Electronic Journal of Qualitative Theory of Differential Equations
2010, No. **72**, 1-14; **http://www.math.u-szeged.hu/ejqtde/**

EXISTENCE OF HOMOCLINIC ORBIT FOR SECOND-ORDER NONLINEAR DIFFERENCE EQUATION*

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Abstract: By using the Mountain Pass Theorem, we establish some existence criteria to guarantee the second-order nonlinear difference equation $\Delta\left[p(t)\Delta u(t-1)\right]+f(t,u(t))=0$ has at least one homoclinic orbit, where $t\in\mathbb{Z},\ u\in\mathbb{R}$.

Keywords: Nonlinear difference equation; Discrete variational methods; Mountain Pass Lemma; Homoclinic orbit

AMS Subject Classification. 39A11; 58E05; 70H05

1. Introduction

In this paper, we shall be concerned with the existence of homoclinic orbit for the second-order difference equation:

$$\Delta\left[p(t)\Delta u(t-1)\right]+f(t,u(t))=0,\quad t\in\mathbb{Z},\ u\in\mathbb{R}, \tag{1.1}$$

where the forward difference operator $\Delta u(t)=u(t+1)-u(t), \Delta^2 u(t)=\Delta\left(\Delta u(t)\right)$, $p(t)>0$, $f:\mathbb{Z}\times\mathbb{R}\to\mathbb{R}$ is a continuous function in the second variables and satisfies $f(t+T,u)=f(t,u)$ for a given positive integer T. As usual, $\mathbb{N},\mathbb{Z}$ and $\mathbb{R}$ denote the set of all natural,

*This work is partially supported by the Outstanding Doctor Degree Thesis Implantation Foundation of Central South University(No: 2010ybfz073).

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integer and real numbers, respectively. For $a, b \in \mathbb{Z}$, denote $\mathbb{N}(a) = \{a, a+1, ...\}, \mathbb{N}(a, b) = \{a, a+1, ...b\}$ when $a \leq b$.

The theory of nonlinear difference equations has been widely used to study discrete models appearing in many fields such as computer science, economics, neural network, ecology, cybernetics, etc. Since the last decade, there has been much literature on qualitative properties of difference equations, those studies cover many of the branches of difference equations, such as [1-3, 10, 11] and references therein. In some recent papers [7-9, 22-24], the authors studied the existence of periodic solutions of second-order nonlinear difference equation by using the critical point theory. These papers show that the critical point method is an effective approach to the study of periodic solutions of second-order difference equations.

In the theory of differential equations, a trajectory which is asymptotic to a constant state as $|t| \to \infty$ (t denotes the time variable) is called a homoclinic orbit. It is well-known that homoclinic orbits play an important role in analyzing the chaos of dynamical systems. (see, for instance, [5, 6, 15, 19-21], and references therein). If a system has the transversely intersected homoclinic orbits, then it must be chaotic. If it has the smoothly connected homoclinic orbits, then it cannot stand the perturbation, its perturbed system probably produce chaotic phenomenon.

In general, Eq.(1.1) may be regarded as a discrete analogue of the following second-order differential equation

$$[p(t)u'(t)]' + f(t, u(t)) = 0, \quad t \in \mathbb{R},\ u \in \mathbb{R}. \tag{1.2}$$

Recently, the following second order self-adjoint difference equation

$$\Delta\left[p(t)\Delta u(t-1)\right] + q(t)u(t) = f(t, u(t)), \quad t \in \mathbb{Z},\ u \in \mathbb{R} \tag{1.3}$$

has been studied by using variational method (see [12]). Ma and Guo obtained homoclinic orbits as the limit of the subharmonics for Eq.(1.3) by applying the Mountain Pass theorem , their results are relying on $q(t) \neq 0$. If $q(t) = 0$, the traditional ways in [13] are inapplicable to our case.

Some special cases of (1.1) have been studied by many researchers via variational methods, (see, for example, [7] and references therein). However, to our best knowledge, results on homoclinic solutions for Eq.(1.1) has not been studied. Motivated by [6, 12], the main purpose of this paper is to give some sufficient conditions for the existence of homoclinic and even homoclinic solutions to Eq.(1.1).

