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Note on an anisotropic p-Laplacian equation in $\mathbb{R}^n$*

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Abstract

In this paper, we study a kind of anisotropic p-Laplacian equations in $\mathbb{R}^n$. Nontrivial solutions are obtained using mountain pass theorem given by Ambrosetti-Rabinowitz [1].

Key Words and Phrases: anisotropic p-Laplacian equations, nontrivial solutions.

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1 Introduction

Consider the following anisotropic p-Laplacian problem in $\mathbb{R}^n$

$$-\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(a_i \left| \frac{\partial u}{\partial x_i} \right|^{p_i-2} \frac{\partial u}{\partial x_i} \right) = f(x,u) \quad \text{in } \mathbb{R}^n \tag{1.1}$$

Each $p_i \in (1,\infty), a_i = a_i(x)$ are measurable real functions satisfying $0 < a_0 < a_i(x) \in L^\infty(\mathbb{R}^n)$ and we assume that the nonlinear function f satisfies the subcritical growth conditions

$$|f(x,u)| \leq g(x)|u|^r \quad \forall (x,u) \in \mathbb{R}^n \times \mathbb{R} \tag{1.2}$$

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for some $r \in (p_+ - 1, \bar{p}^* - 1)$. Herein, $p_+ = \max(p_1, \dots, p_n)$, $\dfrac{1}{\bar{p}} = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} \dfrac{1}{p_i}$ and $\bar{p}^* = \dfrac{n\bar{p}}{n - \bar{p}}$ with $\bar{p} < n$ and $g : \mathbb{R}^n \to \mathbb{R}$ is a nonnegative function satisfying

$$g \in L^{\omega}(\mathbb{R}^n), \qquad \omega = \frac{\bar{p}^*}{\bar{p}^* - (r+1)}. \tag{1.3}$$

For (1.1) we assume that $F \in C^1(\mathbb{R}^n \times \mathbb{R})$, where $F(x,s) = \displaystyle\int_0^s f(x,t)\,dt$ and there exists $\theta > p_+$ such that

$$0 < \theta F(x,s) \leq s f(x,s), \quad \forall x \in \mathbb{R}^n,\ \forall s \in \mathbb{R} \setminus \{0\}. \tag{1.4}$$

In the isotropic case, we can refer the reader to the works by [3] and [7] where existence and regularity results are obtained. As to anisotropic equations with different orders of derivations in different directions involving critical exponents with unbounded nonlinearities, to our knowledge they were not intensively studied before, in passing, we mention the work [2]. Let us mention also that in [4] the authors have studied another class of anisotropic elliptic equations. Via an adaptation of the concentration-compactness lemma of P.-L. Lions to anisotropic operators, they have obtained the existence of multiple nonnegative solutions. Let us point out that in the case of bounded domains, more work in this direction can be found in [5] where the authors proved existence and nonexistence results for some anisotropic quasilinear elliptic equations.

The purpose of this paper is to obtain nontrivial weak solutions using mountain pass theorem (see e.g. [1]).

Our main result is the following.

Theorem 1.1. *Assume* (1.2), (1.3)*, and* (1.4)*. Moreover, if* $g \in L^{\frac{\omega p_1}{p_1 - 1}}(\mathbb{R}^n)$ *for some* $r + 1 < p_1 < \bar{p}^*$*, then problem* (1.1) *has at least one nontrivial solution.*

2 Preliminaries

We let $1 \leq p_1, \dots, p_n < \infty$ be N real numbers. Denote by $\overline{p}$ the harmonic mean of these numbers, i.e., $\frac{1}{\overline{p}} = \frac{1}{n}\sum_{i=1}^{n} \frac{1}{p_i}$, and set $p_+ = \max(p_1, \dots, p_n)$,

$p_- = \min(p_1, \dots, p_n)$. We always have $p_- \le \overline{p} \le p_+$. The Sobolev conjugate of $\overline{p}$ is denoted by $\overline{p}^\star$, i.e., $\overline{p}^\star = \frac{n\overline{p}}{n-\overline{p}}$.

