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Boundedness of solutions to a retarded Liénard equation*

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Abstract

This paper is concerned with the following retarded Liénard equation

$$x''(t) + f_1(x(t))(x'(t))^2 + f_2(x(t))x'(t) + g_1(x(t)) + g_2(x(t-\tau(t))) = e(t).$$

We prove a new theorem which ensures that all solutions of the above Liénard equation satisfying given initial conditions are bounded. As one will see, our results improve some earlier results even in the case of $f_1(x) \equiv 0$.

Keywords: Liénard equation, boundedness.
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1 Introduction

The study on boundedness of solutions to all kinds of Liénard equations has been of interest for many mathematicians (cf. [2, 3, 5–7, 9, 10] and references therein).

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Recently, the authors in [7] studied the boundedness of solutions to the following Liénard equation with a deviating argument:

$$x''(t) + f(x(t))x'(t) + g_1(x(t)) + g_2(x(t - \tau(t))) = e(t), \tag{1.1}$$

where f, g_1 and g_2 are continuous functions on $\mathbb{R}$, $\tau(t) \geq 0$ is a bounded continuous function on $\mathbb{R}$, and $e(t)$ is a bounded continuous function on $\mathbb{R}^+ = [0, +\infty)$. Under the condition

(A0) There exists a constant $d > 1$ such that $d|u| \leq \operatorname{sgn}(u)\varphi(u)$ for all $u \in \mathbb{R}$, where

$$\varphi(u) = \int_0^u [f(x) - 1]dx.$$

and other assumptions, the authors in [7] established a theorem which ensures that all solutions of (1.1) are bounded. Very recently, in [8], the assumption (A0) is weakened into

(A1) $|u| < \operatorname{sgn}(u)\varphi(u)$ for all $u \in \mathbb{R}$.

In this paper, we will study the following more general equation:

$$x''(t) + f_1(x(t))(x'(t))^2 + f_2(x(t))x'(t) + g_1(x(t)) + g_2(x(t - \tau(t))) = e(t), \tag{1.2}$$

where f_1, f_2, g_1 and g_2 are continuous functions on $\mathbb{R}$, $\tau(t) \geq 0$ is a bounded continuous function on $\mathbb{R}$, and $e(t)$ is a bounded continuous function on $\mathbb{R}^+ = [0, +\infty)$. Under weaker assumption than (A0) and (A1) (see Remark 2.4), we prove that all solutions of (1.2) are bounded, and thus improve the results in [7, 8] even in the case of $f_1(x) \equiv 0$.

2 Main results

Throughout the rest of this paper, we denote

$$F(x) = \exp\left(\int_0^x f_1(u)du\right);$$

and assume that there is a constant $\lambda > 0$ satisfying

$$\operatorname{sgn}(u)\int_0^u F^2(x)dx \le \lambda|u|, \quad u \in \mathbb{R}. \tag{2.1}$$

Moreover, assume that there exist a constant $\varepsilon > 0$ and two nondecreasing functions G, Φ defined on $\mathbb{R}^+$ such that

$$\lambda\varepsilon < \liminf_{u\to\pm\infty} \frac{\operatorname{sgn}(u)\int_0^u F(x)f_2(x)dx}{|u|} - 1, \tag{2.2}$$

$$|g_1(u) - \varepsilon\phi(u)| \le \Phi(|u|), \quad |g_2(u)| \le G(|u|), \quad \forall u \in \mathbb{R}, \tag{2.3}$$

and

$$\limsup_{x\to+\infty} \frac{\Phi(x) + G(x)}{x} < \varepsilon, \tag{2.4}$$

where

$$\phi(x) = \int_0^x F(u)[f_2(u) - \varepsilon F(u)]du.$$

Denote

$$y = F(x)\frac{dx}{dt} + \phi(x).$$

Then Eq. (1.2) is transformed into the following system:

$$\begin{cases} \dfrac{dx(t)}{dt} = \dfrac{-\phi(x(t)) + y(t)}{F(x(t))}, \\ \dfrac{dy(t)}{dt} = F(x(t))\big\{-\varepsilon y(t) - [g_1(x(t)) - \varepsilon\phi(x(t))] - g_2(x(t-\tau(t))) + e(t)\big\}. \end{cases} \tag{2.5}$$

