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A NOTE ON A LINEAR SPECTRAL THEOREM FOR A CLASS OF FIRST ORDER SYSTEMS IN $\mathbb{R}^{2N}$

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Abstract

Along the lines of Atkinson [3], a spectral theorem is proved for the boundary value problem

$$\begin{cases} Jz' + f(t)Jz + P(t)z = \lambda B(t)z \\ x(0) = x(T) = 0, \end{cases} \qquad t \in [0,T], \quad z = (x,y) \in \mathbb{R}^N \times \mathbb{R}^N,$$

where $f(t)$ is real-valued and $P(t), B(t)$ are symmetric matrices, with $B(t)$ positive definite. A suitable rotation index associated to the system is used to highlight the connections between the eigenvalues and the nodal properties of the corresponding eigenfunctions.

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1 Introduction

In the past few years, there has been some interest for the linear spectral theory for planar first order systems with various boundary conditions, this often being a first step towards bifurcation results for nonlinear systems. Starting from the simplest case, that is, the autonomous Hamiltonian problem

$$\begin{cases} Jz' = \lambda z \\ x(0) = x(T) = 0, \end{cases} \qquad t \in [0,T], \quad z = (x,y) \in \mathbb{R}^2 = \mathbb{R} \times \mathbb{R},$$

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with $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, there have been many generalizations in order to deal with more general nonautonomous linear operators (see [4, 23, 24]). In particular, in [4] it is considered the so called *Dirac operator*, given by

$$\tau z := B(t)^{-1}[2q(t)Jz' + q'(t)Jz + P(t)z], \tag{1}$$

where $B(t), P(t)$ are symmetric 2×2 matrices with $B(t)$ positive definite, and $q(t) > 0$. Such an operator plays a central role in Quantum Mechanics, arising in a natural way after separation of variables in the Dirac equation (see [13, 25] for a complete discussion about this subject). The common feature of the above mentioned results is the existence of a two-sided sequence of (simple) eigenvalues, unbounded both from below and from above; the proof typically relies on the use of classical polar coordinates in the plane, which allow to establish a sharp connection between the eigenvalues and the nodal properties of the associated eigenfunctions.

When passing to higher dimension, a more delicate analysis has to be performed, since the classical rotation number approach is no longer available in a simple way, so that some more sophisticated tools have to be introduced. Starting from the works by Conley [9] and Greenberg [16] (this last one, however, dealing with the very different case of systems of n-th order equations), there have been several contributions, essentially depending on the symplectic structure of the linear equation in consideration.

With such preliminaries, it seems natural to look for a general statement which, at the same time, unifies the planar results and extends them to the higher dimensional case. We stress that this goal has to be achieved maintaining the crucial connection between the eigenvalues of the problem and some kind of nodal properties of the associated eigenfunctions. As our starting point, we will follow [3, Chapter 10], where the Hamiltonian problem

$$\begin{cases} Jz' + P(t)z = \lambda B(t)z \\ x(0) = x(T) = 0, \end{cases} \qquad t \in [0, T], \quad z = (x, y) \in \mathbb{R}^{2N} = \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}} \tag{2}$$

is studied. Here, $B(t), P(t)$ are symmetric $2N \times 2N$ matrices with $B(t)$ positive definite and $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ is the standard $2N \times 2N$ symplectic matrix. Incidentally, we remark that, in Ekeland [12], such boundary conditions are referred to as Bolza boundary conditions (see also [11]). Atkinson's analysis is based on the definition of suitable angular coordinates associated to system (2), which are strictly connected with the symplectic structure of the problem (see Remark 2.4).

The main aim of this paper is to show that Atkinson's angular coordinates can be used to establish a spectral theorem which, actually, holds true for systems having the more general form

$$\begin{cases} Jz' + f(t)Jz + P(t)z = \lambda B(t)z \\ x(0) = x(T) = 0, \end{cases} \qquad t \in [0,T], \quad z = (x,y) \in \mathbb{R}^{2N} = \mathbb{R}^N \times \mathbb{R}^N, \tag{3}$$

where $f(t)$ is a continuous function, and $B(t), P(t)$ are as above. We observe that the existence of a two-sided sequence of eigenvalues for system (3) when $f(t) \equiv 0$ is essentially proved in [3, Section 10.9]. However, we remark again that the main novelty is the highlighting of the relationship between the eigenvalues of the problem and the nodal properties of the associated eigenfunctions. Such a connection is made possible by the use of a suitable *rotation index* associated to system (3), based on the above mentioned angular coordinates and generalizing the classical concept of rotation number for planar systems. Moreover, in Remark 2.4 we will show that the rotation index is deeply related with the Maslov index [22] of the symplectic path given by the fundamental matrix associated to (3).

Our main result (Theorem 3.1) contains the theorems proved in [4, 23, 24], extending them to higher dimension. We point out that our class of systems seems to be the widest for which a spectral analysis can be carried out with the mentioned symplectic tools.

The plan of the article is the following. In Section 2 we introduce the rotation index associated to system (3), while in Section 3 we state and prove the spectral theorem. Finally, in Section 4 we survey some possible applications and we discuss some open questions concerning nonlinear systems.

Notation. We denote by $\mathcal{L}(\mathbb{R}^d)$ (resp. $\mathcal{L}(\mathbb{C}^d)$) the set of the real (resp. complex) $d \times d$ matrices. Given A in one of such sets of matrices, we denote by A^t its transpose and A^* its adjoint. Moreover, by $\mathcal{L}_s(\mathbb{R}^d)$ (resp. $\mathcal{L}_s(\mathbb{C}^d)$) we mean the subspace of the real symmetric, i.e., $A^t = A$ (resp. complex Hermitian, i.e., $A^* = A$) matrices. If $A, B \in \mathcal{L}_s(\mathbb{R}^d)$, we write $A < B$ (resp. $A \leq B$) if $B - A$ is positive definite (resp. positive semidefinite). By $\mu_{\min}(A)$ (resp. $\mu_{\max}(A)$) we denote the minimum (resp. maximum) of the eigenvalues of A.

2 The rotation index

In this section, we introduce the rotation index and prove its main properties. To deal with the non-Hamiltonian systems of the kind (3), we think that it is of some interest to carry out the discussion from the following "abstract" point of view. From now on, we will denote by

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

the standard $2N \times 2N$ symplectic matrix.

2.1 Hamiltonian-like matrices

In this subsection we introduce two classes of matrices which will play a crucial role throughout the paper.

Definition 2.1. *We will say that* $A \in \mathcal{L}(\mathbb{R}^{2N})$ *is* Hamiltonian-like *if there exists* $c \in \mathbb{R}$ *such that*

$$(AJ)^t = AJ + cJ.$$

The set of Hamiltonian-like matrices will be denoted by $\mathfrak{m}(\mathbb{R}^{2N})$.