Anisotropic Sobolev spaces were introduced and studied by Nikol'skiĭ [8], Slobodeckiĭ [9], Troisi [10], and later by Trudinger [11] in the framework of Orlicz spaces.

Let $(W, \|\cdot\|)$ be the anisotropic Sobolev space defined by

$$W = \left\{u \in W^{1,\bar{p}}(\mathbb{R}^n) \,:\, a_i^{\frac{1}{p_i}} \frac{\partial u}{\partial x_i} \in L^{p_i}(\mathbb{R}^n),\ i = 1, \dots, n\right\},$$

with the dual $(W^*, \|\cdot\|^*)$ and the duality pairing $\langle\cdot,\cdot\rangle$. W is a real reflexive Banach under the norm

$$\|u\| = \sum_{i=1}^{n} \left(\int_{\mathbb{R}^n} a_i \left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx\right)^{\frac{1}{p_i}} = \sum_{i=1}^{n} \left\|a_i^{\frac{1}{p_i}} \frac{\partial u}{\partial x_i}\right\|_{L^{p_i}(\mathbb{R}^n)}.$$

Let us recall that a weak solution of the equation (1.1) is any $u \in W$ such that

$$\sum_{i=1}^{n} \int_{\mathbb{R}^n} a_i \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i}\, dx = \int_{\mathbb{R}^n} f(x,u) v \, dx, \tag{2.1}$$

for all $v \in W$.

They coincide with the critical points of the C^1-energy functional corresponding to problem (1.1)

$$\Phi(u) = \sum_{i=1}^{n} \frac{1}{p_i} \int_{\mathbb{R}^n} a_i(x) \left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx - \int_{\mathbb{R}^n} F(x,u)\, dx, \tag{2.2}$$

for all $u \in W$.

Remark 2.1. *Let us remark that* (1.3) *implies* $\frac{1}{\omega} + \frac{r}{\overline{p}^*} + \frac{1}{\overline{p}^*} = 1$ *which guarantee that the integral given in right side of* (2.1) *is well defined.*

To deal with the functional framework we apply the following mountain pass theorem [1].

Theorem 2.2. *Let I be a C^1-differentiable functional on a Banach space E and satisfying the Palais-Smale condition (PS), suppose that there exists a neighbourhood U of 0 in E and a positive constant α satisfying the following conditions:*

(I1) $I(0) = 0$.

(I2) $I(u) \geq \alpha$ *on the boundary of* U.

(I3) *There exists an* $e \in W \backslash U$ *such that* $I(e) < \alpha$.

Then

$$c = \inf_{\gamma \in \Gamma} \sup_{y \in [0,1]} I(\gamma(y))$$

is a critical value of I *with* $\Gamma = \{h \in C([0,1]);\, h(0) = 0,\, h(1) = e\}$.

Let us recall that the functional $I : E \to \mathbb{R}$ of class C^1 satisfies the Palais-Smale compactness condition (PS) if every sequence $(u_k)_{k=1}^{\infty} \subset E$ for which there exists $M > 0$ such that: $I(u_k) \leq M$ and $I'(u_k) \to 0$ strongly in $I^\star$ as n goes to infinity (called a (PS) sequence), has a convergent subsequence.

Let us define the functional $J : W \to \mathbb{R}$ by $J(u) = \int_{\mathbb{R}^n} F(x,u)\, dx$ for each $u \in W$. It follows from (1.2),

$$|F(x,u)| \leq \frac{1}{r+1} g(x) |u|^{r+1} \tag{2.3}$$

for all $(x,u) \in \mathbb{R}^n \times \mathbb{R}$. Then by standard argument we have F is in $C^1(\mathbb{R}^n \times \mathbb{R})$, hence we see that J is well defined and continuously Gâteaux differentiable with

$$J^{'}(u)v = \int_{\mathbb{R}^n} f(x,u)v\, dx$$

for all $u, v \in W$.

