

Note on multiplicative perturbation of local C -regularized cosine functions with nondensely defined generators

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Abstract

In this note, we obtain a new multiplicative perturbation theorem for local C -regularized cosine function with a nondensely defined generator A . An application to an integrodifferential equation is given.

Key words : Multiplicative perturbation, local C -regularized cosine functions, second order differential equation

1 Introduction and preliminaries

Let X be a Banach space, A an operator in X . It is well known that the cosine operator function is the main propagator of the following Cauchy problem for a second order differential equation in X :

$$\begin{cases} u''(t) = Au(t), & t \in (-\infty, \infty) \\ u(0) = u_0, u'(0) = u_1, \end{cases}$$

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which controls the behaviors of the solutions of the differential equations in many cases (cf., e.g., [2, 4–10, 13, 15, 16, 19–21]); if A is the generator of a C -regularized cosine function $\{C(t)\}_{t \in \mathbf{R}}$, then $u(t) = C^{-1}C(t)u_0 + C^{-1} \int_0^t C(s)u_1 ds$ is the unique solution of the above Cauchy problem for every pair (u_0, u_1) of initial values in $C(D(A))$ (see [5, 16, 20]). So it is valuable to study deeply the properties of the cosine operator functions.

As a meaningful generalization of the classical cosine operator functions, the C -regularized cosine functions have been investigated extensively (cf., e.g., [2, 4, 5, 9, 10, 13, 15, 16, 20, 21]), where C serves as a regularizing operator which is injective.

Stimulated by these works as well as the works on integrated semigroups and C -regularized semigroups ([3, 11, 14, 17, 18]), we study further the multiplicative perturbation of local C -regularized cosine functions with nondensely defined generators, in the case where (1) the range of the regularizing operator C is not dense in a Banach space X ; (2) the operator C may not commute with the perturbation operator.

Throughout this paper, all operators are linear; $\mathcal{L}(X, Y)$ denotes the space of all continuous linear operators from X to a space Y , and $\mathcal{L}(X, X)$ will be abbreviated to $\mathcal{L}(X)$; $\mathcal{L}_s(X)$ is the space of all continuous linear operators from X to X with the strong operator topology; $\mathbf{C}([0, t], \mathcal{L}_s(X))$ denotes all continuous $\mathcal{L}(X)$ -valued functions, equipped with the norm $\|F\|_\infty = \sup_{r \in [0, t]} \|F(r)\|$. Moreover, we write $D(A)$, $R(A)$, $\rho(A)$, respectively, for the domain, the range and the resolvent set of an operator A . We denote by \tilde{A} the part of A in $\overline{D(A)}$, that is,

$$\tilde{A} \subset A, D(\tilde{A}) = \{x \in D(A) \mid Ax \in \overline{D(A)}\}.$$

We abbreviate C -regularized cosine function to C -cosine function.

Definition 1.1. Assume $\tau > 0$. A one-parameter family $\{C(t); |t| \leq \tau\} \subset \mathcal{L}(X)$ is called a local C -cosine function on X if

- (i) $C(0) = C$ and $C(t+s)C + C(t-s)C = 2C(t)C(s) \quad (\forall |s|, |t|, |s+t| \leq \tau)$,
- (ii) $C(\cdot)x : [-\tau, \tau] \rightarrow X$ is continuous for every $x \in X$.

The associated sine operator function $S(\cdot)$ is defined by $S(t) := \int_0^t C(s)ds \quad (|t| \leq \tau)$.

The operator A defined by

$$\begin{aligned} D(A) &= \{x \in X; \lim_{t \rightarrow 0^+} \frac{2}{t^2}(C(t)x - Cx) \text{ exists and is in } R(C)\}, \\ Ax &= C^{-1} \lim_{t \rightarrow 0^+} \frac{2}{t^2}(C(t)x - Cx), \quad \forall x \in D(A), \end{aligned}$$

is called the generator of $\{C(t); |t| \leq \tau\}$. It is also called that A generates $\{C(t); |t| \leq \tau\}$.

