

# Non-Simultaneous Blow-Up for a Reaction-Diffusion System with Absorption and Coupled Boundary Flux\*

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**Abstract.** This paper deals with non-simultaneous blow-up for a reaction-diffusion system with absorption and nonlinear boundary flux. We establish necessary and sufficient conditions for the occurrence of non-simultaneous blow-up with proper initial data.

**Keywords.** non-simultaneous blow-up; reaction-diffusion system; nonlinear absorption; nonlinear boundary flux; blow-up rate

**Mathematics Subject Classification (2000).** 35K55; 35B33

## 1 Introduction

In this paper, we study non-simultaneous blow-up for the following reaction-diffusion system

$$\begin{aligned} u_t &= u_{xx} - a_1 e^{\alpha_1 u}, & v_t &= v_{xx} - a_2 e^{\beta_1 v}, & (x, t) &\in (0, 1) \times (0, T), \\ u_x(1, t) &= e^{\alpha_2 u(1, t) + p v(1, t)}, & v_x(1, t) &= e^{q u(1, t) + \beta_2 v(1, t)}, & t &\in (0, T), \\ u_x(0, t) &= 0, & v_x(0, t) &= 0, & t &\in (0, T), \\ u(x, 0) &= u_0(x), & v(x, 0) &= v_0(x), & x &\in [0, 1], \end{aligned} \tag{1.1}$$

where  $p, q, a_i > 0$ ,  $\alpha_i, \beta_i \geq 0$ ,  $i = 1, 2$ . The initial data satisfy  $u_0, v_0 \geq 0$ ,  $u'_0, v'_0 \geq 0$ ,  $u''_0 - a_1 e^{\alpha_1 u_0}, v''_0 - a_2 e^{\beta_1 v_0} \geq \delta > 0$ , as well as the compatibility conditions on  $[0, 1]$ . By comparison principle, it follows that  $u_t, v_t > 0$ ,  $u_x, v_x \geq 0$  and  $u, v \geq 0$  for  $(x, t) \in [0, 1] \times [0, T)$ .

The reaction-diffusion system (1.1) can be used to describe heat propagations in mixed solid media with nonlinear absorption and nonlinear boundary flux [1-3, 5, 9, 11, 16]. The

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nonlinear Neumann boundary values in (1.1), coupling the two heat equations, represent some cross-boundary flux.

The problem of heat equations

$$u_t = \Delta u, \quad v_t = \Delta v, \quad (x, t) \in \Omega \times (0, t), \quad (1.2)$$

coupled via somewhat special nonlinear Neumann boundary conditions

$$\frac{\partial u}{\partial \nu} = v^p, \quad \frac{\partial v}{\partial \nu} = u^q, \quad (x, t) \in \partial\Omega \times (0, T), \quad (1.3)$$

was studied by Deng [7] and Lin and Xie [11], who showed that the solutions globally exist if  $pq \leq 1$  and may blow up in finite time if  $pq > 1$  with the blow-up rates  $O((T-t)^{-(p+1)/2(pq-1)})$  and  $O((T-t)^{-(q+1)/2(pq-1)})$ . Similarly, the blow-up rates for the corresponding scalar case of (1.2) and (1.3) was shown to be  $O((T-t)^{-1/2(p-1)})$  in [10].

The system (coupled via a variational boundary flux of exponential type)

$$\begin{aligned} u_t = \Delta u, \quad v_t = \Delta v, & \quad (x, t) \in \Omega \times (0, t), \\ \frac{\partial u}{\partial \nu} = e^{pv}, \quad \frac{\partial v}{\partial \nu} = e^{qu}, & \quad (x, t) \in \partial\Omega \times (0, t), \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), & \quad x \in \Omega, \end{aligned} \quad (1.4)$$

was studied by Deng [7], and has blow-up rates

$$\begin{aligned} -\frac{1}{2q} \log c(T-t) \leq \max_{\bar{\Omega}} u(\cdot, t) \leq -\frac{1}{2q} \log C(T-t), \\ -\frac{1}{2p} \log c(T-t) \leq \max_{\bar{\Omega}} v(\cdot, t) \leq -\frac{1}{2p} \log C(T-t), \end{aligned} \quad (1.5)$$

for  $t \in (0, T)$ . This is the special case with  $\alpha_i = \beta_i = a_i = 0$ ,  $i = 1, 2$ , in our system (1.1).

