4)\in[0,1]\times[(A\beta)_m,(A\alpha)_M] \times[(B\beta)_m,(B\alpha)_M]\times[\alpha_m,\beta_M]\times[-C,C]\}.$$

Let $v=\lambda Tv$. Then we have

$$\begin{aligned} |v(t)| &= |\lambda(Tv)(t)|\\ &= \lambda\left|\int_0^1 G_2(t,s)F(s,(Av)(s),(Bv)(s),v(s),v'(s))ds\right|\\ &\le \int_0^1\left(K(t,s)+\frac{1}{1-\sum\limits_{i=1}^{m-2}b_i}\sum_{i=1}^{m-2}b_iK(\eta_i,s)\right)\left|F(s,(Av)(s),(Bv)(s),v(s),v'(s))\right|ds\\ &\le \frac{L}{1-\sum\limits_{i=1}^{m-2}b_i}=:R, \ t\in[0,1]\end{aligned}$$

and

$$\begin{aligned}
\left|v'(t)\right| &= \left|\lambda (Tv)'(t)\right| \\
&= \lambda \left| \int_0^1 \frac{\partial G_2(t,s)}{\partial t} F(s,(Av)(s),(Bv)(s),v(s),v'(s))ds \right| \\
&\le \int_t^1 \left| F(s,(Av)(s),(Bv)(s),v(s),v'(s)) \right| ds \\
&\le L \le R,\ t \in [0,1],
\end{aligned}$$

which imply that

$$\|v\| = \max\left\{ \|v\|_\infty, \|v'\|_\infty \right\} \le R.$$

It is now immediate from the Leray-Schauder continuation principle that the operator T has a fixed point v_0, which solves the BVP (3.1).

Now, let us prove that v_0 is a solution of the BVP (2.4). Therefor, we only need to verify that $\alpha(t) \le v_0(t) \le \beta(t)$ and $|v_0'(t)| \le C$ for $t \in [0,1]$.

First, we will verify that $v_0(t) \le \beta(t)$ for $t \in [0,1]$. Suppose on the contrary that there exists $t_0 \in [0,1]$ such that

$$v_0(t_0) - \beta(t_0) = \max_{t\in[0,1]} \{v_0(t) - \beta(t)\} > 0.$$

We consider the following three cases:

Case 1: If $t_0 \in (0,1)$, then $v_0(t_0) > \beta(t_0)$, $v_0'(t_0) = \beta'(t_0)$ and $v_0''(t_0) \le \beta''(t_0)$. Since β is a strict upper solution of the BVP (2.4), one has

$$\begin{aligned}
v_0''(t_0) &= -F(t_0,(Av_0)(t_0),(Bv_0)(t_0),v_0(t_0),v_0'(t_0)) \\
&= -f_3(t_0,(Av_0)(t_0),(Bv_0)(t_0),\beta(t_0),\beta'(t_0)) \\
&= -f_2(t_0,(Av_0)(t_0),(B\beta)(t_0),\beta(t_0),\beta'(t_0)) \\
&= -f_1(t_0,(A\beta)(t_0),(B\beta)(t_0),\beta(t_0),\beta'(t_0)) \\
&= -f(t_0,(A\beta)(t_0),(B\beta)(t_0),\beta(t_0),\beta'(t_0)) \\
&> \beta''(t_0),
\end{aligned}$$

which is a contradiction.

Case 2: If $t_0 = 0$, then $v_0(0) > \beta(0)$. On the other hand, $v_0(0) = \sum_{i=1}^{m-2} b_i v_0(\eta_i) \le \sum_{i=1}^{m-2} b_i \beta(\eta_i) \le \beta(0)$. This is a contradiction.

Case 3: If $t_0 = 1$, then $v_0(1) - \beta(1) = \max_{t\in[0,1]} \{v_0(t) - \beta(t)\} > 0$, which shows that $v_0'(1) \ge \beta'(1)$. On the other hand, $v_0'(1) = 0 \le \beta'(1)$. Consequently, $v_0'(1) = \beta'(1)$, and so, $v_0''(1) \le \beta''(1)$. With the similar arguments as in Case 1, we can obtain a contradiction also.

Thus, $v_0(t) \le \beta(t)$ for $t \in [0,1]$. Similarly, we can prove that $\alpha(t) \le v_0(t)$ for $t \in [0,1]$.

Next, we will show that $|v_0'(t)| \le C$ for $t \in [0,1]$. In fact, since f satisfies the Nagumo condition with respect to α and β, with the similar arguments as in Lemma 2.3, we can obtain that

$$\left|v_0'(t)\right| \le N \le C \text{ for } t \in [0,1].$$

Therefore, v_0 is a solution of the BVP (2.4) and $\alpha(t) \le v_0(t) \le \beta(t)$ for $t \in [0,1]$. □

References

[1] C. Z. Bai, Triple positive solutions of three-point boundary value problems for fourth-order differential equations, *Computers and Mathematics with Applications* 56 (2008) 1364-1371.

[2] C. Z. Bai, D. D. Yang and H. B. Zhu, Existence of solutions for fourth order differential equation with four-point boundary conditions, *Appl. Math. Lett.* 20 (2007) 1131-1136.

[3] Z. B. Bai, The upper and lower solution method for some fourth-order boundary value problems, *Nonlinear Anal.* 67 (2007) 1704-1709.

[4] J. R. Graef, C. X. Qian and B. Yang, A three point boundary value problem for nonlinear fourth order differential equations, *J. Math. Anal. Appl.* 287 (2003) 217-233.

[5] B. Liu, Positive solutions of fourth-order two point boundary value problems, *Appl. Math. Comput.* 148 (2004) 407-420.

[6] R. Ma, Positive solutions of fourth-order two point boundary value problems, *Ann. Differential Equations* 15 (1999) 305-313.

[7] R. Ma, J. Zhang and S. Fu, The method of lower and upper solutions for fourth-order two-point boundary value problems, *J. Math. Anal. Appl.* 215 (1997) 415-422.

[8] Z. L. Wei and C. C. Pang, The method of lower and upper solutions for fourth order singular m-point boundary value problems, *J. Math. Anal. Appl.* 322 (2006) 675-692.

[9] Y. L. Zhong, S. H. Chen and C. P. Wang, Existence results for a fourth-order ordinary differential equation with a four-point boundary condition, *Appl. Math. Lett.* 21 (2008) 465-470.

(Received October 8, 2009)