Lemma 1.2. ([2]) *Let A generate a local C -cosine function $\{C(t); |t| \leq \tau\}$ on X . Then*

(i) *For $x \in D(A)$, $t \in [-\tau, \tau]$, $C(t)x, S(t)x \in D(A)$, $AC(t)x = C(t)Ax$, $AS(t)x = S(t)Ax$;*

(ii) *For $x \in X$, $t \in [0, \tau]$, $\int_0^t S(s)x ds \in D(A)$ and $A \int_0^t S(s)x ds = C(t)x - Cx$;*

(iii) *For $x \in D(A)$, $t \in [0, \tau]$, $\int_0^t S(s)Ax ds = A \int_0^t S(s)x ds = C(t)x - Cx$.*

2 Results and proofs

Definition 2.1. *Let $\{C(t); |t| \leq \tau\}$ be a local C -cosine function on X . If a closed linear operator A in X satisfies*

$$(1) \quad C(t)A \subset AC(t), \quad |t| \leq \tau,$$

$$(2) \quad C(t)x = Cx + A \int_0^t \int_0^s C(\sigma)x d\sigma ds, \quad |t| \leq \tau, \quad x \in X,$$

then we say that A subgenerates a local C -cosine function on X , or A is a subgenerator of a local C -cosine function on X .

Remark 2.2. *The generator \mathcal{G} of a local C -cosine function $\{C(t); |t| \leq \tau\}$ is a subgenerator of $\{C(t); |t| \leq \tau\}$. But for each subgenerator A , one has $A \subset \mathcal{G}$ and $\mathcal{G} = C^{-1}AC$. Moreover, if $\rho(A) \neq \emptyset$, then $C^{-1}AC = A$.*

In fact, for $x \in D(A)$, we have

$$\begin{aligned} \lim_{t \rightarrow 0^+} \frac{2(C(t)x - Cx)}{t^2} &= 2 \lim_{t \rightarrow 0^+} \frac{A \int_0^t \int_0^s C(\sigma)x d\sigma ds}{t^2} = 2 \lim_{t \rightarrow 0^+} \frac{\int_0^t \int_0^s C(\sigma)Ax d\sigma ds}{t^2} \\ &= CAx \in R(C), \end{aligned}$$

that is $x \in D(\mathcal{G})$ and $Ax = \mathcal{G}x$, i.e., $A \subset \mathcal{G}$.

For $x \in D(C^{-1}AC)$, then $Cx \in D(A)$ and $ACx \in R(C)$, since $A \subset \mathcal{G}$ we have $\mathcal{G}Cx = ACx \in R(C)$, then $C^{-1}ACx = C^{-1}\mathcal{G}Cx = \mathcal{G}x$, i.e., $C^{-1}AC \subset \mathcal{G}$. On the other hand, for $x \in D(\mathcal{G})$, noting

$$\lim_{n \rightarrow \infty} 2n^2 \int_0^{\frac{1}{n}} \int_0^s C(\sigma)x d\sigma ds = \lim_{n \rightarrow \infty} \frac{2 \int_0^{\frac{1}{n}} \int_0^s C(\sigma)x d\sigma ds}{\frac{1}{n^2}} = Cx,$$

and

$$\lim_{n \rightarrow \infty} A(2n^2 \int_0^{\frac{1}{n}} \int_0^s C(\sigma)x d\sigma ds) = \lim_{n \rightarrow \infty} 2n^2(C(\frac{1}{n})x - Cx) = C\mathcal{G}x,$$

the closedness of A ensures $Cx \in D(A)$ and $ACx = C\mathcal{G}x$, therefore, we have $\mathcal{G} \subset C^{-1}AC$.

From Proposition 1.4 in [12], we can obtain $C^{-1}AC = A$ if $\rho(A) \neq \emptyset$.