In addtion, in this paper, $C([-h,0],\mathbb{R})$ denotes the space of continuous functions $\alpha : [-h,0] \to \mathbb{R}$ with the supremum norm $\|\cdot\|$, where $h = \sup_{t\in\mathbb{R}} \tau(t) \ge 0$. It is well known (cf. [1, 4]) that for any given continuous initial function $\alpha \in C([-h,0],\mathbb{R})$ and a number y_0, there exists a solution of (2.5) on an interval $[0,T)$ satisfying the initial conditions and (2.5) on $[0,T)$. If the solution remains bounded, then $T = +\infty$. We denote such a solution by $x(t) = x(t,\alpha,y_0), y(t) = y(t,\alpha,y_0)$.

Definition 2.1. [2, 7] *Solutions of* (2.5) *are called uniformly bounded if for each* $B_1 > 0$ *there is a* $B_2 > 0$ *such that* $(\alpha, y_0) \in C([-h,0],\mathbb{R}) \times \mathbb{R}$ *and* $\|\alpha\| + |y_0| \le B_1$ *implies that* $|x(t,\alpha,y_0)| + |y(t,\alpha,y_0)| \le B_2$ *for all* $t \in \mathbb{R}^+$.

Theorem 2.2. *Suppose that* (2.1)–(2.4) *hold. Then, solutions of* (2.5) *are uniformly bounded.*

Proof. Let $x(t) = x(t, \alpha, y_0)$, $y(t) = y(t, \alpha, y_0)$ be a solution of (2.5). Without loss of generality, one can assume that $x(t), y(t)$ is defined on $\mathbb{R}^+$ since the following proof gives that $x(t), y(t)$ are bounded.

By (2.2) and (2.4), there is a constant $M > 0$ such that

$$\frac{\operatorname{sgn}(u) \int_0^u F(x) f_2(x) dx}{|u|} > 1 + \lambda\varepsilon, \quad |u| \geq M, \tag{2.6}$$

and

$$\frac{\Phi(x) + G(x) + \overline{e}}{x} < \varepsilon, \quad x \geq M, \tag{2.7}$$

where $\overline{e} = \sup\limits_{t \in \mathbb{R}^+} |e(t)|$. It follows from (2.6) and (2.1) that

$$\frac{\operatorname{sgn}(u)\phi(u)}{|u|} = \frac{\operatorname{sgn}(u) \int_0^u F(x) f_2(x) dx}{|u|} - \frac{\operatorname{sgn}(u) \int_0^u \varepsilon F^2(x) dx}{|u|} > 1, \quad |u| \geq M. \tag{2.8}$$

We denote

$$V(t) = \max_{-h \leq s \leq t} \{\max\{|x(s)|, |y(s)|\}\}, \quad t \geq 0.$$

For any given $t_0 \geq 0$, we consider five cases.

Case (i): $V(t_0) > \max\{|x(t_0)|, |y(t_0)|\}$.

By the continuity of $x(t)$ and $y(t)$, there exists $\delta_1 > 0$ such that

$$\max\{|x(t)|, |y(t)|\} < V(t_0), \quad \forall t \in (t_0, t_0 + \delta_1).$$

Thus, one can conclude

$$V(t) = V(t_0), \quad \forall t \in (t_0, t_0 + \delta_1).$$

Case (ii): $V(t_0) = \max\{|x(t_0)|, |y(t_0)|\} < M$.

Also, by the continuity of $x(t)$ and $y(t)$, there exists $\delta_2 > 0$ such that

$$\max\{|x(t)|, |y(t)|\} < M, \quad \forall t \in (t_0, t_0 + \delta_2).$$

Therefore,

$$V(t) < M, \quad \forall t \in (t_0, t_0 + \delta_2).$$

Case (iii): $V(t_0) = \max\{|x(t_0)|, |y(t_0)|\} = |x(t_0)| \geq M$, and $|x(t_0)| > |y(t_0)|$.