This class extends the usual notion of Hamiltonian matrix, which corresponds to the previous definition with $c = 0$ (i.e., AJ is symmetric). Roughly speaking, if A is a Hamiltonian-like matrix, AJ is not necessarily symmetric, but it is allowed to "move away" from its transpose only along the J-direction. In fact, it is easily seen that $\mathfrak{m}(\mathbb{R}^{2N})$ is generated by the Hamiltonian matrices and the identity matrix. Moreover, writing $A \in \mathcal{L}(\mathbb{R}^{2N})$ in the block structure

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix},$$

it turns out that A is Hamiltonian-like if and only if

$$A_1 = -A_4^t + cI, \quad A_2 = A_2^t, \quad A_3 = A_3^t.$$

Let us list some simple properties of Hamiltonian-like matrices.

Proposition 2.2. *The following properties hold true:*

- $\mathfrak{m}(\mathbb{R}^{2N})$ *is a linear subspace of* $\mathcal{L}(\mathbb{R}^{2N})$, *having dimension* $2N^2 + N + 1$;
- *if* $A \in \mathfrak{m}(\mathbb{R}^{2N})$, *then* $A^t \in \mathfrak{m}(\mathbb{R}^{2N})$;
- *if* $A, B \in \mathfrak{m}(\mathbb{R}^{2N})$, *then* $[A, B] := AB - BA \in \mathfrak{m}(\mathbb{R}^{2N})$.

The easy proof is omitted. As a consequence of the third property, $\mathfrak{m}(\mathbb{R}^{2N})$ is naturally endowed with a Lie algebra structure.
Next, we introduce a second class of matrices which extends the notion of symplectic matrix.

Definition 2.3. *We will say that* $C \in \mathcal{L}(\mathbb{R}^{2N})$ *is* symplectic-like *if there exists* $d \in \mathbb{R} \setminus \{0\}$ *such that*

$$C^t JC = dJ. \tag{4}$$

The set of symplectic-like matrices will be denoted by $\mathfrak{M}(\mathbb{R}^{2N})$.

Notice that, by Binet formula, (4) implies that $d > 0$. Clearly, the symplectic group $Sp(\mathbb{R}^{2N})$ corresponds to the choice $d = 1$.

Proposition 2.4. *The following properties hold true:*

- $\mathfrak{M}(\mathbb{R}^{2N})$ *is a group with respect to the usual product of matrices;*
- *if* $A \in \mathfrak{M}(\mathbb{R}^{2N})$*, then* $A^t \in \mathfrak{M}(\mathbb{R}^{2N})$*.*

Lastly, let us observe that $\mathfrak{M}(\mathbb{R}^{2N})$ has a natural differentiable structure of dimension $2N^2 + N + 1$ such that the map

$$Sp(\mathbb{R}^{2N}) \times (0, +\infty) \to \mathfrak{M}(\mathbb{R}^{2N}), \quad (A, k) \mapsto \sqrt{k}A$$

is a diffeomorphism, with inverse map $A \mapsto \left(\frac{A}{\sqrt{k}}, k\right)$, being $(AJ)^t = AJ + kJ$. Moreover,

$$\mathrm{T}_I\mathfrak{M}(\mathbb{R}^{2N}) = \mathfrak{m}(\mathbb{R}^{2N}),$$

being $\mathrm{T}_I\mathfrak{M}(\mathbb{R}^{2N})$ the tangent space to the manifold $\mathfrak{M}(\mathbb{R}^{2N})$ at the point $I \in \mathfrak{M}(\mathbb{R}^{2N})$.

2.2 Phase angles and definition of the index

In this subsection, we consider the problem

$$\begin{cases} Jz' + A(t)Jz = S(t)z \\ x(0) = x(T) = 0, \end{cases} \qquad t \in [0, T], \quad z = (x, y) \in \mathbb{R}^{2N} = \mathbb{R}^N \times \mathbb{R}^N, \tag{5}$$

with $A \in C([0, T]; \mathfrak{m}(\mathbb{R}^{2N}))$ and $S \in C([0, T]; \mathcal{L}_s(\mathbb{R}^{2N}))$. From now on, we will write

$$(A(t)J)^t = A(t)J + c(t)J. \tag{6}$$

Our aim is to define an index associated to problem (5). Let

$$\Psi(t) = \begin{pmatrix} X_0(t) & X(t) \\ Y_0(t) & Y(t) \end{pmatrix} \in \mathcal{L}(\mathbb{R}^{2N})$$

be the solution of the Cauchy problem

$$\begin{cases} J\Psi' + A(t)J\Psi = S(t)\Psi \\ \Psi(0) = I. \end{cases} \tag{7}$$

Proposition 2.5. *The matrix* $\Psi(t)$ *is symplectic-like for every* $t \in [0, T]$*; in particular,*

$$\Psi(t)^t J \Psi(t) = e^{\int_0^t c(s)\,ds} J. \tag{8}$$

Proof. Using (6) and (7), we get

$$(\Psi(t)^t J \Psi(t))' = \Psi'(t)^t J \Psi(t) + \Psi(t)^t J \Psi'(t) = c(t)\Psi(t)^t J \Psi(t),$$

from which the claim follows. □

Notice that, in the particular case $A(t) \equiv 0$ (and hence $c(t) \equiv 0$), namely in the Hamiltonian case, we recover the well known fact (see, for instance, [12, Proposition 3]) that $\Psi(t)$ is a symplectic matrix for every $t \in [0, T]$.

We now observe that relation (8) implies that

$$Y(t)^t X(t) = X(t)^t Y(t), \tag{9}$$

as it is easily seen writing explicitly the expression of $\Psi(t)$. As a consequence, if we define the complex matrix

$$\Theta(t) := (Y(t) + iX(t))(Y(t) - iX(t))^{-1} \in \mathcal{L}(\mathbb{C}^N), \tag{10}$$

it turns out that $\Theta(t)$ is unitary for every $t \in [0, T]$. This fact is fundamental in order to define the rotation index. Indeed, it is possible to associate to every unitary matrix some kind of angular coordinates, as shown in the definition below.

Definition 2.6. *According to [3, Appendix V.4], we define the* phase angles *associated to system (5) as the unique N continuous functions $\theta_1(t), \ldots, \theta_N(t) : [0, T] \to \mathbb{R}$ such that:*

1) for every $t \in [0, T]$, the numbers $e^{i\theta_1(t)}, \ldots, e^{i\theta_N(t)}$ are the eigenvalues of $\Theta(t)$;

2) $\theta_1(0) = \cdots = \theta_N(0) = 0$;

3) for every $t \in [0, T]$, it holds

$$\theta_1(t) \le \theta_2(t) \le \cdots \le \theta_N(t) \le \theta_1(t) + 2\pi.$$

We stress that the fact that $\Theta(t)$ is unitary also for $A(t) \in \mathfrak{m}(\mathbb{R}^{2N}) \setminus \{0\}$ is the key point for the extension of Atkinson's analysis to the non-Hamiltonian class of systems (5).