Zhao and Zheng [17] studied the following nonlinear parabolic system:

$$\begin{aligned} u_t = \Delta u, \quad v_t = \Delta v, & \quad (x, t) \in \Omega \times (0, t), \\ \frac{\partial u}{\partial \nu} = e^{\alpha_2 u + pv}, \quad \frac{\partial v}{\partial \nu} = e^{qu + \beta_2 v}, & \quad (x, t) \in \partial\Omega \times (0, t), \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), & \quad x \in \Omega. \end{aligned} \quad (1.6)$$

The blow-up rates for (1.6) were shown to be

$$\max_{\bar{\Omega}} u(\cdot, t) = O(\log(T-t)^{\alpha/2}), \quad \max_{\bar{\Omega}} v(\cdot, t) = O(\log(T-t)^{\beta/2}), \quad (1.7)$$

as  $t \rightarrow T$ , where  $(\alpha, \beta)^T$  is the only positive solution of

$$\begin{pmatrix} \alpha_2 & p \\ q & \beta_2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

namely,

$$\alpha = \frac{p - \beta_2}{pq - \alpha_2 \beta_2}, \quad \beta = \frac{q - \alpha_2}{pq - \alpha_2 \beta_2}.$$

Clearly the blow-up rate estimate (1.5) is just the special case of (1.7) with  $\alpha_2 = \beta_2 = 0$ .

The phenomenon of non-simultaneous blow-up is researched extensively [see 4, 13-15]. Recently Zheng and Qiao [20] consider the non-simultaneous blow-up phenomenon of following reaction-diffusion problem

$$\begin{aligned} u_t &= u_{xx} - \lambda_1 u^{\alpha_1}, & v_t &= v_{xx} - \lambda_2 v^{\beta_1}, & (x, t) &\in (0, 1) \times (0, T), \\ u_x(1, t) &= u^{\alpha_2} v^p, & v_x(1, t) &= u^q v^{\beta_2}, & t &\in (0, T), \\ u_x(0, t) &= 0, & v_x(0, t) &= 0, & t &\in (0, T), \\ u(x, 0) &= u_0(x), & v(x, 0) &= v_0(x), & x &\in [0, 1], \end{aligned} \tag{1.8}$$

and they get the following conclusions:

(1) If  $q < \alpha_2 - 1$  with either  $\alpha_2 > \mu$  or  $\alpha_2 = \mu > 1$ , then there exists initial data  $(u_0, v_0)$  such that  $u$  blows up at a finite time  $T$  while  $v$  remains bounded.

(2) If  $u$  blows up at time  $T$  and  $v$  remains bounded up to that time, then  $q < \alpha_2 - 1$  with either  $\alpha_2 > \mu$  or  $\alpha_2 = \mu > 1$ .

(3) Under the condition of (1), if in addition either (i)  $\beta_2 \leq 1$ , or (ii)  $1 < \beta_2 < \gamma$  and  $q < \frac{(\alpha_2-1)(\gamma-\beta_2)}{\gamma-1}$  hold, then any blow-up must be non-simultaneous, namely,  $u$  blows up at a finite time  $T$  while  $v$  remains bounded.

The critical exponents for the system (1.1) were studied in [18] by Zheng and Li, where the following characteristic algebraic system was introduced:

$$\begin{pmatrix} \alpha_2 - \frac{1}{2}\alpha_1 & p \\ q & \beta_2 - \frac{1}{2}\beta_1 \end{pmatrix} \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{1.9}$$

namely,

$$\tau_1 = \frac{p + \frac{1}{2}\beta_1 - \beta_2}{pq - (\frac{1}{2}\alpha_1 - \alpha_2)(\frac{1}{2}\beta_1 - \beta_2)}, \quad \tau_2 = \frac{q + \frac{1}{2}\alpha_1 - \alpha_2}{pq - (\frac{1}{2}\alpha_1 - \alpha_2)(\frac{1}{2}\beta_1 - \beta_2)}. \tag{1.10}$$

To state their main result, first we give some information about eigenfunction for Laplace's equation.