Noticing that $x(t), y(t)$ is a solution to (2.5), it follows from (2.8) that

$$D^+(|x(s)|)|_{s=t_0} = \operatorname{sgn}(x(t_0)) \cdot \frac{-\phi(x(t_0)) + y(t_0))}{F(x(t_0))}$$

$$
\begin{aligned}
&< \frac{-|x(t_0)|+|y(t_0)|}{F(x(t_0))} \\
&< \frac{-|x(t_0)|+|x(t_0)|}{F(x(t_0))} = 0.
\end{aligned}
$$

Then, there exists $\delta_3' > 0$ such that

$$|x(t)| < |x(t_0)| = V(t_0). \quad \forall t \in (t_0, t_0 + \delta_3').$$

On the other hand, by the continuity of $y(t)$, there exists $\delta_3'' > 0$ such that

$$|y(t)| < |x(t_0)| = V(t_0), \quad \forall t \in (t_0, t_0 + \delta_3'').$$

Let $\delta_3 = \min\{\delta_3', \delta_3''\}$. Then

$$\max\{|x(t)|, |y(t)|\} < V(t_0), \quad \forall t \in (t_0, t_0 + \delta_3),$$

which means that

$$V(t) = V(t_0), \quad \forall t \in (t_0, t_0 + \delta_3).$$

Case (iv): $V(t_0) = \max\{|x(t_0)|, |y(t_0)|\} = |y(t_0)| \geq M$, and $|x(t_0)| < |y(t_0)|$.

In view of (2.3), (2.7) and $x(t), y(t)$ being a solution to (2.5), we have

$$
\begin{aligned}
& D^+(|y(s)|)|_{s=t_0} \\
= \; & F(x(t_0))\mathrm{sgn}(y(t_0))\big\{-\varepsilon y(t_0) - [g_1(x(t_0)) - \varepsilon\phi(x(t_0))] - g_2(x(t_0 - \tau(t_0))) + e(t_0)\big\} \\
\leq \; & F(x(t_0))\big\{-\varepsilon|y(t_0)| + \Phi(|x(t_0)|) + G(|x(t_0 - \tau(t_0))|) + \overline{e}\big\} \\
\leq \; & F(x(t_0))\big\{-\varepsilon V(t_0) + \Phi(V(t_0)) + G(V(t_0)) + \overline{e}\big\} < 0,
\end{aligned}
$$

which yields that there exists $\delta_4 > 0$ such that

$$|y(t)| < |y(t_0)| = V(t_0), \quad \forall t \in (t_0, t_0 + \delta_4).$$

On the other hand, without loss, by the continuity of $x(t)$, one can assume that

$$|x(t)| < |y(t_0)| = V(t_0), \quad \forall t \in (t_0, t_0 + \delta_4).$$

So one can conclude

$$\max\{|x(t)|, |y(t)|\} < V(t_0), \quad \forall t \in (t_0, t_0 + \delta_4).$$

Thus $V(t) = V(t_0)$ for all $t \in (t_0, t_0 + \delta_4)$.

Case (v): $V(t_0) = \max\{|x(t_0)|, |y(t_0)|\} = |x(t_0)| = |y(t_0)| \geq M$.

Similar to the proof of Case (iii) and Case (iv), one can show that

$$D^+(|x(s)|)|_{s=t_0} < 0, \quad D^+(|y(s)|)|_{s=t_0} < 0.$$

Then, there exists $\delta_5 > 0$ such that

$$|x(t)| < |x(t_0)| = V(t_0), \quad |y(t)| < |y(t_0)| = V(t_0) \quad \forall t \in (t_0, t_0 + \delta_5).$$

Therefore, $V(t) = V(t_0)$ for all $t \in (t_0, t_0 + \delta_5)$.