Without loss of generality, we assume that $u = 0$ is an equilibrium for (1.1), we say that a solution $u(t)$ of (1.1) is a homoclinic orbit if $u \neq 0$ and $u \to 0$ as $t \to \pm\infty$.

Our main results are the following theorems.

Theorem 1.1. *Assume that the following conditions are satisfied:*

(F1) $F(t,u) = -K(t,u) + W(t,u)$, *where* K, W *is* T*-periodic with respect to* $t, T > 0$, $K(t,u), W(t,u)$ *are continuously differentiable in* u*;*

(F2) *There are constants* $b_1, b_2 > 0$ *such that for all* $(t,u) \in \mathbb{Z} \times \mathbb{R}$

$$b_1|u|^2 \leq K(t,u) \leq b_2|u|^2;$$

(F3) *For all* $(t,u) \in \mathbb{Z} \times \mathbb{R}$, $K(t,u) \leq uK_u(t,u) \leq 2K(t,u)$*;*

(F4) $W_u(t,u) = o(|u|)$, $(|u| \to 0)$ *uniformly in* $t \in \mathbb{Z}$*;*

(F5) *There is a constant* $\mu > 2$ *such that for every* $t \in \mathbb{Z}, u \in \mathbb{R} \backslash \{0\}$,

$$0 < \mu W(t,u) \leq uW_u(t,u).$$

Then Eq.(1.1) possesses at least one nontrivial homoclinic solution.

Theorem 1.2. *Assume that* F *satisfies (F1), (F2), (F3), (F4), (F5) and the following assumption:*

(F6) $p(t) = p(-t), \quad F(t,u) = F(-t,u)$.

Then Eq.(1.1) *possesses a nontrivial even homoclinic orbit.*

2. Preliminaries

In this section, we will establish the corresponding variational framework for (1.1).

Let S be the vector space of all real sequences of the form

$$u = \{u(t)\}_{t\in\mathbb{Z}} = (..., u(-t), u(-t+1), ..., u(-1), u(0), u(1), ..., u(t), ...),$$

namely

$$S = \{u = \{u(t)\} : u(t) \in \mathbb{R}, \quad t \in \mathbb{Z}\}.$$

For each $k \in \mathbb{N}$, let $E_k = \{u \in S | u(t) = u(t+2kT), t \in \mathbb{Z}\}$. It is clear that E_k is isomorphic to $\mathbb{R}^{2kT}$, E_k can be equipped with inner product

$$\langle u, v\rangle_k = \sum_{t=-kT}^{kT-1} [p(t)\Delta u(t-1)\Delta v(t-1) + u(t)v(t)], \quad \forall\ u \in E_k,$$

by which the norm $\|u\|_k$ can be induced by

$$\|u\|_k = \left[\sum_{t=-kT}^{kT-1} \left[p(t)(\Delta u(t-1))^2 + (u(t))^2 \right] \right]^{\frac{1}{2}}, \quad \forall\ u \in E_k. \tag{2.1}$$

It is obvious that E_k is a Hilbert space of $2kT$-periodic functions on $\mathbb{Z}$ with values in $\mathbb{R}$ and linearly homeomorphic to $\mathbb{R}^{2kT}$.

In what follows, l_k^2 denotes the space of functions whose second powers are summable on the interval $\mathbb{N}[-kT, kT-1]$ equipped with the norm

$$\|u\|_{l_k^2} = \left(\sum_{t\in\mathbb{N}[-kT,\ kT-1]} |u(t)|^2 \right)^{\frac{1}{2}}, \quad u \in l_k^2.$$

Moreover, l_k^∞ denotes the space of all bounded real functions on the interval $\mathbb{N}[-kT, kT-1]$ endowed with the norm

$$\|u\|_{l_k^\infty} = \max_{t\in\mathbb{N}[-kT,\ kT-1]} \{|u(t)|\}, \quad u \in l_k^\infty.$$