Remark 2.1. Condition 3) is the main point of Atkinson's definition. In fact, the existence of N continuous functions $\varphi_1, \ldots, \varphi_N$ satisfying conditions 1) and 2) of Definition 2.6 is ensured by Kato's selection Theorem [17, Chapter II, Theorem 5.2] and the Unique Path Lifting Theorem. In [3, Appendix V.5], it is shown that there is a unique way of 'relabelling" and "interchanging" such functions in order to have 3) fulfilled. For a concrete example, see Remark 2.3.

Definition 2.7. *We call* rotation index *of problem (5) the integer*

$$i(A, S) := \sum_{j=1}^{N} \left\lceil \frac{\theta_j(T)}{2\pi} \right\rceil,$$

where $\lceil a \rceil$, for $a \in \mathbb{R}$, is the greatest integer less or equal to a.

We remark that a similar object, in the case $A(t) \equiv 0$, was defined in [19, Theorem 3.3].

Remark 2.2. Let $N = 1$. Denoting by $\varphi(t)$ the angular coordinate of the unique solution of the Cauchy problem

$$\begin{cases} Jz' + A(t)Jz = S(t)z \\ z(0) = (0,1), \end{cases} \tag{11}$$

satisfying $\varphi(0) = \frac{\pi}{2}$, i.e., $z(t) = \rho(t)e^{i\varphi(t)}$, it is easy to see that the relation involving the (unique) phase angle $\theta(t)$ and $\varphi(t)$ is given by

$$\varphi(t) = \frac{\pi - \theta(t)}{2}.$$

In particular, denoting by

$$\mathrm{Rot}\,(z) := -\frac{1}{2\pi}\int_0^T \frac{Jz'(t)\cdot z(t)}{|z(t)|^2}\,dt$$

the classical rotation number of a C^1-path $z : [0,T] \to \mathbb{R}^2 \setminus \{0\}$, it holds that

$$i(A,S) = \lceil -2\mathrm{Rot}\,(z^*)\rceil, \tag{12}$$

where $z^*(t)$ is the solution of (11). Hence, the rotation index is the natural generalization of the classical rotation number to higher dimension. By the way, notice that, since $\mathfrak{m}(\mathbb{R}^2) = \mathcal{L}(\mathbb{R}^2)$, every first order linear planar system is of the type $Jz' + A(t)Jz = S(t)z$.

We conclude this subsection with two remarks: the first one is about the definition of the phase angles, while the second one highlights the symplectic structure of the considered problem.

Remark 2.3. Some attention is needed when dealing with the phase angles. Indeed, for instance in the case of a system of uncoupled second order equations, the phase angles obtained through Atkinson's construction do not coincide with other "natural" angular functions. To better explain this concept, we consider the following example. Let $N = 2$ and $\omega_1, \omega_2 > 0$, and consider the system

$$\begin{cases} u'' + \omega_1^2 u = 0 \\ v'' + \omega_2^2 v = 0. \end{cases} \tag{13}$$

System (13) is equivalent to

$$\begin{cases} x_1' = y_1 \\ y_1' = -\omega_1^2 x_1 \\ x_2' = y_2 \\ y_2' = -\omega_2^2 x_2, \end{cases}$$

which is of the form

$$Jz' + A(t)Jz = S(t)z, \qquad z = (x_1, x_2, y_1, y_2) \in \mathbb{R}^4,$$

with

$$A(t) \equiv 0, \quad S(t) \equiv S = \operatorname{diag}(\omega_1^2, \omega_2^2, 1, 1).$$

A straight computation gives

$$\Theta(t) = \operatorname{diag}\left(\frac{\cos(\omega_1 t) + \frac{i}{\omega_1}\sin(\omega_1 t)}{\cos(\omega_1 t) - \frac{i}{\omega_1}\sin(\omega_1 t)}, \frac{\cos(\omega_2 t) + \frac{i}{\omega_2}\sin(\omega_2 t)}{\cos(\omega_2 t) - \frac{i}{\omega_2}\sin(\omega_2 t)} \right).$$

Set $T = 4\pi$, $\omega_1 = \frac{1}{4}$ and $\omega_2 = 1$. Defining, in a natural way,

$$\varphi_1(t) := \operatorname{arccot}(\omega_1 \cot(\omega_1 t)), \qquad \varphi_2(t) := t,$$

we see that conditions 1) and 2) of Definition 2.6 are satisfied. However, condition 3) is not fulfilled, since, for instance, $\varphi_1(4\pi) + 2\pi = 3\pi < \varphi_2(4\pi) = 4\pi$. The representation for $\varphi_1(t)$ and $\varphi_2(t)$ is plotted in Figure 1 with MAPLE® software.

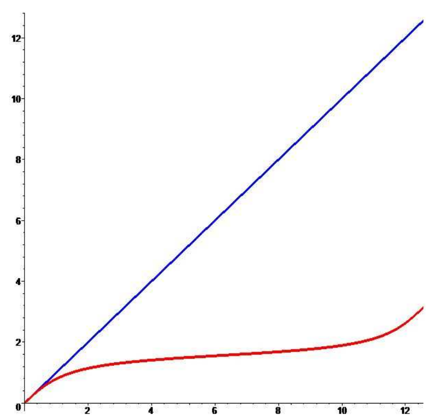

Figure 1: The "wrong" phase angles φ_1 (below) and φ_2 (above)

Letting now $t^* \in [0, T]$ such that $\varphi_1(t^*) + 2\pi = \varphi_2(t^*)$, set

$$\theta_1(t) := \begin{cases} \varphi_1(t) & 0 \le t \le t^* \\ \varphi_2(t) - 2\pi & t^* \le t \le T, \end{cases} \qquad \theta_2(t) := \begin{cases} \varphi_2(t) & 0 \le t \le t^* \\ \varphi_1(t) + 2\pi & t^* \le t \le T. \end{cases}$$

With such a definition, the conditions of Definition 2.6 are all satisfied; by uniqueness, $\theta_1(t)$ and $\theta_2(t)$ are the phase angles associated to system (13). The right representation for the phase angles is given in Figure 2.

The idea is that one has to "interchange" the graphs of $\varphi_1(t)$ and $\varphi_2(t)$ whenever $\varphi_2 - \varphi_1 = 2\pi$ (relabelling the two functions, if necessary). The same reasoning works for $N > 2$, comparing φ_1 and φ_N.