Proposition 2.3. *J' is a compact map from W to W^*.*

Proof. Let u_k be a sequence in W which converges weakly to u. On one hand, in view of Hölder's inequality and Sobolev embedding, we obtain for all $0 \leq R \leq +\infty$,

$$\int_{|x| \geq R} f(x,u)v\, dx \leq \left(\int_{|x| \geq R} |g|^{\omega}\, dx \right)^{\frac{1}{\omega}} (C_1 \|u\|)^r \|v\|^{\bar{p}^*},$$

for all $u, v \in W$. Since $g \in L^{\omega}(\mathbb{R}^n)$, we have $\lim\limits_{R \to +\infty} \int_{|x| \geq R} |g|^{\omega}\, dx = 0$. This implies with the fact that u_k is a bounded sequence, for any ε, there exists $R_\varepsilon > 0$ such that

$$\int_{|x| \geq R_\varepsilon} f(x,u_k)v\, dx \leq \varepsilon \quad \text{and} \quad \int_{|x| \geq R_\varepsilon} f(x,u)v\, dx \leq \varepsilon \tag{2.4}$$

holds for all k.
On the other hand, since $g \in L^{\frac{\omega p_1}{p_1-1}}(\mathbb{R}^n)$ with $r+1 < p_1 < \bar{p}^*$, applying Young's inequality, we get

$$\begin{aligned} f^{\frac{p_1}{p_1-1}}(x,t) &\leq m(x)^{\frac{p_1}{p_1-1}} t^{\frac{rp_1}{p_1-1}} \\ &\leq \frac{\bar{p}^*-(r+1)}{\bar{p}^*} m(x)^{\frac{p_1}{p_1-1}(\frac{\bar{p}^*}{\bar{p}^*-(r+1)})} + \frac{r+1}{\bar{p}^*} t^{\frac{rp_1}{p_1-1}\frac{\bar{p}^*}{r+1}}, \end{aligned}$$

for all $t \in \mathbb{R}$ and a.e. $x \in B_\varepsilon = \{x \in \mathbb{R}^n; |x| < R_\varepsilon\}$. A simple calculation shows that $\frac{rp_1}{p_1-1}\frac{\bar{p}^*}{r+1} < \bar{p}^*$. Using the compact imbedding $W^{1,\bar{p}}(B_\varepsilon)$ into $L^q(B_\varepsilon)$ for all $q \in [1, \bar{p}^*)$ and the continuity of the Nemytskii's operator N_f associated with $f^{\frac{p_1}{p_1-1}}$ from $L^{\frac{rp_1}{p_1-1}\frac{\bar{p}^*}{r+1}}(B_\varepsilon)$ in $L^1(B_\varepsilon)$, we conclude that

$$\int_{|x|<R_\varepsilon} f(x,u_k)^{\frac{p_1}{p_1-1}}\, dx \to \int_{|x|<R_\varepsilon} f(x,u)^{\frac{p_1}{p_1-1}}\, dx,$$

which implies that $f(x,u_k)$ converges to $f(x,u)$ in $L^{\frac{p_1}{p_1-1}}(B_\varepsilon)$. Hence since $L^{\bar{p}^*}(B_\varepsilon) \subset L^{p_1}(B_\varepsilon)$, we have $f(x,u_k)v$ converges to $f(x,u)v$ in $L^1(B_\varepsilon)$, i.e.

$$\int_{|x|<R_\varepsilon} (f(x,u_k) - f(x,u))v\, dx \to 0, \tag{2.5}$$

for all $v \in L^{p_1}(B_\varepsilon)$. Finally, in view of (2.4) and (2.5), we get

$$\int_{\mathbb{R}^n} f(x,u_k)v\, dx \to \int_{\mathbb{R}^n} f(x,u)v\, dx,$$

for all $v \in W$. This completes the proof of Proposition 2.3 and consequently J' is compact. $\square$

3 Proof of Theorem 1.1

First we need the following Lemmas to show that the functional Φ satisfies the geometric conditions of Theorem 2.2.

Lemma 3.1. *Suppose* (1.2), (1.3) *and* (1.4) *hold, then there exist constants* α *and* ρ, *such that* $\Phi(u) \geq \alpha$ *for* $||u|| = \rho$.