Let  $\varphi_0$  be the first eigenfunction of

$$\varphi'' + \lambda\varphi = 0 \quad \text{in } (-1, 1); \quad \varphi(-1) = \varphi(1) = 0, \tag{1.11}$$

with the first eigenvalue  $\lambda_0$ , normalized by  $\|\varphi_0\|_\infty = 1$ . It is well know that [6]  $\varphi_0 > 0$  in  $(-1, 1)$ , and there are positive constants  $c_i$  ( $i=1,2,3,4$ ) and  $\varepsilon_0$  such that

$$\begin{aligned} c_1 &\leq \varphi_0'(-1), \quad -\varphi_0'(1) \leq c_2 \leq \max_{[-1,1]} |\varphi_0'| = c_4, \\ |\varphi_0'| &\geq \frac{c_1}{2} \text{ on } \{x \in (-1, 1) : \text{dist}(x, -1) \leq \varepsilon_0\} \cup \{x \in (-1, 1) : \text{dist}(x, 1) \leq \varepsilon_0\}, \\ \varphi_0 &\geq c_3 \text{ on } \{x \in (-1, 1) : \text{dist}(x, -1) \geq \varepsilon_0\} \cap \{x \in (-1, 1) : \text{dist}(x, 1) \geq \varepsilon_0\}. \end{aligned} \tag{1.12}$$

Now we can state the main result of [18]

**Proposition 1.1** (1) If  $1/\tau_1 > 0$  or  $1/\tau_2 > 0$ , then the solutions of (1.1) blow up in finite time with large initial data.

(2) If  $1/\tau_i < 0$ ,  $i=1,2$ , then the solutions of (1.1) are globally bounded.

(3) Assume that  $1/\tau_1 = 1/\tau_2 = 0$ .

(i) If  $\alpha_2 > \frac{1}{2}\alpha_1$  and  $\beta_2 > \frac{1}{2}\beta_1$ , then the solutions of (1.1) blow up in finite time with large initial data.

(ii) If  $a_1 \geq 2^{\alpha_1} \left( \frac{\lambda_0}{c_1} + \frac{3c_4^2}{c_1^2} \right)$ ,  $a_2 \geq 2^{\beta_1} \left( \frac{\lambda_0}{c_1} + \frac{3c_4^2}{c_1^2} \right)$  with  $\alpha_2 < \frac{1}{2}\alpha_1$ ,  $\beta_2 < \frac{1}{2}\beta_1$ , then the solutions of (1.1) are globally bounded.

(iii) If  $a_1 \leq \min \left\{ \frac{c_1^2 M^2}{4\alpha_1}, \frac{\lambda_0 c_3^2 M^2}{\alpha_1} \right\}$ ,  $a_2 \leq \min \left\{ \frac{c_1^2 M^2}{4\beta_1}, \frac{\lambda_0 c_3^2 M^2}{\beta_1} \right\}$  with  $\alpha_2 < \frac{1}{2}\alpha_1$ ,  $\beta_2 < \frac{1}{2}\beta_1$ ,  $M = \min\{\alpha_1/(2c_2), \beta_1/(2c_2)\}$ , then the solutions of (1.1) blow up in finite time for large initial data.

Intrigued by [18-20], we consider the non-simultaneous blow-up of (1.1). The main results of this paper are the following two Theorems for non-simultaneous blow-up. Without loss of generality, we only deal with the case where  $u$  blows up while  $v$  remains bounded.

**Theorem 1.1** If  $q < \alpha_2$  with either  $2\alpha_2 > \alpha_1$  or  $2\alpha_2 = \alpha_1$ ,  $a_1 \leq \frac{\alpha_1}{4} \min\{c_1^2/(4c_2^2), \lambda_0 c_3^2/c_2^2\}$ , then there exists initial data  $(u_0, v_0)$  such that  $u$  blows up at a finite time  $T$  while  $v$  remains bounded up to that time.

**Theorem 1.2** If  $u$  blows up at a finite time  $T$  while  $v$  remains bounded up to that time, then  $q < \alpha_2$  with either  $2\alpha_2 > \alpha_1$  or  $2\alpha_2 = \alpha_1$ ,  $a_1 \leq \frac{\alpha_1}{4} \min\{c_1^2/(4c_2^2), \lambda_0 c_3^2/c_2^2\}$ .