Theorem 2.3. *Let nondensely defined operator A generate a local C -cosine function $\{C(t); |t| \leq \tau\}$ on X , $S(t) = \int_0^t C(s)ds$, and $B \in \mathcal{L}(\overline{D(A)})$. Then*

(1) *there exists an operator family $\{E(t); |t| \leq \tau\} \subset \mathcal{L}(X)$ such that*

$$E(t)x = Cx + A(I + B) \int_0^t \int_0^s E(\sigma)x d\sigma ds, \quad |t| \leq \tau, \quad x \in \overline{D(A)},$$

provided that

(H1)

$$\left\| A \int_0^t S(t-s)C^{-1}B\Phi(s)ds \right\| \leq M \int_0^t \sup_{0 \leq s \leq \sigma} \|\Phi(s)\| d\sigma, \quad t \in [0, \tau],$$

where $\Phi \in \mathbf{C}([0, \tau], X)$, and $M > 0$ is a constant.

(2) $(I + B)\tilde{A}$ *generates a local C_1 -cosine function on $\overline{D(A)}$ provided that*

(H1')

$$\left\| \int_0^t \Phi(s)C^{-1}BAS(t-s)x ds \right\| \leq M\|x\| \int_0^t \sup_{0 \leq s \leq \sigma} \|\Phi(s)\| d\sigma, \quad t \in [0, \tau],$$

where $x \in D(A)$, $\Phi \in \mathbf{C}([0, \tau], \mathcal{L}_s(X))$, and $M > 0$ is a constant,

(H2) *there exists an injective operator $C_1 \in \mathcal{L}(\overline{D(A)})$ such that $R(B) \subset R(C_1) \subset C(\overline{D(A)})$, $C_1(I + B)\tilde{A} \subset (I + B)\tilde{A}C_1$, and $C^{-1}C_1(D(\tilde{A}))$ is a dense subspace in $D(A)$,*

(H3) $\rho((I + B)\tilde{A}) \neq \emptyset$.

(3) $\tilde{A}(I + B)$ *subgenerates a C_1 -cosine function on $\overline{D(A)}$ provided that $C_1B = BC_1$, and (H1'), (H2), and (H3) hold.*

Proof. First, we prove the conclusion (2).

Define the operator functions $\{\overline{C}_n(t)\}_{t \in [0, \tau]}$ as follows:

$$\begin{cases} \overline{C}_0(t)x = C(t)x, \\ \overline{C}_n(t)x = \int_0^t \overline{C}_{n-1}(s)C^{-1}BAS(t-s)x ds, \quad x \in D(A), t \in [0, \tau], n = 1, 2, \dots \end{cases}$$

By induction, we obtain:

(i) $\overline{C}_n(t) \in \mathbf{C}([0, \tau], \mathcal{L}_s(\overline{D(A)}))$;

(ii) $\|\overline{C}_n(t)\| \leq \frac{M^n t^n}{n!} \sup_{s \in [0, \tau]} \|C(s)\|, \quad t \in [0, \tau], \forall n \geq 0.$

It follows that the series $\sum_{n=0}^{\infty} \frac{M^n t^n}{n!}$ converges uniformly on $[0, \tau]$ and consequently,

$$\overline{C}(t)x := \sum_{n=0}^{\infty} \overline{C}_n(t)x \in \mathbf{C}([0, \tau], \overline{D(A)}), \quad \forall x \in \overline{D(A)},$$

and satisfies

$$\overline{C}(t)x = C(t)x + \int_0^t \overline{C}(s)C^{-1}BAS(t-s)x ds, \quad x \in D(A), t \in [0, \tau]. \quad (2.1)$$

Using (H1') and Gronwall's inequality, we can see the uniqueness of solution of (2.1).