Proof. From (2.3), one can easily deduce that

$$\lim_{t\to 0}\frac{F(x,t)}{u^{p_+}} = 0 \quad \text{and} \quad \lim_{t\to\infty}\frac{F(x,t)}{u^{\bar{p}^*}} = 0.$$

This implies that

$$\left|\frac{F(x,t)}{u^{p_+}}\right| \le \varepsilon \quad \forall |t| < \delta_1; \quad \left|\frac{F(x,t)}{u^{\bar{p}^*}}\right| \le \varepsilon \quad \forall |t| > \delta_2,$$

for some $\delta_1, \delta_2 > 0$. Now, using the fact that the function $\frac{F(x,t)}{u^{\bar{p}^*}}$ is continuous over $[\delta_1, \delta_2]$, we conclude that for all $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that

$$F(x,t) \le \varepsilon |t|^{p_+} + c_\varepsilon |t|^{\bar{p}^*} \quad \forall (x,t) \in \mathbb{R}^n \times \mathbb{R}. \tag{3.1}$$

Using again (2.3), the relation (3.1) and continuous imbedding of W in $L^{p_+}(\mathbb{R}^n)$ and $L^{\bar{p}^*}(\mathbb{R}^n)$, we get for $||u||$ small enough that

$$\begin{aligned}\Phi(u) &= \sum_{i=1}^n \frac{1}{p_i}\int_{\mathbb{R}^n} a_i \left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx - \int_{\mathbb{R}^*} F(x,u)\,dx,\\ &\ge \frac{1}{p_+}\,||u||^{p_+} - C\varepsilon\,||u||^{p_+} - C'c_\varepsilon\,||u||^{\bar{p}^*}\\ &\ge ||u||^{p_+}\left((\frac{1}{p_+} - \varepsilon C) - C'c_\varepsilon\,||u||^{\bar{p}^* - p_+}\right),\end{aligned}$$

for some constants $C, C' > 0$. Therefore, for $0 < \varepsilon < \frac{1}{Cp_+}$, there exist α and ρ small enough positive constants such that $\Phi(u) \ge \alpha > 0$ for all $||u|| = \rho$. $\square$

Lemma 3.2. *Suppose* (1.2) *and* (1.4) *hold. Then* $\Phi(tu) \to -\infty$ *as* $t \to +\infty$.

Proof. We have

$$\Phi(t^{1/p_+}u) = \sum_{i=1}^n \frac{t^{\frac{p_i}{p_+}}}{p_i}\int_{\mathbb{R}^n} a_i \left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx - \int_{\mathbb{R}^n} F(x, t^{1/p_+}u)\,dx.$$

Remark that $F(x,tu) \ge t^\theta F(x,u)$ for any $t \ge 1$, this is due to the fact that the function $\frac{F(x,tu)}{t^\theta}$ is increasing for all $t > 0$, we obtain

$$\Phi(t^{1/p_+}u) = \sum_{i=1}^n \frac{t^{\frac{p_i}{p_+}}}{p_i}\int_{\mathbb{R}^n} a_i \left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx - t^{\theta/p_+}\int_{\mathbb{R}^n} F(x,u)\,dx.$$

Then

$$\Phi(t^{1/p_+}u) \to -\infty \quad \text{as } t \to +\infty$$

follows immediately since $\theta > p_+$. $\square$

Lemma 3.3. *Let u_k be a Palais-Smale sequence of Φ. Then u_k possesses a subsequence converging strongly to some $u \in W$.*