Let $\bar{b}_1 = \min\{1, 2b_1\}, \bar{b}_2 := \max\{1, 2b_2\}$ and $\eta_k : E_k \to [0, +\infty)$ be such that

$$\eta_k(u) = \left(\sum_{t=-kT}^{kT-1} \left[p(t)(\Delta u(t-1))^2 + 2K(t,u) \right] \right)^{\frac{1}{2}}. \tag{2.2}$$

By (F2),

$$\bar{b}_1\|u\|_k^2 \le \eta_k^2(u) \le \bar{b}_2\|u\|_k^2, \tag{2.3}$$

let

$$I_k(u) = \sum_{t=-kT}^{kT-1} \left[\frac{1}{2} p(t)(\Delta u(t-1))^2 - F(t, u(t)) \right] \tag{2.4}$$

$$= \frac{1}{2}\eta_k^2(u) - \sum_{t=-kT}^{kT-1} W(t, u(t)), \tag{2.5}$$

where $F(t,u) = \int\limits_0^u f(t,s)ds$. Then $I_k \in C^1(E_k, \mathbb{R})$ and it is easy to check that

$$I_k'(u)v = \sum_{t=-kT}^{kT-1} \left[p(t)\Delta u(t-1)\Delta v(t-1) - f(t,u)v \right],$$

by (F5),

$$I_k'(u)u \le \eta_k^2(u) - \sum_{t=-kT}^{kT-1} W_u(t,u)u, \tag{2.6}$$

by using

$$u(-kt-1)=u(kT-1), \quad u(-kT)=u(kT), \tag{2.7}$$

we can compute the Fréchet derivative of (2.4) as

$$\frac{I_k(u)}{\partial u(t)} = -\Delta\left[p(t)\Delta u(t-1)\right] - f(t,u),\ t\in\mathbb{Z}.$$

Thus, u is a critical point of I_k on E_k if and only if

$$\Delta\left[p(t)\Delta u(t-1)\right] + f(t,u(t)) = 0, \quad t\in\mathbb{Z},\ u\in\mathbb{R}, \tag{2.8}$$

so the critical points of I_k in E_k are classical $2kT$-periodic solutions of (1.1). That is, the functional I_k is just the variational framework of (1.1).

3. Proofs of theorems

At first, let us recall some properties of the function $W(t,u)$ from Theorem 1.1. They are all necessary to the proof of Theorems .

Fact 3.1[6]**.** *For every $t\in[0,T]$, the following inequalities hold*:

$$W(t,u)\le W\left(t,\frac{u}{|u|}\right)|u|^{\mu}, \quad if\ 0<|u|\le 1, \tag{3.1}$$

$$W(t,u)\ge W\left(t,\frac{u}{|u|}\right)|u|^{\mu}, \quad if\ |u|\ge 1. \tag{3.2}$$

It is an immediate consequence of (F5).

Fact 3.2**.** *Set $m := \inf\{W(t,u) : t\in[0,T], |u|=1\}$. Then for every $\zeta\in\mathbb{R}\backslash\{0\}$, $u\in E_k\backslash\{0\}$, we have*

$$\sum_{t=-kT}^{kT-1} W(t,\zeta u(t)) \ge m|\zeta|^{\mu}\sum_{t=-kT}^{kT-1}|u(t)|^{\mu} - 2kTm. \tag{3.3}$$

Proof. Fix $\zeta\in\mathbb{R}\backslash\{0\}$ and $u\in E_k\backslash\{0\}$. Set

$$A_k=\{t\in[-kT,kT-1] : |\zeta u(t)|\le 1\},\ B_k=\{t\in[-kT,kT-1] : |\zeta u(t)|\ge 1\}.$$

From (3.2) we have

$$
\begin{aligned}
\sum_{t=-kT}^{kT-1} W(t,\zeta u(t)) &\geq \sum_{t\in B_k} W(t,\zeta u(t)) \geq \sum_{t\in B_k} W\left(t,\frac{\zeta u(t)}{|\zeta u(t)|}\right)|\zeta u(t)|^{\mu} \\
&\geq m\sum_{t\in B_k}|\zeta u(t)|^{\mu} \\
&\geq m\sum_{t=-kT}^{kT-1}|\zeta u(t)|^{\mu} - m\sum_{t\in A_k}|\zeta u(t)|^{\mu} \\
&\geq m|\zeta|^{\mu}\sum_{t=-kT}^{kT-1}|u(t)|^{\mu} - 2kTm.
\end{aligned}
$$

