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The upper and lower solution method for nonlinear third-order three-point boundary value problem*

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Abstract

This paper is concerned with the following nonlinear third-order three-point boundary value problem

$$\begin{cases} u'''(t) + f\left(t, u\left(t\right), u'\left(t\right)\right) = 0, \ t \in [0,1], \\ u\left(0\right) = u'\left(0\right) = 0, \ u'\left(1\right) = \alpha u'\left(\eta\right), \end{cases}$$

where $0 < \eta < 1$ and $0 \le \alpha < 1$. A new maximum principle is established and some existence criteria are obtained for the above problem by using the upper and lower solution method.
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1 Introduction

Third-order differential equations arise in a variety of different areas of applied mathematics and physics, e.g., in the deflection of a curved beam having a constant or varying cross section, a three-layer beam, electromagnetic waves or gravity driven flows and so on [6].

Recently, third-order boundary value problems (BVPs for short) have received much attention. For instance, [3, 4, 5, 8, 9, 13] discussed some third-order two-point BVPs, while [2, 7, 10, 11, 12] studied some third-order three-point BVPs. In particular, Feng and Liu [5] employed the upper and lower solution method to prove the existence of solution for the third-order two-point BVP

$$\begin{cases} u'''(t) + f\left(t, u\left(t\right), u'\left(t\right)\right) = 0, \ t \in [0,1], \\ u\left(0\right) = u'\left(0\right) = u'\left(1\right) = 0. \end{cases}$$

In 2008, Guo, Sun and Zhao [7] established some existence results for at least one positive solution to the third-order three-point BVP

$$\begin{cases} u'''(t) + a\left(t\right) f\left(u\left(t\right)\right) = 0, \ t \in (0,1), \\ u\left(0\right) = u'\left(0\right) = 0, \ u'\left(1\right) = \alpha u'\left(\eta\right). \end{cases}$$

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Their main tool was the well-known Guo-Krasnoselskii fixed point theorem.

Motivated greatly by [5, 7], in this paper, we will investigate the following nonlinear third-order three-point BVP

$$\begin{cases} u'''(t) + f(t, u(t), u'(t)) = 0, \ t \in [0,1], \\ u(0) = u'(0) = 0, \ u'(1) = \alpha u'(\eta), \end{cases} \tag{1.1}$$

where $0 < \eta < 1$ and $0 \le \alpha < 1$. A new maximum principle is established and some existence criteria are obtained for the BVP (1.1) by using the upper and lower solution method. In order to obtain our main results, we need the following fixed point theorem [1].

Theorem 1.1 *Let (E, K) be an ordered Banach space and $[a, b]$ be a nonempty interval in E. If $T : [a, b] \to E$ is an increasing compact mapping and $a \le Ta$, $Tb \le b$, then T has a fixed point in $[a, b]$.*

2 Preliminaries

In this section, we will present some fundamental definitions and several important lemmas.

Definition 2.1 *If $x \in C^3[0,1]$ satisfies*

$$\begin{cases} x'''(t) + f(t, x(t), x'(t)) \ge 0, \ t \in [0,1], \\ x(0) = 0, \ x'(0) \le 0, \ x'(1) \le \alpha x'(\eta), \end{cases}$$

then x is called a lower solution of the BVP (1.1) .

Definition 2.2 *If $y \in C^3[0,1]$ satisfies*

$$\begin{cases} y'''(t) + f(t, y(t), y'(t)) \le 0, \ t \in [0,1], \\ y(0) = 0, \ y'(0) \ge 0, \ y'(1) \ge \alpha y'(\eta), \end{cases}$$

then y is called an upper solution of the BVP (1.1).

Let $G(t, s)$ be the Green′s function of the second-order three-point BVP

$$\begin{cases} -u''(t) = 0, \ t \in [0,1], \\ u(0) = 0, \ u(1) = \alpha u(\eta). \end{cases}$$

Then

$$G(t,s) = \frac{1}{1-\alpha\eta} \begin{cases} s(1-\alpha\eta) + st(\alpha - 1), & s \le \min\{\eta, t\}, \\ t(1-\alpha\eta) + st(\alpha-1), & t \le s \le \eta, \\ s(1-\alpha\eta) + t(\alpha\eta - s), & \eta \le s \le t, \\ t(1-s), & \max\{\eta, t\} \le s. \end{cases}$$

For $G(t, s)$, we have the following two lemmas.

Lemma 2.1 $G(t, s) \ge 0$ *for* $(t, s) \in [0,1] \times [0,1]$.