Put

$$\widehat{C}(t) := \overline{C}(t)C^{-1}C_1, \quad t \in [0, \tau].$$

It follows from (2.1) and $C^{-1}C_1 \in \mathcal{L}(\overline{D(A)})$ that for $x \in \overline{D(A)}$,

$$\widehat{C}(t)x \in \mathbf{C}([0, \tau], \overline{D(A)}),$$

and satisfies

$$\widehat{C}(t)x = C(t)C^{-1}C_1x + \int_0^t \widehat{C}(s)C_1^{-1}BAS(t-s)C^{-1}C_1x ds, \quad x \in D(A), t \in [0, \tau]. \quad (2.2)$$

Note that $D(\widetilde{A}) \subset D(C_1^{-1}B\widetilde{A}C_1)$ and $C^{-1}C_1$ maps $D(\widetilde{A})$ into $D(\widetilde{A})$. So, for $x \in D(\widetilde{A})$, by (2.2), we have

$$\begin{aligned} & \int_0^t \int_0^s \widehat{C}(\sigma)C_1^{-1}B\widetilde{A}C_1x d\sigma ds \\ &= \int_0^t \int_0^s C(\sigma)C^{-1}B\widetilde{A}C_1x d\sigma ds \\ &+ \int_0^t \int_0^s \widehat{C}(\sigma)C_1^{-1}B[C(s-\sigma)C^{-1}B\widetilde{A}C_1x - B\widetilde{A}C_1x] d\sigma ds. \end{aligned} \quad (2.3)$$

Therefore, for $x \in D(\tilde{A})$, we have

$$\begin{aligned}
& \int_0^t \int_0^s \widehat{C}(\sigma)(I+B)\tilde{A}x d\sigma ds \\
= & \int_0^t \int_0^s C(\sigma)C^{-1}(I+B)\tilde{A}C_1x d\sigma ds \\
& + \int_0^t \int_0^s \widehat{C}(\sigma)C_1^{-1}B[C(s-\sigma)C^{-1}(I+B)\tilde{A}C_1x - (I+B)\tilde{A}C_1x] d\sigma ds \\
= & C(t)C^{-1}C_1x - C_1x + \int_0^t \int_0^s C(\sigma)C^{-1}B\tilde{A}C_1x d\sigma ds \\
& + \int_0^t \int_0^s \widehat{C}(\sigma)C_1^{-1}BC(s-\sigma)C^{-1}\tilde{A}C_1x d\sigma ds - \int_0^t \int_0^s \widehat{C}(\sigma)C_1^{-1}B\tilde{A}C_1x d\sigma ds \\
& + \int_0^t \int_0^s \widehat{C}(\sigma)C_1^{-1}B[C(s-\sigma)C^{-1}B\tilde{A}C_1x - B\tilde{A}C_1x] d\sigma ds \\
\stackrel{(2.3)}{=} & \widehat{C}(t)x - C_1x. \tag{2.4}
\end{aligned}$$

Now we consider the integral equation

$$v(t)x = C_1x + \int_0^t \int_0^s v(\sigma)(I+B)\tilde{A}x d\sigma ds, \quad x \in D(\tilde{A}), t \in [0, \tau], \tag{2.5}$$

where $v(t) \in \mathbf{C}([0, \tau], \mathcal{L}_s(\overline{D(\tilde{A})}))$. Let $\tilde{v}(t)$ satisfy the equation (2.5). Then from (2.5), we obtain, for $x \in D(\tilde{A})$,

$$\begin{aligned}
& \int_0^t \tilde{v}(s)S(t-s)C^{-1}C_1x ds - C_1 \int_0^t S(s)C^{-1}C_1x ds \\
= & \int_0^t \int_0^s \tilde{v}(\sigma)(I+B)\tilde{A} \int_0^{s-\sigma} S(r)C^{-1}C_1x dr d\sigma ds \\
= & \int_0^t \tilde{v}(s)S(t-s)C^{-1}C_1x ds - \int_0^t \int_0^s \tilde{v}(\sigma)C_1x d\sigma ds \\
& + \int_0^t \int_0^s \tilde{v}(\sigma)B\tilde{A} \int_0^{s-\sigma} S(r)C^{-1}C_1x dr d\sigma ds.
\end{aligned}$$

Hence,

$$\int_0^t \int_0^s \tilde{v}(\sigma)C_1x d\sigma ds = C_1 \int_0^t S(s)C^{-1}C_1x ds + \int_0^t \int_0^s \tilde{v}(\sigma)B\tilde{A} \int_0^{s-\sigma} S(r)C^{-1}C_1x dr d\sigma ds,$$

that is,

$$(\tilde{v}(t)C)C^{-1}C_1x = C_1C(t)C^{-1}C_1x + \int_0^t (\tilde{v}(s)C)C^{-1}B\tilde{A}S(t-s)C^{-1}C_1x ds.$$