We will prove Theorem 1.1 and 1.2 in the next two sections.

## 2 Proof of Theorem 1.1

At first, we consider the scalar problem of the form

$$\begin{aligned} u_t &= u_{xx} - a_1 e^{\alpha_1 u}, & (x, t) &\in (0, 1) \times (0, T), \\ u_x(1, t) &= e^{\alpha_2 u(1, t)} e^{ph(t)}, \quad u_x(0, t) = 0, & t &\in (0, T), \\ u(x, 0) &= u_0(x), & x &\in [0, 1], \end{aligned} \tag{2.1}$$

with  $a_1, \alpha_i$  in (1.1),  $i=1,2$  and  $h(t)$  continuous, non-decreasing,  $0 \leq h(t) \leq K$ . Similarly to Theorem 3.2 in [19], we can prove the following Lemma, where and in the sequel  $C$  is used to represent positive constants independent of  $t$ , and may change from line to line.

**Lemma 2.1** Let  $u$  be a solution of (2.1). Assume (i)  $2\alpha_2 > \alpha_1$  or (ii)  $2\alpha_2 = \alpha_1$  with  $a_1 \leq \frac{\alpha_1}{4} \min\{c_1^2/(4c_2^2), \lambda_0 c_3^2/c_2^2\}$ . Then  $u$  blows up in a finite time  $T$  for sufficiently large

initial value, and moreover

$$u(1, t) = \max_{[0,1]} u(\cdot, t) \leq \log C(T - t)^{-\frac{1}{2\alpha_2}}, \quad 0 < t < T. \quad (2.2)$$

**Proof.** Let  $w$  solve

$$\begin{aligned} w_t &= w_{xx} - a_1 e^{\alpha_1 w}, & (x, t) &\in (0, 1) \times (0, T), \\ w_x(1, t) &= e^{\alpha_2 w(1, t)}, \quad w_x(0, t) = 0, & t &\in (0, T), \\ w(x, 0) &= u_0(x), & x &\in [0, 1]. \end{aligned} \quad (2.3)$$

Then,  $w \leq u$  in  $(0, 1) \times [0, T)$  by the comparison principle. Notice that the two assumptions  $2\alpha_2 > \alpha_1$  or  $2\alpha_2 = \alpha_1$  with  $a_1 \leq \frac{\alpha_1}{4} \min\{c_1^2/(4c_2^2), \lambda_0 c_3^2/c_2^2\}$  are corresponding to the blow-up conditions by Proposition 1.1 with  $\alpha_1 = \beta_1$ ,  $\alpha_2 = \beta_2$ ,  $p = q = 0$  and  $u_0 = v_0$ ,  $a_1 = a_2$  in (1.1). So there exists initial data such that  $w$  blows up in finite time  $T'$ . Then  $u$  blows up in finite time  $t = T$ .

To establish the desired blow-up rate, we exploit the method used in [19]. From the assumptions on initial data, we know that  $w_t > 0$  and  $w_x \geq 0$  for  $(x, t) \in [0, 1) \times [0, T)$ . Set  $J(x, t) = \sqrt{w_t} - \varepsilon w_x$  for  $(x, t) \in (0, 1) \times [0, T)$ . Let  $\varepsilon$  be sufficiently small such that

$$J(x, 0) = \sqrt{w_t(x, 0)} - \varepsilon w_x(x, 0) \geq 0, \quad x \in [0, 1], \quad (2.4)$$

a simple computation yields

$$\begin{aligned} J_x(1, t) &- \left[ \left( \frac{1}{2} \alpha_2 e^{\alpha_2 w} - \varepsilon w_t^{\frac{1}{2}} - \varepsilon^2 e^{\alpha_2 w} \right) J \right] (1, t) \\ &= \varepsilon \left( \frac{1}{2} \alpha_2 e^{2\alpha_2 w} - a_1 e^{\alpha_1 w} - \varepsilon^2 e^{2\alpha_2 w} \right) (1, t) \geq 0, \quad t \in (0, T), \end{aligned} \quad (2.5)$$

when (i)  $2\alpha_2 > \alpha_1$  or (ii)  $2\alpha_2 = \alpha_1$  with  $a_1 \leq \frac{\alpha_1}{4} \min\{c_1^2/(4c_2^2), \lambda_0 c_3^2/c_2^2\}$ .