By the above proof, $\forall t_0 \geq 0$, there exists a constant $\delta > 0$ such that

$$V(t) \leq \max\{V(t_0), M\}, \quad \forall t \in (t_0, t_0 + \delta).$$

Now, we claim that

$$V(t) \leq \max\{V(0),\ M\}, \quad \forall t \geq 0. \tag{2.9}$$

In fact, if this is not true, then

$$\alpha := \inf\{t \geq 0 : V(t) > \max\{V(0), M\}\} < +\infty.$$

By the definition of α and the continuity of $V(t)$, we have

$$V(t) \leq \max\{V(0),\ M\},\ \forall t \in [0, \alpha]. \tag{2.10}$$

In addition, it follows from the above proof that there is a constants $\delta^{'} > 0$ such that

$$V(t) \leq \max\{V(\alpha),\ M\}, \quad \forall t \in (\alpha, \alpha + \delta^{'}). \tag{2.11}$$

Combing (2.10) and (2.11), we have

$$V(t) \leq \max\{V(0),\ M\}, \quad \forall t \in [0, \alpha + \delta^{'}),$$

which contradicts with the definition of α. Thus, (2.9) holds. Then, it follows that solutions of (2.5) are uniformly bounded. $\square$

Remark 2.3. Theorem 2.2 yields that all solutions to (1.2) with any given initial conditions are uniformly bounded, i.e., for any given initial conditions (ϕ, y_0), there is a constant $B > 0$ such that any solution $x(t)$ to (1.2) with initial conditions (ϕ, y_0) satisfies

$$|x(t)| \leq B, \quad t \in \mathbb{R}^+.$$

Remark 2.4. In the case of $f_1(x) \equiv 0$, the assumption (2.2) is equivalent to

$$\liminf_{u\to\infty} \frac{\operatorname{sgn}(u)\varphi(u)}{|u|} > \lambda\varepsilon,$$

where

$$\varphi(u) = \int_0^u [f_2(x) - 1]dx.$$

This means that (2.2) is weaker than (A0) and (A1) to some extent.

Next, we give two example to illustrate our results.

Example 2.5. Consider the following Liénard equation:

$$x''(t) + f(x(t))x'(t) + g(x(t)) = e(t), \tag{2.12}$$

where

$$f(x) = \frac{e^{-x} - xe^{-x} + 3}{2},\ g(x) = \frac{1}{6}xe^{-x} + \frac{1}{3}x,\ e(t) = \cos t.$$

Noticing that $F(x) \equiv 1$, one can easily verify that (2.1)–(2.4) hold with $\varepsilon = \frac{1}{3}$, $\lambda = 1$, $\Phi(x) = \frac{x}{18}$ and $G(x) \equiv 0$. Then, Theorem 2.2 yields that all solutions to (2.12) with any given initial conditions are uniformly bounded.

Remark 2.6. In the above example,

$$\varphi(x) = \int_0^x [f(u) - 1]du = \frac{1}{2}xe^{-x} + \frac{1}{2}x,$$

Obviously, neither (A0) nor (A1) hold. Thus, the results in [7, 8] can not be applied to the above example.

Example 2.7. Consider the following Liénard equation:

$$x''(t) + f_1(x(t))(x'(t))^2 + f_2(x(t))x'(t) + g_1(x(t)) + g_2(x(t - \tau(t))) = e(t), \tag{2.13}$$

where

$$f_1(x) = \frac{\cos x}{2 + \sin x},\ f_2(x) = 8e^{x^2},\ g_2(x) = \frac{1}{2}x,\ \tau(t) = 1 + \cos t,\ e(t) = \sin t,$$

and

$$g_1(x) = \int_0^x (1 + \frac{1}{2}\sin u)(8e^{u^2} - 1 - \frac{1}{2}\sin u)du.$$

In view of

$$F(x) = \exp\left(\int_0^x f_1(u)du\right) = 1 + \frac{1}{2}\sin x,$$

it is not difficult to verify that (2.1)–(2.4) hold with $\varepsilon = 1$, $\lambda = \frac{9}{4}$, $\Phi(x) \equiv 0$ and $G(x) = \frac{x}{2}$. Then, by Theorem 2.2, all solutions to (2.13) with any given initial conditions are uniformly bounded.

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