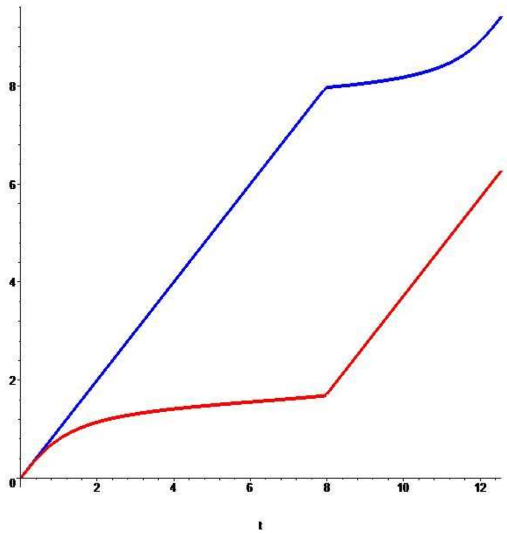

Figure 2: The "correct" phase angles θ_1 (below) and θ_2 (above)

Remark 2.4. First of all, remember that $\mathbb{R}^{2N}$ is canonically endowed with the sympletic structure given by the bilinear antisymmetric form

$$\omega(z_1, z_2) = Jz_1 \cdot z_2 = x_1 \cdot y_2 - x_2 \cdot y_1,$$

where $z_1 = (x_1, y_1) \in \mathbb{R}^N \times \mathbb{R}^N$ and $z_2 = (x_2, y_2) \in \mathbb{R}^N \times \mathbb{R}^N$; moreover, a N-dimensional subspace $L \subset \mathbb{R}^{2N}$ is called *Lagrangian* if $\omega(z_1, z_2) = 0$ for every $z_1, z_2 \in L$. We now recall the definition of the Maslov index for a path of Lagrangian subspaces, referring to [22] for further details. Let W be a fixed Lagrangian subspace of $\mathbb{R}^{2N}$, and $L(t)$, $t \in [0, T]$, a path (smooth in the Lagrangian Grassmanian) of Lagrangian subspaces, too; we say that $t_* \in [0, T]$ is a *crossing* for the pair $(L(t), W)$ if

$$L(t_*) \cap W \neq \{0\}.$$

For every crossing t_*, we fix a Lagrangian complement $M(t_*)$ of $L(t_*)$ (i.e., a Lagrangian subspace such that $L(t_*) \cap M(t_*) = \{0\}$) and we define the quadratic form $\Gamma(L(t), W, t_*) : L(t_*) \cap W \to \mathbb{R}$ as

$$\Gamma(L(t), W, t_*)(z) := \frac{d}{dt}\omega(z, m(t))\Big|_{t=t_*},$$

where, for t in a neighborhood of t_*, $m(t)$ is the unique vector belonging to $M(t_*)$ such that $z + m(t) \in L(t)$. It can be seen that Γ does not depend on the choice of $M(t_*)$. If $L(t)$ and W always intersect transversally (i.e., if $\Gamma(L(t), W, t_*)$ is nondegenerate for every crossing t_*), the *Maslov index* of the path $L(t)$, relative to

W, is defined by

$$\mu(L(t), W, [0,T]) := \frac{1}{2}\,\mathrm{sign}\,\Gamma(L(0), W, 0) + \sum_{t_* \in\,]0,T[} \mathrm{sign}\,\Gamma(L(t_*), W, t_*)$$
$$+ \frac{1}{2}\,\mathrm{sign}\,\Gamma(L(T), W, T),$$

where "sign" denotes the signature of a quadratic form, and the sum is taken only over the crossing instants. Of course, we agree that $\mathrm{sign}\,\Gamma(L(0), W, 0) = 0$ if 0 is not a crossing, and the same for $\mathrm{sign}\,\Gamma(L(T), W, T)$. The assumption of transverse intersection is of generic type, and the definition can be extended in a standard way to the general case (even for a nonsmooth path).

With such preliminaries, we now focus on the symplectic structure of problem (5), and we show the relation between the Maslov index and the phase angles. Observe first that the boundary conditions in (5) can be expressed as

$$z(0), z(T) \in V,$$

being $V = \{0\} \times \mathbb{R}^N$ the vertical Lagrangian subspace. Such a Lagrangian structure is preserved by the flow associated to (5), in the sense that $\Psi(t)V$ is still a Lagrangian subspace for every t, thanks to the fact that $\Psi(t)$ is a symplectic-like matrix for every t (use relation (9)). Then, we can consider the Maslov index $\mu(\Psi(t)V, V, [0,T])$. With the same arguments in [7, Proposition 3.12], it is possible to see that, defining

$$\Lambda(t) := \Big\{ j \in \{1, \ldots, N\} \mid \theta_j(t) \equiv 0 \bmod 2\pi \Big\},$$

we have

$$\dim(\Psi(t)V \cap V) = \#\,\Lambda(t).$$

In particular, t_* is a crossing for the pair $(\Psi(t)V, V)$ if and only if $\Lambda(t_*) \neq \emptyset$, that is,

$$\theta_j(t_*) \equiv 0 \bmod 2\pi$$

for at least one $j \in \{1, \ldots, N\}$.

Moreover, the shape of the graph of such θ_j's in a neighborhood of t_* is strictly related with the signature of $\Gamma(\Psi(t)V, V, t_*)$. Precisely, if $\theta_j(t_*) = 2r_j(t_*)\pi$ for $j \in \Lambda(t_*)$, with $r_j(t_*) \in \mathbb{Z}$, from [8, Proposition 5] it follows that

$$\mu(\Psi(t)V, V, [0,T]) = \frac{1}{2}\Big[\kappa^+(0) + \sum_{t_* \in\,]0,T[} \Big(\kappa^+(t_*) + \kappa^-(t_*)\Big) + \kappa^-(T)\Big], \tag{14}$$

with

$$\kappa^+(t_*) := \lim_{t \to t_*^+} \#\{j \in \Lambda(t_*) \mid \theta_j(t) > 2r_j(t_*)\pi\}$$
$$- \lim_{t \to t_*^+} \#\{j \in \Lambda(t_*) \mid \theta_j(t) < 2r_j(t_*)\pi\},$$

$$\kappa^-(t_*) := \lim_{t\to t_*^-} \#\{j \in \Lambda(t_*) \mid \theta_j(t) < 2r_j(t_*)\pi\}$$
$$- \lim_{t\to t_*^-} \#\{j \in \Lambda(t_*) \mid \theta_j(t) > 2r_j(t_*)\pi\},$$

(of course, $\kappa^+(0)$ is defined like this if 0 is a crossing, and is set equal to 0 otherwise; the same for $\kappa^-(T)$). Finally, referring to (5), we remark that, if $A(t) \equiv 0$ and $S(t)$ is positive definite (namely, when we consider a convex linear Hamiltonian system), it follows from [3, Theorem 10.8.1] that each phase angle $\theta_j(t)$ is strictly increasing on $[0,T]$, so that (14) implies

$$\begin{aligned}\mu(\Psi(t)V, V, [0,T]) &= \frac{N}{2} + \Big(\sum_{t_* \in]0,T[} \dim(\Psi(t_*)V \cap V)\Big) + \frac{1}{2}\dim(\Psi(T)V \cap V)\\ &= \frac{N}{2} - \frac{1}{2}\dim(\Psi(T)V \cap V) + i(A,S).\end{aligned}$$

This relation shows that, in the convex case, the Maslov index, the rotation index and the number of moments of verticality (see [2] and [7, Definition 3.6]) are "the same". In this framework, it is also well known that such objects are deeply related with the Morse index of a suitable operator in an infinite-dimensional Hilbert space (see [12, Theorem 6]).