For  $(x, t) \in (0, 1) \times [0, T)$ , a simple computation shows

$$J_t - J_{xx} + \frac{1}{2} a_1 \alpha_1 e^{\alpha_1 w} J = \frac{1}{4} w_t^{-\frac{3}{2}} w_{tx}^2 + \frac{1}{2} \varepsilon a_1 \alpha_1 e^{\alpha_1 w} w_x \geq 0. \quad (2.6)$$

By the comparison principle [12], we have  $J \geq 0$  and hence

$$w_t(1, t) \geq \varepsilon^2 w_x^2(1, t) = \varepsilon^2 e^{2\alpha_2 w(1, t)}, \quad t \in [0, T). \quad (2.7)$$

Integrating (2.7) from  $t$  to  $T$ , we get (2.2) immediately.  $\square$

**Proof of Theorem 1.1.** It suffices to choose initial data  $(u_0, v_0)$  such that  $u$  blows up while  $v$  remains bounded. At first, fix  $v_0 \geq 0$  and take  $K = \max_{[0,1]} v_0 = v_0(1)$ ,  $N = \frac{1}{K} e^{2\beta_2 K} + 3$ . Thus,  $w(x, t)$  which solves (2.3) is a subsolution of  $u$ . Since Proposition 1.1, there exists initial data  $u_0$  such that  $w$  blows up at a finite time  $T'$ . Now, for the fixed  $v_0$ , retake  $u_0(x) = w(x, T' - \varepsilon)$ , the  $u$  blows up in a finite time  $T \leq \varepsilon$ .

If  $v$  remain bounded by  $v < 2K$  for  $t \in [0, T]$ , the proof is complete.

Otherwise, Let  $t_0$  be the first time such that  $\max_{[0,1]} v(\cdot, t_0) = v(1, t_0) = 2K$ . Now, we introduce the following cut-off function:

$$\tilde{v}(x, t) = \begin{cases} v(x, t), & (x, t) \in [0, 1] \times [0, t_0], \\ 2K, & (x, t) \in [0, 1] \times [t_0, T]. \end{cases} \quad (2.8)$$

Corresponding, let  $\tilde{u}(x, t)$  solve

$$\begin{aligned} \tilde{u}_t &= \tilde{u}_{xx} - a_1 e^{\alpha_1 \tilde{u}}, & (x, t) \in (0, 1) \times (0, \tilde{T}), \\ \tilde{u}_x(1, t) &= e^{\alpha_2 \tilde{u}(1, t)} e^{p\tilde{v}(1, t)}, \quad \tilde{u}_x(0, t) = 0, & t \in (0, \tilde{T}), \\ \tilde{u}(x, 0) &= u_0(x), & x \in [0, 1], \end{aligned} \quad (2.9)$$

where  $\tilde{T}$  is the blow-up time of  $\tilde{u}$  satisfying  $\tilde{T} \geq T$ . By Lemma 2.1,

$$\tilde{u}(1, t) = \max_{[0,1]} \tilde{u}(\cdot, t) \leq \log C(\tilde{T} - t)^{-\frac{1}{2\alpha_2}}, \quad 0 < t < \tilde{T}. \quad (2.10)$$

Therefore,

$$u(1, t) = \tilde{u}(1, t) \leq \log C(\tilde{T} - t)^{-\frac{1}{2\alpha_2}} \leq \log C(T - t)^{-\frac{1}{2\alpha_2}}, \quad 0 < t \leq t_0. \quad (2.11)$$

Let  $\Gamma(x, t)$  be the fundamental solution of the heat equation in  $[0, 1]$ , namely

$$\Gamma(x, t) = \frac{1}{2\sqrt{\pi t}} \exp\left\{\frac{-x^2}{4t}\right\}. \quad (2.12)$$