Lemma 2.2 *Let* $M =: \max_{t\in[0,1]} \int_0^1 G(t, s)\, ds$. *Then* $M < \dfrac{1}{2}$.

Proof. Since a simple computation shows that

$$\int_0^1 G(t,s)\,ds = -\frac{1}{2}t^2 + \frac{1-\alpha\eta^2}{2(1-\alpha\eta)}t,$$

it is easy to obtain that

$$M = \begin{cases} \dfrac{1}{8}\left(\dfrac{1-\alpha\eta^2}{1-\alpha\eta}\right)^2, & \alpha\eta(2-\eta) \le 1, \\ \dfrac{\alpha\eta(1-\eta)}{2(1-\alpha\eta)}, & \alpha\eta(2-\eta) \ge 1, \end{cases}$$

which implies that $M < \frac{1}{2}$. □

Lemma 2.3 *Assume that λ_1 and λ_2 are two nonnegative constants with $\lambda_1 + \lambda_2 \le 2$. If $m \in C^2[0,1]$ satisfies*

$$m''(t) \ge \lambda_1 \int_0^t m(s)\,ds + \lambda_2 m(t) \ \textit{ for } t \in [0,1]$$

and

$$m(0) \le 0,\ m(1) \le \alpha m(\eta),$$

then $m(t) \le 0$ for $t \in [0,1]$.

Proof. We consider two cases: $\lambda_1 = 0$ and $\lambda_1 \ne 0$.

Case 1. $\lambda_1 = 0$. If $\lambda_2 = 0$, then $m''(t) \ge 0$ for $t \in [0,1]$, which implies that the graph of $m(t)$ is concave up. Since $m(0) \le 0$, we only need to prove $m(1) \le 0$. Suppose on the contrary that $m(1) > 0$. Then $m(0) \le 0 < m(1) \le \alpha m(\eta) < m(\eta)$. So,

$$0 < \frac{m(\eta) - m(0)}{\eta} \le \frac{m(1) - m(\eta)}{1-\eta} < 0,$$

which is a contradiction. Therefore, $m(1) \le 0$.

If $\lambda_2 \ne 0$, then $m''(t) \ge \lambda_2 m(t)$ for $t \in [0,1]$. Suppose on the contrary that there exists $t_0 \in [0,1]$ such that $m(t_0) = \max_{t\in[0,1]} m(t) > 0$. Obviously, $t_0 \ne 0$. If $t_0 = 1$, then $0 < m(1) \le \alpha m(\eta) < m(\eta) \le m(1)$, which is a contradiction. Consequently, $t_0 \in (0,1)$, $m'(t_0) = 0$ and $m''(t_0) \le 0$, which contradicts $m''(t_0) \ge \lambda_2 m(t_0) > 0$.

Case 2. $\lambda_1 \ne 0$. Suppose on the contrary that there exists $t_0 \in [0,1]$ such that $m_0 = m(t_0) = \max_{t\in[0,1]} m(t) > 0$. Similarly, we can obtain that $t_0 \in (0,1)$, $m'(t_0) = 0$ and $m''(t_0) \le 0$. Consequently, $0 \ge m''(t_0) \ge \lambda_1 \int_0^{t_0} m(s)\,ds + \lambda_2 m(t_0)$, which implies that $\int_0^{t_0} m(s)\,ds \le 0$. So, there exists $t_1 \in [0,t_0)$ such that $m_1 = m(t_1) = \min_{t\in[0,t_0]} m(t) < 0$. It follows from Taylor's formula that there exists $\xi \in (t_1, t_0)$ such that

$$m_1 = m(t_1) = m(t_0) + m'(t_0)(t_1 - t_0) + \frac{m''(\xi)}{2}(t_1 - t_0)^2.$$

Noting that $m_1 < 0$, we obtain

$$m''(\xi) = \frac{2(m_1 - m_0)}{(t_1 - t_0)^2} < \frac{2m_1}{(t_1 - t_0)^2} < 2m_1.$$

And so,

$$2m_1 > m''(\xi) \geq \lambda_1 \int_0^{\xi} m(s)\,ds + \lambda_2 m(\xi) \geq \lambda_1 \xi m_1 + \lambda_2 m_1,$$

which implies that $\lambda_1 + \lambda_2 > 2$. This contradicts the fact that $\lambda_1 + \lambda_2 \leq 2$. $\square$

3 Main results

In the remainder of this paper, we always assume that the following condition is satisfied:

(H) $f : [0,1] \times R^2 \to R$ is continuous and there exist two nonnegative constants λ_1 and λ_2 with $\lambda_1 + \lambda_2 \leq 2$ such that

$$f(t, u_1, v_1) - f(t, u_2, v_2) \geq -\lambda_1 (u_1 - u_2) - \lambda_2 (v_1 - v_2)$$

for $t \in [0,1]$, $u_1 \geq u_2$ and $v_1 \geq v_2$.