Note that $C^{-1}C_1(D(\tilde{A})) \subset D(\tilde{A})$ is dense in $\overline{D(\tilde{A})}$, and the solution $\bar{w}(t)$ of the equation

$$w(t)y = C_1C(t)y + \int_0^t w(s)C^{-1}B\tilde{A}S(t-s)y ds, \quad y \in C^{-1}C_1(D(\tilde{A})), t \in [0, \tau]$$

in $\mathbf{C}([0, \tau], \mathcal{L}_s(\overline{D(A)}))$ is unique, we can see the solution of (2.5) is also unique.

By the uniqueness of solution of (2.5), we can obtain that

$$\widehat{C}(-t)x = \widehat{C}(t)x, \quad \widehat{C}(t)C_1x = C_1\widehat{C}(t)x, \quad \text{for each } x \in \overline{D(A)}, t \in [0, \tau].$$

Moreover, for $t, h, t \pm h \in [0, \tau]$, we have

$$\begin{aligned} & \widehat{C}(t+h)C_1x + \widehat{C}(t-h)C_1x \\ = & \int_0^{t+h} \int_0^s \widehat{C}(\sigma)(I+B)AC_1x d\sigma ds + \int_0^{t-h} \int_0^s \widehat{C}(\sigma)(I+B)AC_1x d\sigma ds + 2C_1^2x \\ = & \int_0^h \int_0^s \widehat{C}(t+\sigma)(I+B)AC_1x d\sigma ds + \int_0^t \int_0^{t-s} \widehat{C}(\sigma)(I+B)AC_1x d\sigma ds \\ & + \int_0^h \int_0^t \widehat{C}(t-\sigma)(I+B)AC_1x d\sigma ds + \int_0^h \int_0^s \widehat{C}(t-\sigma)(I+B)AC_1x d\sigma ds \\ & + \int_0^t \int_0^{t-s} \widehat{C}(\sigma)(I+B)AC_1x d\sigma ds - \int_0^h \int_0^t \widehat{C}(t-\sigma)(I+B)AC_1x d\sigma ds \\ & + 2C_1^2x \\ = & \int_0^h \int_0^s \left[\widehat{C}(t+\sigma)C_1 + \widehat{C}(t-\sigma)C_1 \right] (I+B)Ax d\sigma ds + 2 \int_0^t (t-s)\widehat{C}(s)(I+B)AC_1x ds \\ & + 2C_1^2x, \end{aligned}$$

and for all $x \in D(\widetilde{A})$, $t, h \in [0, \tau]$, we have

$$\begin{aligned} 2\widehat{C}(t)\widehat{C}(h)x &= 2\widehat{C}(t) \left[\int_0^h \int_0^s \widehat{C}(\sigma)(I+B)\widetilde{A}x d\sigma ds + C_1x \right] \\ &= \int_0^h \int_0^s 2\widehat{C}(t)\widehat{C}(\sigma)(I+B)Ax d\sigma ds + 2 \int_0^t (t-s)\widehat{C}(s)(I+B)\widetilde{A}C_1x ds \\ &\quad + 2C_1^2x. \end{aligned}$$

Therefore, for $x \in D(\widetilde{A})$, $t \in [0, \tau]$,

$$\begin{aligned} & [\widehat{C}(t+h)C_1x + \widehat{C}(t-h)C_1x] - 2\widehat{C}(t)\widehat{C}(h)x \\ = & \int_0^t \int_0^s \left\{ [\widehat{C}(\sigma+h)C_1 + \widehat{C}(\sigma-h)C_1] - 2\widehat{C}(\sigma)\widehat{C}(h) \right\} (I+B)\widetilde{A}x d\sigma ds. \end{aligned}$$

It follows from the uniqueness of solution of (2.5) and the denseness of \widetilde{A} in $\overline{D(A)}$ that

$$2\widehat{C}(t)\widehat{C}(h) = \widehat{C}(t+h)C_1 + \widehat{C}(t-h)C_1$$

on $\overline{D(A)}$, for $t, h, t \pm h \in [0, \tau]$. Therefore, $\{\widehat{C}(t)\}_{t \in [-\tau, \tau]}$ is a local C_1 -cosine operator function on $\overline{D(A)}$.