Fact 3.3[6]**.** *Let $Y:[0,+\infty)\to[0,+\infty)$ be given as follows: $Y(0)=0$ and*

$$
Y(s)=\max_{t\in[0,T],0<|u|\leq s}\frac{uW_u(t,u)}{|u|^2}, \tag{3.4}
$$

for $s>0$. Then Y is continuous, nondecreasing, $Y(s)>0$ for $s>0$ and $Y(s)\to+\infty$ as $s\to+\infty$.

It is easy to verify this fact applying (F4), (F5) and (3.2).

We will obtain a critical point of I_k by use of a standard version of the Mountain Pass Theorem(see [17]). It provides the minimax characterization for the critical value which is important for what follows. Therefore, we state this theorem precisely.

Lemma 3.1. *(Mountain Pass Lemma [14, 17]). Let E be a real Banach space and $I\in C^1(E,\mathbb{R})$ satisfy (PS)-condition. Suppose that I satisfies the following conditions:*

(i) $I(0)=0$;

(ii) There exist constants $\rho,\alpha>0$ such that $I|_{\partial B_\rho(0)}\geq\alpha$;

(iii) There exists $e\in E\setminus\bar{B}_\rho(0)$ such that $I(e)\leq 0$.

Then I possesses a critical value $c\geq\alpha$ given by

$$
c=\inf_{g\in\Gamma}\max_{s\in[0,1]}I(g(s)),
$$

where $B_\rho(0)$ is an open ball in E of radius ρ centered at 0, and

$$
\Gamma=\{g\in C([0,1],E)\ :\ g(0)=0, g(1)=e\}.
$$

Lemma 3.2. *I_k satisfies the (PS) condition.*

Proof. In our case it is clear that $I_k(0) = 0$. We show that I_k satisfies the (PS) condition. Assume that $\{u_j\}_{j\in\mathbb{N}}$ in E_k is a sequence such that $\{I_k(u_j)\}_{j\in\mathbb{N}}$ is bounded and $I'_k(u_j) \to 0,\ j \to +\infty$. Then there exists a constant $C_k > 0$ such that

$$|I_k(u_j)| \leq C_k, \qquad \|I'_k(u_j)\|_{k^*} \leq C_k \tag{3.5}$$

for every $j \in \mathbb{N}$. We first prove that $\{u_j\}_{j\in\mathbb{N}}$ is bounded. By (2.5) and (F5)

$$\eta_k^2(u_j) \leq 2I_k(u_j) + \sum_{t=-kT}^{kT-1} W_u(t,u)u, \tag{3.6}$$

From (3.6) and (2.6) we have

$$\left(1 - \frac{2}{\mu}\right)\eta_k^2(u_j) \leq 2I_k(u_j) - \frac{2}{\mu}I'_k(u_j)u_j, \tag{3.7}$$

by (3.7) and (2.3) we have

$$\begin{aligned}\left(1 - \frac{2}{\mu}\right)\bar{b}_1\|u_j\|_k^2 &\leq 2I_k(u_j) - \frac{2}{\mu}I'_k(u_j)u_j \\ &\leq 2I_k(u_j) + \frac{2}{\mu}\|I'_k(u_j)\|_{k^*}\|u_j\|_k.\end{aligned}$$

It follows from (3.6) that

$$\left(1 - \frac{2}{\mu}\right)\bar{b}_1\|u_j\|_k^2 - \frac{2}{\mu}C_k\|u_j\|_k - 2C_k \leq 0. \tag{3.8}$$

Since $\mu > 2$, (3.8) implies that $\{u_j\}_{\mathbb{N}}$ is bounded in E_k. Thus, $\{u_j\}$ possesses a convergent subsequence in E_k. The desired result follows.