Theorem 3.1 *If the BVP (1.1) has a lower solution x and an upper solution y with $x'(t) \leq y'(t)$ for $t \in [0,1]$, then the BVP (1.1) has a solution $u \in C^3[0,1]$, which satisfies*

$$x'(t) \leq u'(t) \leq y'(t) \;\; \textit{for } t \in [0,1].$$

Proof. Let $v(t) = u'(t)$. Then the BVP (1.1) is equivalent to the following BVP

$$\begin{cases} v''(t) + f\left(t, \int_0^t v(s)\,ds, v(t)\right) = 0, \;\; t \in [0,1], \\ v(0) = 0, \;\; v(1) = \alpha v(\eta). \end{cases} \tag{3.1}$$

Let $E = C[0,1]$ be equipped with the norm $\|v\|_\infty = \max_{t\in[0,1]} |v(t)|$ and

$$K = \{v \in E : v(t) \geq 0 \text{ for } t \in [0,1]\}.$$

Then K is a cone in E and (E, K) is an ordered Banach space. Now, if we define operators $L : D \subset E \to E$ and $N : E \to E$ as follows:

$$Lv = -v''(t) + \lambda_1 \int_0^t v(s)\,ds + \lambda_2 v(t)$$

and

$$Nv = f\left(t, \int_0^t v(s)\,ds, v(t)\right) + \lambda_1 \int_0^t v(s)\,ds + \lambda_2 v(t),$$

where $D = \{v \in E \; : \; v'' \in E, \; v(0) = 0 \text{ and } v(1) = \alpha v(\eta)\}$, then it is easy to see that the BVP (3.1) is equivalent to the operator equation

$$Lv = Nv. \tag{3.2}$$

Now, we shall show that the operator equation (3.2) is solvable. The proof will be given in several steps.

Step 1. $L : D \subset E \to E$ is invertible.

Suppose $h \in E$. We will find unique $v \in D$ such that $Lv = h$. Since $Lv = h$ is equivalent to the integral equation

$$v(t) = \int_0^1 G(t,s)\left[h(s) - \left(\lambda_1 \int_0^s v(r)\,dr + \lambda_2 v(s)\right)\right] ds,\ t \in [0,1],$$

we define a mapping $A : E \to E$ by

$$(Av)(t) = \int_0^1 G(t,s)\left[h(s) - \left(\lambda_1 \int_0^s v(r)\,dr + \lambda_2 v(s)\right)\right] ds,\ t \in [0,1].$$

Noting that $M < \frac{1}{2}$ and $0 \leq \lambda_1 + \lambda_2 \leq 2$, it is easy to verify that $A : E \to E$ is a contraction mapping. And so, there exists unique $v \in D$ such that $Av = v$, which implies that $Lv = h$. This shows that L is invertible.

Step 2. $L^{-1} : E \to E$ is continuous.

Assume that $\{h_n\}_{n=1}^{\infty} \subset E$, $h \in E$ and $\lim_{n\to\infty} h_n = h$. Denote $L^{-1}h_n = v_n$ and $L^{-1}h = v$. Then

$$v_n(t) = \int_0^1 G(t,s)\left[h_n(s) - \left(\lambda_1 \int_0^s v_n(r)\,dr + \lambda_2 v_n(s)\right)\right] ds,\ t \in [0,1]$$

and

$$v(t) = \int_0^1 G(t,s)\left[h(s) - \left(\lambda_1 \int_0^s v(r)\,dr + \lambda_2 v(s)\right)\right] ds,\ t \in [0,1].$$

So,

$$\begin{aligned}
\|v_n - v\| &= \max_{t\in[0,1]} \left| \int_0^1 G(t,s)\left[(h_n(s) - h(s)) - \lambda_1 \int_0^s (v_n(r) - v(r))\,dr - \lambda_2 (v_n(s) - v(s))\right] ds \right| \\
&\leq \max_{t\in[0,1]} \int_0^1 G(t,s)\left[\|h_n - h\| + (\lambda_1 + \lambda_2)\|v_n - v\|\right] ds \\
&\leq M\|h_n - h\| + 2M\|v_n - v\|.
\end{aligned}$$

This shows that $\|v_n - v\| \leq \dfrac{M}{1-2M}\|h_n - h\|$, which together with $\lim_{n\to\infty} h_n = h$ implies that $\lim_{n\to\infty} v_n = v$. This indicates that $L^{-1} : E \to E$ is continuous.