2.3 Properties of the rotation index

In this subsection we establish some properties of the rotation index; similar computations can be found in [3].

Lemma 2.8. *Let $A \in C([0,T]; \mathfrak{m}(\mathbb{R}^{2N}))$. The following properties hold true:*

- Normalization*: if $S \in C([0,T]; \mathcal{L}_s(\mathbb{R}^{2N}))$, then*

$$i(A,S) = i\Big(0, S - AJ - \frac{c}{2}J\Big),$$

with $c(t)$ as in (6);

- Monotonicity*: if $S_1, S_2 \in C([0,T]; \mathcal{L}_s(\mathbb{R}^{2N}))$, then*

$$S_1(t) \le S_2(t) \ \text{ for every } t \in [0,T] \quad \Longrightarrow \quad i(A,S_1) \le i(A,S_2);$$

- Divergence*: if $(S_k)_k \subset C([0,T]; \mathcal{L}_s(\mathbb{R}^{2N}))$, then*

$$\begin{cases} \mu_{\min}(S_k(t)) \ge 0 \text{ for every } t \in [0,T] \\ \displaystyle\int_0^T \mu_{\min}(S_k(t))\, dt \to +\infty \end{cases} \quad \Longrightarrow \quad i(A,S_k) \to +\infty,$$

and

$$\left\{\begin{array}{l}\mu_{\max}(S_k(t)) \leq 0 \text{ for every } t \in [0,T] \\ \displaystyle\int_0^T \mu_{\max}(S_k(t))\,dt \to -\infty\end{array}\right. \qquad \Longrightarrow \qquad i(A, S_k) \to -\infty.$$

Notice that the term appearing on the right-hand side of the normalization formula is well defined, since $S(t) - A(t)J - \frac{c(t)}{2}J$ is a symmetric matrix for every $t \in [0,T]$. Hence, the normalization property allows to compute the index reducing to the Hamiltonian case.

Proof. Let $\Psi(t)$, $\widetilde{\Psi}(t)$ be the solutions of the Cauchy problems

$$\left\{\begin{array}{l} J\Psi' + A(t)J\Psi = S(t)\Psi \\ \Psi(0) = I, \end{array}\right. \qquad \left\{\begin{array}{l} J\widetilde{\Psi}' = \Big(S(t) - A(t)J - \frac{c}{2}J\Big)\widetilde{\Psi} \\ \widetilde{\Psi}(0) = I, \end{array}\right.$$

respectively; then, it is easy to verify that

$$\Psi(t) = e^{\frac{1}{2}\int_0^t c(s)\,ds}\widetilde{\Psi}(t).$$

Letting

$$\Psi(t) = \begin{pmatrix} X_0(t) & X(t) \\ Y_0(t) & Y(t) \end{pmatrix}, \qquad \widetilde{\Psi}(t) = \begin{pmatrix} \widetilde{X_0}(t) & \widetilde{X}(t) \\ \widetilde{Y_0}(t) & \widetilde{Y}(t) \end{pmatrix},$$

it follows that

$$\Theta(t) := (Y(t) + iX(t))(Y(t) - iX(t))^{-1} = (\widetilde{Y}(t) + i\widetilde{X}(t))(\widetilde{Y}(t) - i\widetilde{X}(t))^{-1} =: \widetilde{\Theta}(t).$$

Hence, the phase angles of $\widetilde{\Theta}(t)$ are the same as the ones for $\Theta(t)$, and the normalization formula follows.

As a consequence, throughout the proofs of the monotonicity and divergence properties of the index, we will suppose, without loss of generality, that $A(t) \equiv 0$.

First, we prove the monotonicity property. Let $S_1, S_2 \in \mathcal{L}_s(\mathbb{R}^{2N})$ with $S_1 \leq S_2$; we define, for $\sigma \in [0,1]$,

$$f(\sigma) := i(0, S_1 + \sigma(S_2 - S_1)).$$

It is sufficient to prove that f is nondecreasing. To this aim, let

$$\Psi(t,\sigma) = \begin{pmatrix} X_0(t,\sigma) & X(t,\sigma) \\ Y_0(t,\sigma) & Y(t,\sigma) \end{pmatrix}$$

be the solution of the Cauchy problem

$$\left\{\begin{array}{l} J\Psi' = (S_1(t) + \sigma(S_2(t) - S_1(t)))\Psi \\ \Psi(0,\sigma) = I, \end{array}\right.$$

and

$$\Theta(t,\sigma) := (Y(t,\sigma) + iX(t,\sigma))(Y(t,\sigma) - iX(t,\sigma))^{-1}.$$

According to [3, Appendix V.5], the phase angles $\theta_j(t,\sigma)$ are continuous functions of (t,σ). A simple computation yields

$$\Theta^* D_\sigma \Theta = 2i[(Y - iX)^{-1}]^* (Y^t D_\sigma X - X^t D_\sigma Y)(Y - iX)^{-1}, \tag{15}$$

and it is easy to see that

$$\Psi^t J D_\sigma \Psi = \begin{pmatrix} * & * \\ * & (Y^t D_\sigma X - X^t D_\sigma Y) \end{pmatrix}.$$

Being

$$(\Psi^t J D_\sigma \Psi)' = -(J\Psi')^t D_\sigma \Psi + \Psi^t J (D_\sigma \Psi)', \tag{16}$$

we evaluate

$$\begin{aligned} (D_\sigma \Psi)' = D_\sigma(\Psi') &= D_\sigma[-J(S_1(t) + \sigma(S_2(t) - S_1(t)))\Psi] \\ &= -J(S_2(t) - S_1(t))\Psi - J(S_1(t) + \sigma(S_2(t) - S_1(t)))D_\sigma \Psi. \end{aligned}$$

Substituting into (16) yields

$$(\Psi^t J D_\sigma \Psi)' = \Psi^t (S_2(t) - S_1(t))\Psi,$$

which, together with the initial condition $\Psi^t J D_\sigma \Psi(0,\sigma) = 0$, implies

$$\Psi(t,\sigma)^t J D_\sigma \Psi(t,\sigma) = \int_0^t \Psi(s,\sigma)^t (S_2(s) - S_1(s)) \Psi(s,\sigma)\, ds,$$

and hence

$$\begin{aligned} &(Y(t,\sigma)^t D_\sigma X(t,\sigma) - X(t,\sigma)^t D_\sigma Y(t,\sigma)) = \\ &= \int_0^t \begin{pmatrix} X(s,\sigma) \\ Y(s,\sigma) \end{pmatrix}^t (S_2(s) - S_1(s)) \begin{pmatrix} X(s,\sigma) \\ Y(s,\sigma) \end{pmatrix} ds. \end{aligned}$$