Next, we show that the subgenerator of $\{\widehat{C}(t)\}_{t \in [-\tau, \tau]}$ is operator $(I + B)\widetilde{A}$.

By the equality (2.4), (H3), the uniqueness of solution of (2.5), we obtain on $\overline{D(A)}$

$$(\lambda - (I + B)\widetilde{A})^{-1}\widehat{C}(t) = \widehat{C}(t)(\lambda - (I + B)\widetilde{A})^{-1}, \quad t \in [0, \tau], \quad \lambda \in \rho((I + B)\widetilde{A}),$$

therefore,

$$(I + B)\widetilde{A}\widehat{C}(t)x = \widehat{C}(t)(I + B)\widetilde{A}x, \quad x \in D(\widetilde{A}), \quad t \in [0, \tau], \quad (2.6)$$

that is,

$$\widehat{C}(t)(I + B)\widetilde{A} \subset (I + B)\widetilde{A}\widehat{C}(t), \quad t \in [0, \tau].$$

Moreover, since $\rho((I + B)\widetilde{A}) \neq \emptyset$, $(I + B)\widetilde{A}$ is a closed operator. It follows from (2.4) and the closedness of $(I + B)\widetilde{A}$ that $\int_0^t \int_0^s \widehat{C}(\sigma)x d\sigma ds \in D(\widetilde{A})$ and

$$\widehat{C}(t)x = C_1x + (I + B)\widetilde{A} \int_0^t \int_0^s \widehat{C}(\sigma)x d\sigma ds, \quad (2.7)$$

for each $x \in \overline{D(A)}$, $t \in [0, \tau]$. Therefore, $(I + B)\widetilde{A}$ is a subgenerator of $\{\widehat{C}(t)\}_{t \in [-\tau, \tau]}$. By (H3) and remark 2.2, we can see that $(I + B)\widetilde{A}$ is the generator of $\{\widehat{C}(t)\}_{t \in [-\tau, \tau]}$. This completes the proof of statement (2).

By a combination of similar arguments as above and those given in the proof of [11, Theorem 2.1], we can obtain the conclusion (1).

Next, we prove the conclusion (3).

In view of statement (1) just proved, we can see that $(I + B)\widetilde{A}$ subgenerates $\{\widehat{C}(t)\}_{t \in [-\tau, \tau]}$ on $\overline{D(A)}$. Set

$$Q(t)x = C_1x + \widetilde{A} \int_0^t \int_0^s \widehat{C}(\sigma)(I + B)x d\sigma ds, \quad |t| \leq \tau, \quad x \in \overline{D(A)}.$$

Obviously, by (2.7) and the fact that the graph norms of \widetilde{A} and $(I + B)\widetilde{A}$ are equivalent, we can see that $\{Q(t)\}_{t \in [-\tau, \tau]}$ is a strongly continuous operator family of bounded linear operators on $\overline{D(A)}$. Moreover, by (2.6) and (2.7), for $|t| \leq \tau$, $x \in \overline{D(A)}$, we obtain

$$\begin{aligned} Q(t)\widetilde{A}(I + B)x &= C_1\widetilde{A}(I + B)x + \widetilde{A} \int_0^t \int_0^s \widehat{C}(\sigma)(I + B)\widetilde{A}(I + B)x d\sigma ds, \\ &= \widetilde{A}(I + B)C_1x + \widetilde{A}(I + B)\widetilde{A} \int_0^t \int_0^s \widehat{C}(\sigma)(I + B)x d\sigma ds \\ &= \widetilde{A}(I + B)Q(t)x \end{aligned}$$

and

$$\begin{aligned} (I + B) \int_0^t \int_0^s Q(\sigma)x d\sigma ds &= \int_0^t \int_0^s (I + B)C_1x + \int_0^t \int_0^s [\widehat{C}(\sigma)(I + B)x - C_1(I + B)x] d\sigma ds \\ &= \int_0^t \int_0^s \widehat{C}(\sigma)(I + B)x d\sigma ds. \end{aligned}$$