Step 3. $L^{-1}N : E \to E$ is completely continuous.

Since f and L^{-1} are continuous, we only need to prove that $L^{-1} : E \to E$ is compact. Let X be a bounded subset in E. Then there exists a constant $C > 0$ such that $\|h\| \leq C$ for any $h \in X$. For any $v \in L^{-1}(X)$, there exists an $h \in X$ such that $v = L^{-1}h$. So,

$$v(t) = \int_0^1 G(t,s)\left[h(s) - \left(\lambda_1 \int_0^s v(r)\,dr + \lambda_2 v(s)\right)\right] ds,\ t \in [0,1].$$

On the one hand, for any $v \in L^{-1}(X)$, we have

$$\begin{aligned}
\|v\| &= \max_{t\in[0,1]} \left| \int_0^1 G(t,s)\left[h(s) - \left(\lambda_1 \int_0^s v(r)\,dr + \lambda_2 v(s)\right)\right] ds \right| \\
&\leq \max_{t\in[0,1]} \int_0^1 G(t,s)\left[\|h\| + (\lambda_1 + \lambda_2)\|v\|\right] ds \\
&\leq M\|h\| + 2M\|v\|,
\end{aligned}$$

which implies that $\|v\| \leq \frac{M}{1-2M}\|h\| \leq \frac{MC}{1-2M}$. This shows that $L^{-1}(X)$ is uniformly bounded.

On the other hand, in view of the uniform continuity of $G(t,s)$, we know that for any $\epsilon > 0$, there exists a $\delta > 0$ such that for any $t_1, t_2 \in [0,1]$ and $|t_1 - t_2| < \delta$, $|G(t_1,s) - G(t_2,s)| < \frac{1-2M}{C}\epsilon$ for any $s \in [0,1]$. Then for any $v \in L^{-1}(X)$, $t_1, t_2 \in [0,1]$ and $|t_1 - t_2| < \delta$, we have

$$\begin{aligned} |v(t_1) - v(t_2)| &= \left| \int_0^1 (G(t_1,s) - G(t_2,s)) \left[h(s) - \left(\lambda_1 \int_0^s v(r)\,dr + \lambda_2 v(s) \right) \right] ds \right| \\ &\leq \int_0^1 |G(t_1,s) - G(t_2,s)| \left[\|h\| + (\lambda_1 + \lambda_2)\|v\| \right] ds \\ &\leq \frac{C}{1-2M} \int_0^1 |G(t_1,s) - G(t_2,s)|\,ds \\ &< \epsilon, \end{aligned}$$

which shows that $L^{-1}(X)$ is equicontinuous.

By the Arzela-Ascoli theorem, we know that $L^{-1}(X)$ is relatively compact, which implies that $L^{-1} : E \to E$ is a compact mapping.

Step 4. $L^{-1}N : E \to E$ is increasing.

Suppose $h_1,\ h_2 \in E$ and $h_1 \leq h_2$. Then the condition (H) implies that $Nh_1 \leq Nh_2$. Denote $v_1 = L^{-1}Nh_1$ and $v_2 = L^{-1}Nh_2$. Then $Lv_1 = Nh_1 \leq Nh_2 = Lv_2$. It follows from Lemma 2.3 that $v_1 \leq v_2$. Therefore, $L^{-1}N : E \to E$ is increasing.

Step 5. Let $\beta_0 = x'$ and $\gamma_0 = y'$. Then $\beta_0 \leq L^{-1}N\beta_0$ and $L^{-1}N\gamma_0 \leq \gamma_0$.

Since x is a lower solution of the BVP (1.1), we have

$$-\beta_0''(t) + \lambda_1 \int_0^t \beta_0(s)\,ds + \lambda_2\beta_0(t) \leq (N\beta_0)(t),\ t \in [0,1],\ \beta_0(0) \leq 0,\ \beta_0(1) \leq \alpha\beta_0(\eta). \tag{3.3}$$

Let $\beta^* = L^{-1}N\beta_0$. Then $L\beta^* = N\beta_0$, that is,

$$-(\beta^*)''(t) + \lambda_1 \int_0^t \beta^*(s)\,ds + \lambda_2\beta^*(t) = (N\beta_0)(t),\ t \in [0,1],\ \beta^*(0) = 0,\ \beta^*(1) = \alpha\beta^*(\eta). \tag{3.4}$$

Denote $q(t) = \beta_0(t) - \beta^*(t)$. In view of (3.3) and (3.4), we know that

$$-q''(t) + \lambda_1 \int_0^t q(s)\,ds + \lambda_2 q(t) \leq 0,\ t \in [0,1],\ q(0) \leq 0,\ q(1) \leq \alpha q(\eta).$$

By Lemma 2.3, we get $q(t) \leq 0$ for $t \in [0,1]$, i.e., $\beta_0 \leq \beta^* = L^{-1}N\beta_0$. Similarly, we can obtain that $L^{-1}N\gamma_0 \leq \gamma_0$.