Lemma 3.3. *I_k satisfies Mountain Pass Theorem. Then, there exists subharmonics $u_k \in E_k$.*

Proof. By (2.1), we have

$$\|u\|_k^2 = ((P_k + I_k)u, u), \tag{3.9}$$

where $u = (u(-kT), u(-kT+1), ..., u(-1), u(0), u(1), ..., u(kT-1))^T$,

$$P_k = \begin{pmatrix} p_{-kT} + p_{-kT+1} & -p_{-kT+1} & 0 & \cdots & 0 & -p_{-kT} \\ -p_{-kT+1} & p_{-kT+1} + p_{-kT+2} & -p_{-kT+2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & p_{kT-2} + p_{kT-1} & -p_{kT-1} \\ -p_{kT} & 0 & 0 & \cdots & -p_{kT-1} & p_{kT-1} + p_{kT} \end{pmatrix}_{2kT\times 2kT},$$

here $p_i = p(i),\ i \in \mathbb{N}[-kT, kT-1]$ and

$$I_k = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}_{2kT\times 2kT}.$$

By $p(t) > 0$, $P_k + I_k$ is positive definite. Suppose that the eigenvalues of $P_k + I_k$ are $\lambda_{-kT}, \lambda_{-kT+1}, \ldots\lambda_{-1}, \lambda_0, \lambda_1, \ldots\lambda_{kT-1}$, then they are all greater than zero. We define

$$\lambda_{\max} = \max\{\lambda_{-kT}, \lambda_{-kT+1}, \ldots\lambda_{-1}, \lambda_0, \lambda_1, \ldots\lambda_{kT-1}\},$$

$$\lambda_{\min} = \min\{\lambda_{-kT}, \lambda_{-kT+1}, \ldots\lambda_{-1}, \lambda_0, \lambda_1, \ldots\lambda_{kT-1}\}.$$

By (3.9), we have

$$\lambda_{\min}\|u\|^2 \le \|u\|_k^2 \le \lambda_{\max}\|u\|^2. \tag{3.10}$$

For our setting, clearly $I_k(0) = 0$. By (F4), there exists $\rho > 0$ such that $|W(t,x)| \le \frac{1}{4}\bar{b}_1\lambda_{\min}u^2$ for any $|u| \le \frac{\rho}{\sqrt{\lambda_{\min}}},\quad t \in \mathbb{N}[-kT, kT-1]$. Thus, for any $u \in E_k$ and $\|u\|_k \le \rho$, we obtain $|u(t)| \le \|u\| \le \frac{1}{\sqrt{\lambda_{\min}}}\|u\|_k \le \frac{1}{\sqrt{\lambda_{\min}}}\rho,\ \forall\, t \in \mathbb{N}[-kT, kT-1]$, which leads to

$$\begin{aligned} I_k(u) &\ge \frac{1}{2}\bar{b}_1\|u\|_k^2 - \frac{1}{4}\bar{b}_1\lambda_{\min}\sum_{t=-kT}^{kT-1}((u(t))^2 \\ &= \frac{1}{2}\bar{b}_1\|u\|_k^2 - \frac{1}{4}\bar{b}_1\lambda_{\min}\|u\|^2 \\ &\ge \frac{1}{2}\bar{b}_1\|u\|_k^2 - \frac{1}{4}\bar{b}_1\lambda_{\min}\frac{1}{\lambda_{\min}}\|u\|_k^2 = \frac{1}{4}\bar{b}_1\|u\|_k^2. \end{aligned}$$

Take $a = \frac{1}{4}\bar{b}_1\rho^2 > 0$, we get

$$I_k(u)|\,\partial B_\rho \ge a.$$

By Hölder inequality and (3.3), we have $\zeta \in \mathbb{R},\ \omega \in E_k\backslash\{0\}$, which leads to

$$\begin{aligned} I_k(\zeta\omega) &\le \frac{1}{2}\bar{b}_2\zeta^2\|\omega\|^2 - m|\zeta|^\mu\sum_{t=-kT}^{kT-1}|\omega|^\mu + 2kT \\ &\le \frac{1}{2}\bar{b}_2\zeta^2\|\omega\|^2 - m|\zeta|^\mu(2kT)^{\frac{2-\mu}{2}}\|\omega\|^\mu\left(\frac{1}{\lambda_{\max}}\right)^{\frac{\mu}{2}} + 2kT. \end{aligned}$$