It is know that  $\Gamma$  satisfies (see [8])

$$\begin{aligned} \int_0^1 \Gamma(x - y, t - z) dy &\leq 1, \\ \int_0^t \Gamma(1, t - \tau) \frac{1}{2(t - \tau)} d\tau &\leq C^* \sqrt{t - z}, \quad \int_z^t \Gamma(0, t - \tau) d\tau = \frac{1}{\sqrt{\pi}} \sqrt{t - z}, \\ \frac{\partial \Gamma}{\partial \nu_y}(x - y, t - \tau) &= \frac{x - y}{2(t - \tau)} \Gamma(x - y, t - \tau), \quad x, y \in [0, 1], \quad 0 \leq z < t. \end{aligned} \quad (2.13)$$

By the Green's identity with (1.1) for  $v$ ,

$$\begin{aligned} v(x, t) &= \int_0^1 \Gamma(x - y, t - z) v(y, z) dy + \int_z^t \int_0^1 \Gamma(x - y, t - \tau) (-a_2 e^{\beta_1 v(y, \tau)}) dy d\tau \\ &+ \int_z^t \frac{\partial v}{\partial x}(1, \tau) \Gamma(x - 1, t - \tau) d\tau - \int_z^t \frac{\partial \Gamma}{\partial \nu_y}(x - 1, t - \tau) v(1, \tau) d\tau \\ &+ \int_z^t \frac{\partial \Gamma}{\partial \nu_y}(x, t - \tau) v(0, \tau) d\tau, \end{aligned} \quad (2.14)$$

where  $0 \leq z < t < T$ ,  $0 < x < 1$ . With  $z = 0$  and  $x \rightarrow 1$ , it follows that

$$\begin{aligned} v(x, t) &= \int_0^1 \Gamma(1 - y, t) v(y, 0) dy + \int_0^t \int_0^1 \Gamma(x - y, t - \tau) (-a_2 e^{\beta_1 v(y, \tau)}) dy d\tau \\ &+ \int_0^t e^{q_u(1, \tau) + \beta_2 v(1, \tau)} \Gamma(0, t - \tau) d\tau + \int_0^t v(0, \tau) \Gamma(1, t - \tau) \frac{1}{2(t - \tau)} d\tau. \end{aligned} \quad (2.15)$$

By (2.11), we have furthermore

$$v(1, t_0) \leq v_0(1) + C_0 e^{\beta_2 v(1, t_0)} \int_0^{t_0} (t_0 - \tau)^{-\frac{q}{2\alpha_2} - \frac{1}{2}} d\tau + C^* \sqrt{t_0} v(1, t_0). \quad (2.16)$$

Since  $q < \alpha_2$ , the integral term in (2.16) is smaller than  $1/(NC_0)$  with  $\sqrt{t_0} \leq \sqrt{T} \leq 1/(NC^*)$  if we choose  $u_0$  large to make  $T$  sufficiently small. This yields

$$\frac{N-1}{N} v(1, t_0) \leq v_0(1) + \frac{1}{N} e^{\beta_2 v(1, t_0)}. \quad (2.17)$$

Consequently,

$$\frac{2(N-1)}{N} K \leq K + \frac{1}{N} e^{2\beta_2 K}, \quad (2.18)$$

and hence

$$N \leq \frac{1}{K} e^{2\beta_2 K} + 2, \quad (2.19)$$

a contradiction.  $\square$

### 3 Proof of Theorem 1.2

We begin with a Lemma to prove Theorem 1.2.

**Lemma 3.1** *Let  $u$  be a solution of*

$$\begin{aligned} u_t &= u_{xx} - a_1 e^{\alpha_1 u}, & (x, t) &\in (0, 1) \times (0, T), \\ u_x(1, t) &\leq L e^{\alpha_2 u(1, t)}, \quad u_x(0, t) = 0, & t &\in (0, T), \\ u(x, 0) &= u_0(x), & x &\in [0, 1], \end{aligned} \quad (3.1)$$

where  $a_1 > 0$ ,  $\alpha_i \geq 0$ ,  $i=1,2$  and  $L$  is a positive constant. If  $u$  blows up at a finite time, then either  $2\alpha_2 > \alpha_1$  or  $2\alpha_2 = \alpha_1$ ,  $a_1 \leq \frac{\alpha_1}{4} \min\{c_1^2/(4c_2^2), \lambda_0 c_3^2/c_2^2\}$ . Furthermore,

$$u(1, t) = \max_{[0,1]} u(\cdot, t) \geq \log C(T-t)^{-\frac{1}{2\alpha_2}}, \quad \text{as } t \rightarrow T. \quad (3.2)$$

**Proof.** The blow-up of  $u$  implies either  $2\alpha_2 > \alpha_1$  or  $2\alpha_2 = \alpha_1$ ,  $a_1 \leq \frac{\alpha_1}{4} \min\{c_1^2/(4c_2^2), \lambda_0 c_3^2/c_2^2\}$  by Proposition 1.1.