It follows from Theorem 1.1 that $L^{-1}N : E \to E$ has a fixed point $v \in [\beta_0, \gamma_0]$, which solves the BVP (3.1). Therefore, $u(t) = \int_0^t v(s)\,ds, t \in [0,1]$ is a solution of the BVP (1.1) and $x'(t) \leq u'(t) \leq y'(t)$ for $t \in [0,1]$. □

Corollary 3.2 *(1) If* $\min_{t\in[0,1]} f(t,0,0) \geq 0$ *and there exists* $c > 0$ *such that*

$$\max\left\{ f(t,u,v) : (t,u,v) \in [0,1] \times [0,c] \times \left[0, \frac{3c}{2}\right] \right\} \leq 3c,$$

then the BVP (1.1) has a nonnegative solution u with $\|u\| \le c$. Moreover, if there exists $t_n \in (0,1]$ $(n = 1, 2, \cdots)$ satisfying $\lim_{n\to\infty} t_n = 0$ such that $f(t_n, 0, 0) > 0$ $(n = 1, 2, \cdots)$, then $u(t) > 0$ for $t \in (0,1]$.

(2) If $\max\limits_{t\in[0,1]} f(t, 0, 0) \le 0$ and there exists $c > 0$ such that

$$\min\left\{f(t,u,v) : (t,u,v) \in [0,1] \times [-c,0] \times \left[-\frac{3c}{2}, 0\right]\right\} \ge -3c,$$

then the BVP (1.1) has a nonpositive solution u with $\|u\| \le c$. Moreover, if there exists $t_n \in (0,1]$ $(n = 1, 2, \cdots)$ satisfying $\lim_{n\to\infty} t_n = 0$ such that $f(t_n, 0, 0) < 0$ $(n = 1, 2, \cdots)$, then $u(t) < 0$ for $t \in (0,1]$.

(3) If there exists $c > 0$ such that

$$\max\left\{|f(t,u,v)| : (t,u,v) \in [0,1] \times [-c,c] \times \left[-\frac{3c}{2}, \frac{3c}{2}\right]\right\} \le 3c,$$

then the BVP (1.1) has a solution u with $\|u\| \le c$. Moreover, if $f(t,0,0)$ is not identically zero on $[0,1]$, then u is nontrivial.

Proof. Since the proof of (2) and (3) is similar, we only prove (1). Let $x(t) \equiv 0$ and $y(t) = 3c\left(\frac{t^2}{2} - \frac{t^3}{6}\right)$ for $t \in [0,1]$. Then it is easy to verify that x and y are lower and upper solutions of the BVP (1.1), respectively. By Theorem 3.1, we know that the BVP (1.1) has a solution u satisfying $x'(t) \le u'(t) \le y'(t)$ for $t \in [0,1]$, which together with $x(0) = u(0) = y(0) = 0$ implies that u is nonnegative and $\|u\| \le c$.

If there exists $t_n \in (0,1]$ $(n = 1, 2, \cdots)$ satisfying $\lim_{n\to\infty} t_n = 0$ such that $f(t_n, 0, 0) > 0$ $(n = 1, 2, \cdots)$, then it is not difficult to prove that for any $\epsilon \in (0,1)$, $u(t)$ is not identically zero on $[0, \epsilon]$. In view of $u(0) = 0$ and $u'(t) \ge 0$ for $t \in [0,1]$, we know that $u(t) > 0$ for $t \in (0,1]$. □

4 An example

Consider the following BVP:

$$\begin{cases} u'''(t) + te^{u(t)} + \frac{8}{135}(u'(t))^3 = 0, \ t \in [0,1], \\ u(0) = u'(0) = 0, \ u'(1) = \frac{1}{4}u'\left(\frac{1}{2}\right). \end{cases} \tag{4.1}$$

Since $f(t,u,v) = te^u + \frac{8}{135}v^3$, it is easy to verify that the condition (H) is fulfilled with $\lambda_1 = \lambda_2 = 1$. If we let $c = 1$, then all the conditions of Corollary 3.2 (1) are satisfied. It follows from Corollary 3.2 (1) that the BVP (4.1) has a positive solution u and $\|u\| \le 1$.

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