It follows that for any $|t| \leq \tau$, $x \in \overline{D(A)}$, $(I + B) \int_0^t \int_0^s Q(\sigma)x d\sigma ds \in D(\widetilde{A})$ and

$$\widetilde{A}(I + B) \int_0^t \int_0^s Q(\sigma)x d\sigma ds = \widetilde{A} \int_0^t \int_0^s \widehat{C}(\sigma)(I + B)x d\sigma ds = Q(x)x - C_1x.$$

According to Definition 2.1, we see that $\widetilde{A}(I + B)$ subgenerates a local C_1 -cosine function $\{Q(t)\}_{t \in [-\tau, \tau]}$ on $\overline{D(A)}$. \square

Example 2.4. Let Ω be a domain in R^n and write

$$\begin{aligned} C_0(\Omega) := \{f \in C(\Omega) : &\text{for each } \varepsilon > 0 \text{ there is a compact } \Omega_\varepsilon \subset \Omega \\ &\text{such that } |f(s)| < \varepsilon \text{ for all } s \in \Omega \setminus \Omega_\varepsilon\}. \end{aligned}$$

Given $q \in C(\Omega)$ with $q(\eta) \geq 0$, $b \in C_0(\Omega)$ with

$$be^{\tau q}, \quad qbe^{\tau q} \in C_0(\Omega), \tag{2.8}$$

and $K \in L^1(\Omega)$, we consider the following Cauchy problem

$$\begin{cases} \frac{\partial^2 u(t, \eta)}{\partial t^2} &= q(\eta) \left(u(t, \eta) + b(\eta) \int_\Omega K(\sigma)u(t, \sigma) d\sigma \right), \\ u(0, \eta) &= f_1(\eta), \quad u'(0, \eta) = f_2(\eta), \quad \eta \in \Omega, \quad 0 \leq t \leq \tau, \end{cases} \tag{2.9}$$

where $f_1, f_2 \in C_0(\Omega)$.

Set $Af =: qf$ with $D(A) =: \{f \in C_0(\Omega); qf \in C_0(\Omega)\}$.

When q is bounded, A generates a classical cosine function $C(t)$ on $C_0(\Omega)$ (i.e., I -cosine function on $[0, \infty)$), with

$$\|C(t)\| \leq e^{\left(\sup_{\eta \in \Omega} q(\eta)\right)t}, \quad t \geq 0$$

(for the exponential growth bound of a cosine function (which is closely related to a strongly continuous semigroup in some cases), as well as its relation with the spectral bound of the generator, we refer to, e.g., [1, 16]). Nevertheless, when q is unbounded, A

does not generate a global C -cosine function $C(t)$ on $C_0(\Omega)$ for any C . On the other hand, A generates a local C -cosine function $C(t)$ on $C_0(\Omega)$:

$$C(t)f = \left\{ \frac{1}{2} \left[e^{t\sqrt{q}} + e^{-t\sqrt{q}} \right] e^{-\tau\sqrt{q}} f \right\}_{t \in [-\tau, \tau]},$$

with $Cf = e^{-\tau\sqrt{q}}f$. Set

$$(Bf)(\eta) = b(\eta) \int_{\Omega} K(\sigma)f(\sigma)d\sigma, \quad f \in C_0(\Omega).$$

From (2.8), we see the hypothesis (H1) in Theorem 2.3 holds. This means, by Theorem 2.3 (1) and [20, Theorem 2.4], that the Cauchy problem (2.9) has a unique solution in $C^2([0, \tau]; C_0(\Omega))$ for every couple of initial values in a large subset of $C_0(\Omega)$.

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