Since $\mu > 2$, which shows *(iii)* of Lemma 3.1 holds with $e = e_m$, a sufficiently large multiple of any $\omega \in E_k\backslash\{0\}$. Consequently by Lemmas 3.1 and 3.2, I_k possesses a critical value c_k

given by *(iii)* with $E = E_k,\ \Gamma = \Gamma_k$. Let u_k denote the corresponding critical point of I_k on E_k. Note that $\|u_k\| \neq 0$ since $c_k > 0$.

Lemma 3.4. *Suppose that the conditions of Theorem* 1.1 *hold true, then there exists a constant d independent of k such that $\|u_k\|_k \leq d,\ \forall\ k \in \mathbb{N}$.*

Lemma 3.5. *Suppose that the conditions of Theorem* 1.1 *hold true, then there exists a constant d independent of k such that the following inequalities are true:*

$$\|u\|^2 \leq \|u\|_k^2 \leq \bar{\lambda}\|u\|^2, \quad \|u\|_{l_k^\infty}^2 \leq \|u\|_k^2. \tag{3.11}$$

Lemmas 3.6. *Suppose that* (F1) $-$ (F4) *are satisfied, then there exists a constant δ such that*

$$\delta \leq \|u_k\|_{l_k^\infty} \leq d,$$

where $\|u_k\|_{l_k^\infty} = \max_{t\in\mathbb{N}[-kT,kT-1]}\{|u_k(t)|\}$.

By a fashion similar to the proofs in [12], we can prove Lemma 3.4, Lemma 3.5 and Lemma 3.6, respectively. The detailed proofs are omitted.

Proof of Theorem 1.1. We will show that $\{u_k\}_{k\in\mathbb{N}}$ possesses a convergent subsequence $\{u_{k_m}\}$ in $E_{loc}(\mathbb{Z},\mathbb{R})$ and a nontrivial homoclinic orbit u_∞ emanating from 0 such that $u_{k_m} \to u_\infty$ as $k_m \to \infty$.

Since $u_k = \{u_k(t)\}$ is well defined on $\mathbb{N}[-kT,\ kT-1]$ and $\|u_k\|_k \leq d$ for all $k \in \mathbb{N}$, we have the following consequences.

First, let $u_k = \{u_k(t)\}$ be well defined on $\mathbb{N}[-T, T-1]$. It is obvious that $\{u_k\}$ is isomorphic to $\mathbb{R}^{2T}$. Thus there exists a subsequence $\{u_{k_m}^1\}$ and $u^1 \in E^1$ of $\{u_k\}_{k\in\mathbb{N}\setminus\{1\}}$ such that

$$\|u_{k_m}^1 - u^1\|_1 \to 0.$$

Second, let $\{u_{k_m}^1\}$ be restricted to $\mathbb{N}[-2T, 2T-1]$. Clearly, $\{u_{k_m}^1\}$ is isomorphic to $\mathbb{R}^{4T}$. Thus there exists a further subsequence $\{u_{k_m}^2\}$ of $\{u_{k_m}^1\}$ satisfying $u^2 \notin \{u_{k_m}^2\}$ and $u^2 \in E_2$ such that

$$\|u_{k_m}^2 - u^2\|_2 \to 0 \ \ k_m \to \infty.$$

Repeat this procedure for all $k \in \mathbb{N}$. We obtain sequence $\{u_{k_m}^p\} \subset \{u_{k_m}^{p-1}\},\ u^p \notin \{u_{k_m}^p\}$ and there exists $u^p \in E_p$ such that