Consequently, recalling (15), $\Theta(t,\sigma)$ solves the differential equation

$$D_\sigma \Theta(t,\sigma) = i\Theta(t,\sigma)\Omega(t,\sigma),$$

where, being

$$L(t,\sigma) = (Y(t,\sigma) - iX(t,\sigma))^{-1},$$

$\Omega(t,\sigma)$ is the complex Hermitian matrix

$$\Omega(t,\sigma) = 2L(t,\sigma)^* \left[\int_0^t \begin{pmatrix} X(s,\sigma) \\ Y(s,\sigma) \end{pmatrix}^t (S_2(s) - S_1(s)) \begin{pmatrix} X(s,\sigma) \\ Y(s,\sigma) \end{pmatrix} ds \right] L(t,\sigma).$$

¿From the positive semidefiniteness of $S_2(s) - S_1(s)$, we infer the positive semidefiniteness of $\Omega(T, \sigma)$. By [3, Theorem V.6.1], it follows that the phase angles $\theta_j(T, \cdot)$ are nondecreasing, giving the conclusion.
We now prove the divergence property. By standard computations, letting $\Psi_k(t)$ be the solution of the Cauchy problem

$$\begin{cases} J\Psi_k' = S_k(t)\Psi_k \\ \Psi_k(0) = I, \end{cases}$$

and $\Theta_k(t)$ the associated unitary matrix (defined as in (10)), it is easily seen that $\Theta_k(t)$ satisfies the differential equation

$$\Theta_k(t)' = i\Theta_k(t)\Omega_k(t),$$

where $\Omega_k(t)$, in view of (9), is the complex Hermitian $N \times N$ matrix

$$\Omega_k(t) = 2M_k(t)^* S_k(t) M_k(t),$$

being

$$M_k(t) = \begin{pmatrix} X_k(t) \\ Y_k(t) \end{pmatrix} L_k(t),$$

with $L_k(t) = (Y_k(t) - iX_k(t))^{-1}$. By [3, Theorem V.6.3], for every $j = 1, \dots, N$,

$$\int_0^T \mu_{\min}(\Omega_k(t))\, dt \le \theta_{j,k}(T) \le \int_0^T \mu_{\max}(\Omega_k(t))\, dt.$$

We claim that $\int_0^T \mu_{\min}(\Omega_k(t))\, dt \to \infty$ for $k \to \infty$, which implies the conclusion. We first notice that

$$M_k(t) = \frac{1}{2} \begin{pmatrix} i(I - \Theta_k(t)) \\ I + \Theta_k(t) \end{pmatrix}. \tag{17}$$

By variational characterization,

$$\begin{aligned} \int_0^T \mu_{\min}(M_k(t)^* S_k(t) M_k(t))\, dt &= \int_0^T \min_{|v|=1} (M_k(t)^* S_k(t) M_k(t) v, v)\, dt \\ &= \int_0^T \min_{|v|=1} (S_k(t) M_k(t) v, M_k(t) v)\, dt \\ &\ge \int_0^T \mu_{\min}(S_k(t)) \min_{|v|=1} |M_k(t) v|^2\, dt; \end{aligned}$$

thus, it is sufficient to prove that there exists $C > 0$ such that, for every $t \in [0, T]$ and $|v| = 1$, it holds

$$|M_k(t) v| \ge C.$$

By contradiction, suppose that there exist $t_k \in [0,T]$, $v_k \to v$ such that

$$|M_k(t_k)v_k| \to 0.$$

By (17), we then deduce that

$$(I \pm \Theta_k(t_k))v_k \to 0. \tag{18}$$

However, since $\|\Theta_k(t_k)\| = 1$ for every $k \in \mathbb{N}$, there exists $\Theta_\infty \in \mathcal{L}(\mathbb{C}^N)$ such that, up to subsequences, $\Theta_k(t_k) \to \Theta_\infty$ in $\mathcal{L}(\mathbb{C}^N)$. Consequently, by the boundedness of v_k, we have that $\Theta_k(t_k)v_k \to \Theta_\infty v$. In view of (18), this implies both $v = \Theta_\infty v$ and $v = -\Theta_\infty v$, so that $v = 0$, a contradiction.
By a completely symmetric argument, we can show that if $\int_0^T \mu_{\max}(S_k(t))\,dt \to -\infty$, then $\int_0^T \mu_{\max}(\Omega_k(t))\,dt \to -\infty$, giving the conclusion. □

Remark 2.5. It is worth noticing that, if $S_2(t) > S_1(t)$ on a nonempty interval, letting $\theta_j^{S_1}(t)$ and $\theta_j^{S_2}(t)$ be the phase angles associated to the systems

$$\begin{cases} Jz' + A(t)Jz = S_1(t)z \\ x(0) = x(T) = 0, \end{cases} \qquad \begin{cases} Jz' + A(t)Jz = S_2(t)z \\ x(0) = x(T) = 0, \end{cases}$$

respectively, the same arguments used above show that $\theta_j^{S_2}(T) > \theta_j^{S_1}(T)$. This fact will be used in the proof of Theorem 3.1.

Remark 2.6. The above lemma is a partial analogous of [4, Lemma 2.1]. However, it is worth noticing that, in general, the continuity property is no longer true and only an upper-semicontinuity property holds; precisely

$$S_k(t) \to S(t), \quad A_k(t) \to A(t) \quad \Longrightarrow \quad i(A,S) \geq \limsup_{k\to+\infty} i(A_k,S_k).$$

Indeed, the continuous dependence for the solutions to the Cauchy problems implies that (with obvious notations) $\Theta_k(t) \to \Theta(t)$ uniformly in $t \in [0,T]$; from [3, Appendix V.5], it follows that $\theta_{k,j}(T) \to \theta_j(T)$ for every $j = 1,\dots,N$. The claim is now a consequence of the upper-semicontinuity of $\lceil\cdot\rceil$.

3 The spectral theorem

In this section we prove the main result of this paper. For a comment about formula (20), which is the main point of our result, we refer to Remark 3.1.