By the Green's identity, similarly to (2.14)

$$\begin{aligned} u(x, t) &\leq \int_0^1 \Gamma(x-y, t-z) u(y, z) dy + L \int_z^t e^{\alpha_2 u(1, \tau)} \Gamma(x-1, t-\tau) d\tau \\ &\quad - \int_z^t \frac{\partial \Gamma}{\partial \nu_y}(x-1, t-\tau) u(1, \tau) d\tau + \int_z^t \frac{\partial \Gamma}{\partial \nu_y}(x, t-\tau) u(0, \tau) d\tau, \end{aligned} \quad (3.3)$$

where  $0 < z < t < T$ ,  $0 < x < 1$ . Let  $x \rightarrow 1$  with the jumping relations to obtain

$$\begin{aligned} \frac{1}{2}u(1, t) &\leq \int_0^1 \Gamma(1-y, t-z)u(y, z)dy + L \int_z^t e^{\alpha_2 u(1, \tau)} \Gamma(0, t-\tau) d\tau \\ &\quad + \int_z^t \frac{\partial \Gamma}{\partial \nu_y}(1, t-\tau)u(0, \tau) d\tau \\ &\leq u(1, z) + \frac{L}{\sqrt{\pi}} \sqrt{T-z} e^{\alpha_2 u(1, t)} + C^* \sqrt{T-z} u(1, t). \end{aligned} \quad (3.4)$$

For any  $z \in (0, T)$  with  $C^* \sqrt{T-z} \leq 1/4$ , choose  $t \in (z, T)$  such that  $\frac{1}{4}u(1, t) - u(1, z) \geq C_0 > 0$ . Then

$$C_0 \leq \frac{L}{\sqrt{\pi}} \sqrt{T-t} e^{\alpha_2 u(1, t)}, \quad (3.5)$$

which implies (3.2).  $\square$

**Proof of Theorem 1.2.** Since  $v \leq K$  for  $(x, t) \in [0, 1] \times [0, T)$ , we have

$$\begin{aligned} u_t &= u_{xx} - a_1 e^{\alpha_1 u}, & (x, t) &\in (0, 1) \times (0, T), \\ u_x(1, t) &\leq e^{pK} e^{\alpha_2 u(1, t)}, \quad u_x(0, t) = 0, & t &\in (0, T), \\ u(x, 0) &= u_0(x), & x &\in [0, 1]. \end{aligned} \quad (3.6)$$

Then, we obtain from Lemma 3.1 that  $2\alpha_2 > \alpha_1$  or  $2\alpha_2 = \alpha_1$  with  $a_1 \leq \frac{\alpha_1}{4} \min \left\{ \frac{c_1^2}{4c_2^2}, \frac{\lambda_0 c_3^2}{c_2^2} \right\}$ , and moreover,

$$u(1, t) = \max_{[0,1]} u(\cdot, t) \geq \log C (T-t)^{-\frac{1}{2\alpha_2}}, \quad \text{as } t \rightarrow T. \quad (3.7)$$

Next, let us show  $q < \alpha_2$ . Due to (2.14), we have by letting  $x \rightarrow 1$  that

$$v(1, t) \geq \int_z^t e^{qu(1, \tau) + \beta_2 v(1, \tau)} \Gamma(0, t-\tau) d\tau - a_2 \int_z^t \int_0^1 \Gamma(1-y, t-\tau) e^{\beta_1 v(y, \tau)} dy d\tau, \quad (3.8)$$

and so,

$$v(1, t) \geq C_1 \int_z^t (T-\tau)^{-\frac{q}{2\alpha_2} - \frac{1}{2}} d\tau - a_2 e^{\beta_1 v(1, \tau)}. \quad (3.9)$$

The boundedness of  $v(1, t)$  as  $t \rightarrow T$  requires that  $q < \alpha_2$ .  $\square$

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