$$\|u_{k_m}^p - u^p\|_p \to 0,\ k_m \to \infty, \ \ p = 1, 2, ...$$

Moreover, we have

$$\|u^{p+1}-u^p\|_p \le \|u^{p+1}_{k_m}-u^{p+1}\|_p + \|u^{p+1}_{k_m}-u^p\|_p \to 0,$$

which leads to

$$u^{p+1}(s)=u^p(s), s\in \mathbb{N}[-pT,pT-1].$$

So, for the sequence $\{u^p\}$, we have $u^p \to u_\infty$, $p\to\infty$, where $u_\infty(s)=u^p(s)$ for $s\in \mathbb{N}[-pT,pT-1]$ and $p\in\mathbb{N}$. Then take a diagonal sequence $\{u_{k_m}\}:\ u^1_{k_1},\ u^2_{k_2}, \ldots\ u^m_{k_m}, \ldots$, since $\{u^m_{k_m}\}$ is a sequence of $\{u^p_{k_m}\}$ for any $p\ge 1$, it follows that

$$\|u^m_{k_m}-u_\infty\| = \|u^m_{k_m}-u^m\|_m \to 0,\ m\in\mathbb{N}.$$

It shows that

$$u_{k_m}\to u_\infty \ \text{ as } k_m\to\infty, \text{ in } \ E_{loc}(\mathbb{Z},\mathbb{R}),$$

where $u_\infty \in E_\infty(\mathbb{Z},\ \mathbb{R})$, $E_\infty(\mathbb{Z},\ \mathbb{R})=\{u\in S|\|u\|_\infty = \sum\limits_{t=-\infty}^{+\infty}[p(t)(\Delta u(t-1))^2+(u(t))^2]<\infty\}$.

By series convergence theorem, u_∞ satisfy

$$u_\infty(t)\to 0,\ \triangle u_\infty(t-1)\to 0,$$

and

$$\sum_{t=-pT}^{pT-1}\left\{\left[p(t)(\Delta u^m_{k_m}(t-1))^2+(u^m_{k_m}(t))^2\right]<\infty\right\} = \|u^m_{k_m}\|_p,$$

as $|t|\to\infty$.

Letting $t\to\infty$, $\forall\ p\ge 1$, we have

$$\sum_{t=-pT}^{pT-1}\left[\frac{1}{2}p(t)(\Delta u_{k_m}(t-1))^2 - F(t,u^m_{k_m}(t))\right]\le d_1,$$

as $m\ge p$, $k_m\ge p$, where d_1 is independent of k, $\{k_m\}\subset\{k\}$ are chosen as above, we have

$$\sum_{t=-pT}^{pT-1}\left[\frac{1}{2}p(t)(\Delta u_\infty(t-1))^2 - F(t,u_\infty(t))\right]\le d_1.$$

Letting $p\to\infty$, by the continuity of $F(t,u)$ and I'_k, which leads to

$$I_\infty(u_\infty)=\sum_{t=-\infty}^{+\infty}\left[\frac{1}{2}p(t)(\Delta u_\infty(t-1))^2 - F(t,u_\infty(t))\right]\le d_1,\ \forall\ u\in E_\infty,$$

and

$$I'_\infty(u_\infty) = 0.$$

Clearly, u_∞ is a solution of (1.1).

To complete the proof of Theorem 1.1, it remains to prove that $u_\infty \not\equiv 0$.

It follows from (3.4), (3.11) that

$$\sum_{t=-kT}^{kT-1} uW_u(t,u) \le Y\left(\|u\|_{l_k^\infty}\right)\|u_k\|_k^2, \tag{3.12}$$

Since $I'_k(u_k)u_k = 0$, we obtain

$$\sum_{t=-kT}^{kT-1} uW_u(t,u) = \sum_{t=-kT}^{kT-1} p(t)(\Delta u(t-1))^2 + \sum_{t=-kT}^{kT-1} K_u(t,u)u, \tag{3.13}$$