Theorem 3.1. *Let us assume that $f \in C([0,T],\mathbb{R})$ and $B, P \in C([0,T];\mathcal{L}_s(\mathbb{R}^{2N}))$, with $B(t)$ positive semidefinite for every $t \in [0,T]$, and positive definite on a nonempty interval. Then the linear boundary value problem*

$$\begin{cases} Jz' + f(t)Jz + P(t)z = \lambda B(t)z \\ x(0) = x(T) = 0, \end{cases} \qquad t \in [0,T], \quad z = (x,y) \in \mathbb{R}^{2N} = \mathbb{R}^N \times \mathbb{R}^N, \tag{19}$$

has a countable set of eigenvalues, with no accumulation points. Moreover, the eigenvalues can be ordered, if repeated according to their multiplicity, in a two-sided sequence $(\lambda_k)_{k\in\mathbb{Z}}$ *such that*

1) $\cdots \leq \lambda_{-1} \leq \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$*;*

2) $\lambda_k \to \pm\infty$ *for* $k \to \pm\infty$*;*

3) for every $k \in \mathbb{Z}$*,*

$$i(fI - PJ, \lambda_k B) = \max_{\lambda_k=\lambda_j} j. \tag{20}$$

Notice that system (19) is of the form (5), since $f(t)I - P(t)J$ is Hamiltonian-like for every $t \in [0, T]$. Actually, the two forms are equivalent, but here we choose the form (19) in order to better compare Theorem 3.1 with the results in [4, 23, 24] (see Remark 3.1 below).

Proof. We introduce the phase angles $\theta_1(t, \lambda), \ldots, \theta_N(t, \lambda)$ associated to (19) as in the previous section; from [3, Appendix V.5], it follows that $\theta_j(t, \lambda)$ is continuous on $[0, T] \times \mathbb{R}$, for every $j = 1, \ldots, N$. Arguing as in [7, Proposition 3.12], it is possible to see that λ is an eigenvalue of problem (19), with multiplicity m, if and only if there exist exactly m indexes $j_1, \ldots, j_m \in \{1, \ldots, N\}$ such that, for every $r = 1, \ldots, m$,

$$\theta_{j_r}(T, \lambda) \equiv 0 \bmod 2\pi.$$

We set, for $\lambda \in \mathbb{R}$,

$$g(\lambda) := i(fI - PJ, \lambda B) = \sum_{j=1}^{N} \left\lceil \frac{\theta_j(T, \lambda)}{2\pi} \right\rceil.$$

Clearly, g is integer-valued and, in view of Remark 2.6, upper semicontinuous. Moreover, by the monotonicity and divergence properties of the rotation index proved in Lemma 2.8, g is nondecreasing and satisfies

$$\lim_{\lambda\to\pm\infty} g(\lambda) = \pm\infty. \tag{21}$$

Define, for $k \in \mathbb{Z}$,

$$A_k := \{\lambda \in \mathbb{R} \mid g(\lambda) \geq k\}.$$

In view of (21), each A_k is nonempty and lower bounded; moreover, $A_{k+1} \subseteq A_k$ for every $k \in \mathbb{Z}$. We set

$$\lambda_k := \inf A_k;$$

by the previous remarks, $-\infty < \lambda_k \leq \lambda_{k+1}$ and $\lambda_k \to \pm\infty$ for $k \to \pm\infty$. We begin to prove formula (20); by upper semicontinuity of g, for every $j \in \mathbb{Z}$ it holds $g(\lambda_j) \geq j$; in particular, this implies

$$g(\lambda_k) \geq \max_{\lambda_k=\lambda_j} j.$$

On the other hand, if $\lambda_k = \lambda_{k+1} = \cdots = \lambda_{k+m} < \lambda_{k+m+1}$ and $g(\lambda_k) > k+m$, then

$$g(\lambda_k) \geq k+m+1,$$

so $\lambda_k \in A_{k+m+1}$, yielding the contradiction $\lambda_{k+m+1} \leq \lambda_k$.
We now prove that $\{\lambda_k\}_{k\in\mathbb{Z}}$ is exactly the set of the eigenvalues of problem (19), each of them repeated according to its multiplicity. Suppose that $\lambda_{k-m} < \lambda_{k-m+1} = \cdots = \lambda_k < \lambda_{k+1}$; we claim that λ_k is an eigenvalue of multiplicity m. Indeed, $g(\lambda_k) - g(\lambda_k^-) = m$; writing, for $j = 1, \dots, N$,

$$\theta_j(T, \lambda_k) = 2k_j\pi + \alpha_j, \quad k_j \in \mathbb{Z}, \quad 0 \leq \alpha_j < 2\pi,$$

by continuity and strict monotonicity of $\theta_j(T, \cdot)$ (see Remark 2.5), we get that

$$m = g(\lambda_k) - g(\lambda_k^-) = \#\{\alpha_j = 0\}.$$

Hence, for exactly m integers $j_1, \dots, j_m$ we have

$$\theta_{j_r}(T, \lambda_k) \equiv 0 \bmod 2\pi,$$

which implies the conclusion. Conversely, if λ_* is an eigenvalue of problem (19) with multiplicity m, with a similar argument we see that $g(\lambda_*) - g(\lambda_*^-) = m$, which implies that

$$\inf A_{g(\lambda_*)-m} < \lambda_* = \inf A_{g(\lambda_*)-m+1} = \ldots = \inf A_{g(\lambda^*)} < \inf A_{g(\lambda*)+1}\,.$$

Hence $\lambda_{g(\lambda_*)-m} < \lambda_* = \lambda_{g(\lambda_*)-m+1} = \ldots = \lambda_{g(\lambda_*)} < \lambda_{g(\lambda_*)+1}$. □

Remark 3.1. For $N = 1$, that is, in the planar case, formula (20), according to (12), reduces to

$$\mathrm{Rot}\,(z_k) = -\frac{k}{2},$$

where $z_k(t)$ is the (unique, up to a multiplicative constant) eigenfunction associated to the eigenvalue λ_k. Hence, Theorem 3.1 contains, and extends to higher dimension, [24, Theorems 2.7-2.8], as well as [4, Theorem 2.2] and the results in [25] concerning the general Dirac operator (1). When $N > 1$, formula (20) has still to be meant as an information about the nodal properties of the eigenfunctions, since the Maslov index (see Remark 2.4) can also be viewed as a winding number of the fundamental matrix associated to (19) in the symplectic group, [20].

Remark 3.2. As a consequence of (20), we remark that $\lambda \mapsto i(fI - PJ, \lambda B)$ is strictly increasing, for λ varying in the set of the eigenvalues. In the particular case when all the eigenvalues are simple, for instance in the planar setting, it reduces to $i(fI - PJ, \lambda_k B) = k$.

Remark 3.3. We point out that the entire above discussion can be carried out in the Carathéodory setting, provided that the solutions are meant in the generalized sense (i.e., absolutely continuous functions which satisfy the differential equation almost everywhere, together with the boundary conditions).