Proof. First we claim that the sequence u_k is bounded. Indeed, Arguing by contradiction and consider a subsequence still denoted by u_k such that $||u_k|| \to \infty$. We have in view of (1.4)

$$\begin{aligned}
\Phi(u_k) &= \sum_{i=1}^{n} \frac{1}{p_i} \int_{\mathbb{R}^n} a_i \left| \frac{\partial u_k}{\partial x_i} \right|^{p_i} dx - \int_{\mathbb{R}^n} F(x, u_k)\, dx \\
&\geq \frac{1}{p_+} \int_{\mathbb{R}^n} a_i \left| \frac{\partial u_k}{\partial x_i} \right|^{p_i} dx - \frac{1}{\theta} \int_{\mathbb{R}^n} f(x, u_k) u_k\, dx \\
&= \frac{1}{p_+} \int_{\mathbb{R}^n} a_i \left| \frac{\partial u_k}{\partial x_i} \right|^{p_i} dx + \frac{1}{\theta} \left(\langle \Phi'(u_k); u_k \rangle - \sum_{i=1}^{n} \int_{\mathbb{R}^n} a_i \left| \frac{\partial u_k}{\partial x_i} \right|^{p_i} dx \right) \\
&\geq \left(\frac{1}{p_+} - \frac{1}{\theta} \right) \sum_{i=1}^{n} \int_{\mathbb{R}^n} a_i \left| \frac{\partial u_k}{\partial x_i} \right|^{p_i} dx + \frac{1}{\theta} \langle \Phi'(u_k); u_k \rangle.
\end{aligned} \tag{3.2}$$

Passing to limit in (3.2) as $k \to \infty$, we obtain

$$M \geq \left(\frac{1}{p_+} - \frac{1}{\theta} \right) \sum_{i=1}^{n} \int_{\mathbb{R}^n} a_i \left| \frac{\partial u_k}{\partial x_i} \right|^{p_i} dx \geq \left(\frac{1}{p_+} - \frac{1}{\theta} \right) ||u_k||^{p_-},$$

where M is the constant of Palais-Smale sequence. This gives a contradiction since $p_+ < \theta$. Hence the sequence u_k has a subsequence still denoted by u_k which converges weakly to some $u \in W$. For any pair integer (n, m) we have

$$\begin{aligned}
&\sum_{i=1}^{n} \int_{\mathbb{R}^n} a_i(x) \left(\left| \frac{\partial u_n}{\partial x_i} \right|^{p_i - 2} \frac{\partial u_n}{\partial x_i} - \left| \frac{\partial u_m}{\partial x_i} \right|^{p_i - 2} \frac{\partial u_m}{\partial x_i} \right) \left(\frac{\partial u_n}{\partial x_i} - \frac{\partial u_m}{\partial x_i} \right) \\
&\qquad = \langle \Phi'(u_n) - \Phi'(u_m); (u_n - u_m) \rangle + \int_{\mathbb{R}^n} (f(x, u_n) - f(x, u_m))(u_n - u_m)\, dx.
\end{aligned} \tag{3.3}$$

By Palais-Smale condition and Proposition 2.3, it is easy to see that the right side of (3.3) approaches zero. Finally, using the following algebraic relation

$$|\xi_1 - \xi_2|^r \leq ((|\xi_1|^{r-2}\xi_1 - |\xi_2|^{r-2}\xi_2)(\xi_1 - \xi_2))^{\rho/2}(|\xi_1|^r + |\xi_2|^r)^{1-\rho/2},$$

with $\rho = r$ if $1 < r \leq 2$ and $\rho = 2$ if $2 < r$, the monotonicity of the anisotropic operator of problem (1.1) now gives the result. This concludes the proof of Lemma 3.3. □

Therefore lemmas 3.1, 3.2 and 3.3 fit into conditions setting of Theorem 2.2 of section 2, this guarantees the existence of at least a nontrivial weak solution for (1.1).

Remark 3.4.

1- One can prove that each solution u of problem (1.1) *satisfies $u \in L^\sigma(\mathbb{R}^n)$ with $\bar{p}^* \leq \sigma \leq \infty$. This regularity result is based on an iterative procedure given in the works [3] where similar results are obtained for the case of degenerate isotropic p-Laplacian problems.*
2- Let us also mention that since $\omega + \varepsilon < \frac{\omega p_1}{p_1 - 1}$, with $0 < \varepsilon < 1$ small enough, the restrictive integrability condition $g \in L^{\omega+\varepsilon}(\mathbb{R}^n)$, suffices to proceed with the iterating method and to bound the maximal norm of the solution.

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