by (3.12) and (3.13), we have

$$Y\left(\|u\|_{l_k^\infty}\right)\|u_k\|_k^2 \ge \min\{1, b_1\}\|u_k\|_k^2,$$

thus,

$$Y\left(\|u_k\|_{l_k^\infty}\right) \ge \min\{1, b_1\} > 0. \tag{3.14}$$

If $\|u\|_{l_k^\infty} \to 0,\ k \to +\infty$, we would have $Y(0) \ge \min\{1, b_1\} > 0$, which is a contradiction to fact 3.3. So there exists $\gamma > 0$ such that

$$\|u_k\|_{l_k^\infty} \ge \gamma \tag{3.15}$$

for any $j \in \mathbb{N}$, $u_k(t+jT)$, so, if necessary, by replacing $u_k(t)$ earlier, if necessary by $u_k(t+jT)$ for some $j \in \mathbb{N}[-k,k]$, it can be assumed that the maximum of $|u_k(t)|$ occurs in $\mathbb{N}[0,T]$. Thus if $u_\infty \equiv 0$, then by lemma 3.6, we have

$$\|u_{k_m}\|_{l_{k_m}^\infty} = \max_{t\in[0,T]} |u_{k_m}(t)| \to 0,$$

which contradicts (3.15) The proof is complete.

Proof of Theorem 1.2. Consider the following boundary problem on finite interval:

$$\begin{cases} \Delta[p(t)\Delta u(t-1)] + f(t,u(t)) = 0, & t \in \mathbb{N}[-kT,kT], \\ u(-kT) = u(kT) = 0 & \\ u(-t) = u(t), & t \in \mathbb{N}[-kT,kT]. \end{cases} \tag{3.16}$$

where $t, k, T \in \mathbb{N}$.

Let S be the vector space of all real sequence of the form

$$u = \{u(t)\}_{t\in\mathbb{Z}} = (..., u(-t), u(-t+1), ..., u(-1), u(0), u(1), ..., u(t), ...)\,,$$

namely

$$S = \{u = \{u(t)\} : u(t) \in R, \quad t \in \mathbb{Z}\}\,.$$

Define

$$E_{kT} = \{u \in S | u(-t) = u(t), t \in \mathbb{Z}\}.$$

Then space E_{kT} is a Hilbert space with the inner product

$$\langle u, v\rangle = \sum_{t=-kT}^{kT} \left[(p(t)\Delta u(t-1)\Delta v(t-1)) + u(t)v(t)\right],$$

for any $u, v \in E_{kT}$, the corresponding norm can be induced by

$$\|u\|_{E_{kT}}^2 = \sum_{t=-kT}^{kT} \left[(p(t)(\Delta u(t-1))^2 + (u(t))^2\right], \quad \forall\ u \in E_{kT}.$$

It is obvious that E_{kT} is Hilbert space with $2kT+1$-periodicity and linearly homeomorphic to $\mathbb{R}^{2kT+1}$.

By a fashion similar to the proofs of Theorem 1.1, we can prove Theorem 1.2. The detailed proofs are omitted.

4. Example

In this section, we give an example to illustrate our results.

Example 4.1. Consider the difference equation

$$\Delta\left[\left(a+\cos\frac{2\pi}{T}t\right)\Delta u(t-1)\right] + \left(\sin\frac{2\pi}{T}t + c\right)\left(u|u|^{\gamma-2} - u\right) = 0, \quad t \in \mathbb{Z}, \tag{4.1}$$

where $a, c > 1$ and $f(t,u) = \left(\sin\frac{2\pi}{T}t + c\right)(u|u|^{\gamma-2} - u)$,

$$p(t) = a + \cos\frac{2\pi}{T}t, \quad F(t,u) = \left(\sin\frac{2\pi}{T}t + c\right)\left(|u|^\gamma - \frac{u^2}{2}\right).$$

Take

$$K(t,u) = \left(\sin\frac{2\pi}{T}t + c\right)\frac{u^2}{2}, \quad W(t,u) = \left(\sin\frac{2\pi}{T}t + c\right)\frac{|u|^\gamma}{\gamma}.$$

It is easy to verify that the conditions of Theorem 1.1 are all satisfied as $2 < \mu \leq \gamma$. Therefore, Eq.(1.1) possesses at least one nontrivial homoclinic orbit.

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(Received March 16, 2010)