4 Towards nonlinear systems

In this brief section we survey some possible existence (and multiplicity) results about nonlinear systems of the form

$$\begin{cases} Jz' + f(t)Jz + P(t)z = B(t)N(t,z) \\ x(0) = x(T) = 0, \end{cases} \qquad t \in [0,T], \quad z = (x,y) \in \mathbb{R}^{2N}, \quad (22)$$

where $f \in C([0,T];\mathbb{R})$, $P, B \in C([0,T];\mathcal{L}_s(\mathbb{R}^{2N}))$, with $B(t)$ positive definite for every $t \in [0,T]$, and $N \in C([0,T] \times \mathbb{R}^{2N};\mathbb{R}^{2N})$. Incidentally, observe that a second order system with Dirichlet boundary conditions like

$$\begin{cases} x'' + g(t,x) = 0 \\ x(0) = x(T) = 0, \end{cases} \qquad t \in [0,T], \quad x \in \mathbb{R}^N,$$

can be routinely written in the form (22). For further convenience, moreover, observe that the (unbounded) operator

$$L : \mathcal{D}(L) \subset L^2([0,T];\mathbb{R}^{2N}) \to L^2([0,T];\mathbb{R}^{2N}), \qquad z \mapsto B(t)^{-1}(Jz' + f(t)Jz + P(t)z),$$

with domain $\mathcal{D}(L) := \{z = (x,y) \in H^1([0,T];\mathbb{R}^{2N}) \mid x(0) = x(T) = 0\}$, is a Fredholm operator of index zero (see the sketch of the proof of Proposition 4.1 for further details), with spectrum $\sigma(L)$ made up of eigenvalues with finite multiplicities; precisely,

$$\sigma(L) = \{\lambda_k\}_{k\in\mathbb{Z}},$$

where λ_k is as in Theorem 3.1. Our attention will be focused on the interaction between $N(t,z)$ and $\sigma(L)$.

A first group of results is concerned with the interaction at infinity. The simplest case occurs in case of noninteraction, usually called *nonresonance*. In this framework, the following quite general proposition can be proved.

Proposition 4.1. *Assume that*

$$N(t,z) = G(t,z)z + R(t,z),$$

with $G(t,z) \in \mathcal{L}_s(\mathbb{R}^{2N})$ *for every* $t \in [0,T]$, $z \in \mathbb{R}^{2N}$, *and* $R(t,z) = o(|z|)$ *for* $|z| \to \infty$. *If there exist* $\epsilon > 0$ *and* $k \in \mathbb{Z}$ *such that*

$$(\lambda_k + \epsilon)I \leq G(t,z) \leq (\lambda_{k+1} - \epsilon)I, \qquad (23)$$

for every $t \in [0,T]$, $z \in \mathbb{R}^{2N}$, *then problem (22) has at least one solution.*

The statement follows from [14, Theorem 1], and relies essentially on the Schauder fixed point theorem. For the reader's convenience, we give a brief sketch.

Sketch of the proof. It is immediately seen that problem (22) is equivalent to the abstract functional equation

$$\tilde{L}z = \tilde{N}z,$$

where, being $\tilde{H} := L^2([0,T];\mathbb{R}^{2N})$ and $\mathcal{D}(\tilde{L}) := \{z = (x,y) \in H^1([0,T];\mathbb{R}^{2N}) \mid x(0) = x(T) = 0\}$, we have defined

$$\tilde{L}z(t) := B^{-1}(t)(Jz'(t) + f(t)Jz(t) + P(t)z(t)) - (\lambda_k + \epsilon)z(t), \quad z \in \mathcal{D}(\tilde{L})$$

and

$$\tilde{N}z(t) := (\lambda_k + \epsilon + G(t, z(t)))z(t) + R(t, z(t)), \quad z \in \tilde{H}.$$

The operator $\tilde{L}$ is self-adjoint with respect to the scalar product

$$(z_1, z_2) := \int_0^T z_1(t) \cdot z_2(t) e^{2\int_0^t f(s)\,ds}\,dt,$$

and it has closed range and compact resolvent. On the other hand, $\tilde{N} : \tilde{H} \to \tilde{H}$ is continuous and maps bounded sets into bounded sets. Finally, setting

$$\tilde{S}z(t) := (\lambda_{k+1} - \lambda_k - 2\epsilon)z(t), \quad z \in \tilde{H},$$

it is obvious that $\tilde{S}$ is a positive, self-adjoint and invertible linear operator. It is now easy to see that all the hypotheses of [14, Theorem 1] are satisfied, indeed:

1) $\ker \tilde{L} = \{0\}$;

2) $\tilde{L}z = \sigma\tilde{S}z \Rightarrow z = 0$, for $0 < \sigma < \frac{\lambda_{k+1}-\lambda_k-\epsilon}{\lambda_{k+1}-\lambda_k-2\epsilon}$;

3) $\tilde{N}z = (\tilde{\Gamma}z)z + \tilde{R}z$, with $\tilde{\Gamma} : \tilde{H} \to \mathcal{L}_s(\tilde{H})$, $0 \leq \tilde{\Gamma}z \leq \tilde{S}$, and $\tilde{R}z = o(\|z\|_{\tilde{H}})$.

The conclusion follows. □

It is interesting to recall that an abstract kind of nonresonance was considered by Amann in [1], applying to first order Hamiltonian systems [1] in $\mathbb{R}^{2N}$ (see [1, Theorem 4.5]) and generalizing the results for second order systems in [18]. We remark that our linear differential operator fits in the abstract setting of [1], so that some "semi-abstract" results can be obtained for system (22); however, it seems not easy to give a concrete application on the lines of [1, Theorem 4.5], where explicit properties of the spectrum of $Jz' = \lambda z$ are exploited.
Wishing to consider the *resonant* situation, that is, when the nonlinearity interacts (at infinity) with the spectrum, it is known that some further conditions (typically,

[1]We warn the reader that most of the citations in the following are concerned with periodic boundary value problems, but all the results can be adapted, with slight modifications, to Bolza boundary problems.

of Landesman-Lazer type) are required in order to prove the existence of a solution. Referring to (22), with $N(t,z) = G(t,z)z + R(t,z)$, (double) resonance occurs, roughly speaking, when

$$\lambda_k I \leq G(t,z) \leq \lambda_{k+1} I,$$

with the same notations as in (23). For the planar case, some results in this spirit, relying essentially on suitable estimates of an angular-type coordinate, have been recently obtained in [15] for some particular cases of (22) (namely, when $f(t)$ and $B(t)$ are constant and $P(t) \equiv 0$). However, many problems arise when trying to extend them to higher dimension, since the analysis of the behavior of each one of the phase angles introduced in Section 2 is much more difficult. Again, we point out that some "semi-abstract" statements in the resonant case can be obtained from [5, 14] (these papers, however, are mainly oriented to applications to second order systems).

On the other hand, if system (22) has an equilibrium point, in the sense that $N(t,0) \equiv 0$, one usually compares the interaction of the nonlinearity with the spectrum "at 0" and "at infinity", in order to obtain *multiplicity* of solutions. We think, motivated by some results for scalar second order equations (see, for instance, [10]), that the rotation index introduced in Section 2 could be a very useful tool to measure such a gap between 0 and infinity. For first order planar systems, advances in this direction have been obtained in [4, 23, 24] by means of global bifurcation techniques, on the lines of [21]. The higher dimensional case (which leads to the study of the preservation of the index along the bifurcating branches, see [6]) is still to be investigated.

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