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Positive Solutions for Singular Sturm-Liouville Boundary Value Problems with Integral Boundary Conditions*

Xiping Liu†, Yu Xiao, Jianming Chen

College of Science, University of Shanghai for Science and Technology, Shanghai, 200093, China

Abstract: In this paper, we study the second-order nonlinear singular Sturm-Liouville boundary value problems with Riemann-Stieltjes integral boundary conditions

$$\begin{cases} -(p(t)u'(t))' + q(t)u(t) = f(t, u(t)),\ 0 < t < 1, \\ \alpha_1 u(0) - \beta_1 u'(0) = \int_0^1 u(\tau)\mathrm{d}\alpha(\tau), \\ \alpha_2 u(1) + \beta_2 u'(1) = \int_0^1 u(\tau)\mathrm{d}\beta(\tau), \end{cases}$$

where $f(t, u)$ is allowed to be singular at $t = 0, 1$ and $u = 0$. Some new results for the existence of positive solutions of the boundary value problems are obtained. Our results extend some known results from the nonsingular case to the singular case, and we also improve and extend some results of the singular cases.

Keywords: Boundary value problem; Integral boundary conditions; Positive solution; Singularity; Eigenvalue.

1 Introduction

We investigate the existence of positive solutions for the second-order nonlinear singular Sturm-Liouville boundary value problem (BVP) with integral boundary conditions

$$\begin{cases} -(p(t)u'(t))' + q(t)u(t) = f(t, u(t)),\ 0 < t < 1, \\ \alpha_1 u(0) - \beta_1 u'(0) = \int_0^1 u(\tau)\mathrm{d}\alpha(\tau), \\ \alpha_2 u(1) + \beta_2 u'(1) = \int_0^1 u(\tau)\mathrm{d}\beta(\tau), \end{cases} \tag{1.1}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in [0, +\infty)$ are constants such that $\alpha_1\alpha_2 + \alpha_1\beta_2 + \beta_1\alpha_2 > 0$. $\alpha(t), \beta(t)$ are nondecreasing on $[0, 1]$, $\int_0^1 u(\tau)\mathrm{d}\alpha(\tau)$ and $\int_0^1 u(\tau)\mathrm{d}\beta(\tau)$ denote the Riemann-Stieltjes integral of

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†E-mail address: xipingliu@163.com(X. Liu).

u with respect to α and β, respectively. $p \in C^1([0,1],(0,+\infty))$, $q \in C([0,1],[0,+\infty))$, $f \in C((0,1)\times(0,+\infty)),[0,+\infty))$ may be singular at $t=0$, $t=1$ and $u=0$.

In this paper, the integral BVP in (1.1) has a more general form where the nonlinear term $f(t,u)$ is allowed to be singular at $t=0,1$ and $u=0$. We obtain the existence criteria of at least one positive solution for BVP (1.1) in the two cases which are β_1, $\beta_2>0$ and $\beta_1=0$ or $\beta_2=0$.

Boundary value problems (BVPs) arise from applied mathematics, biology, engineering and so on. The existence of positive solutions to nonlocal BVPs has been extensively studied in recent years. There are many results on the existence of positive solutions for three-point BVPs [1, 2], m-point BVPs [3, 4].

It is well known that BVPs with Riemann-Stieltjes integral boundary conditions include two-point, three-point, multi-point and the Riemann integral BVPs as special cases. Such BVPs have attracted the attention of researchers such as [5]-[16]. In [5] and [6], the existence and uniqueness of a solution of BVPs were studied. In [7]-[16], the sufficient conditions for the existence of positive solutions of BVPs were given and many optimal results were obtained. In addition, many papers investigated the existence of solutions for the singular BVPs, for example, [1, 2, 4, 5, 6], [11]-[16]. Especially, in the papers above, [1, 2, 4, 15, 16] studied singularity of the nonlinearity $f(t,u)$ at the point $u=0$.

When $\int_0^1 u(\tau)\mathrm{d}\alpha(\tau)=\sum\limits_{i=1}^{m-2}\alpha_i u(\xi_i)$ and $\int_0^1 u(\tau)\mathrm{d}\beta(\tau)=\sum\limits_{i=1}^{m-2}\beta_i u(\xi_i)$, (1.1) becomes BVP of [3]. If α_1, α_2, β_1, $\beta_2\in[0,1]$, $\alpha_1^2+\alpha_2^2=1$ and $\beta_1^2+\beta_2^2=1$, (1.1) becomes BVPs of [9] and [10] (when $H(x)=x$). The three papers above investigated the existence of solutions for the nonsingular BVPs.

In [12], Webb used the methodology of [13] to study the existence of multiple positive solutions of nonlocal BVP of the form

$$\begin{cases} u''(t)+p(t)u'(t)+q(t)u(t)+g(t)f(t,u(t))=0, \quad 0<t<1,\\ au(0)-bu'(0)=\int_0^1 u(s)\mathrm{d}A(s),\\ cu(1)+du'(1)=\int_0^1 u(s)\mathrm{d}B(s),\end{cases} \tag{1.2}$$

where g, f are non-negative functions, typically f is continuous and $g\in L^1$ may have pointwise singularities. The case when f has no singularity at $u=0$ is covered in [12] for the more general case when the BCs allow Riemann-Stieltjes integrals with sign changing measures. Using the same general method, other nonlocal problems of arbitrary order are studied in [14].

BVP (1.1) includes the three-point problems as special cases, when $\int_0^1 u(\tau)\mathrm{d}\alpha(\tau)=0$ and $\int_0^1 u(\tau)\mathrm{d}\beta(\tau)=\xi u(\eta)$. These were extensively studied by Liu and co-authors (see, for example, [1], [2]). They studied the existence of positive solutions with $\beta_1=0$, $\beta_2=0$, $p\equiv 1$ and $q\equiv 0$ (see [1]). Furthermore, they improved on the results of [1] (see [2]).

If $\beta_1 > 0$, $\beta_2 > 0$, $\int_0^1 u(\tau)\mathrm{d}\alpha(\tau) = \sum\limits_{i=1}^{m-2} \alpha_i u(\xi_i)$ and $\int_0^1 u(\tau)\mathrm{d}\beta(\tau) = \sum\limits_{i=1}^{m-2} \beta_i u(\xi_i)$, then (1.1) becomes BVP of [4]. In this case, we can get the sufficient conditions for the existence of positive solutions of BVP (1.1) under weaker conditions than that in [4].

In [15], by means of the fixed point theorem, Jiang, Liu and Wu concerned with the second-order singular Sturm-Liouville integral BVP

$$\begin{cases} -u''(t) = \lambda h(t) f(t, u(t)), \quad 0 < t < 1, \\ \alpha u(0) - \beta u'(0) = \int_0^1 a(s)u(s)\mathrm{d}s, \\ \gamma u(1) + \delta u'(1) = \int_0^1 b(s)u(s)\mathrm{d}s, \end{cases} \tag{1.3}$$

where h is allowed to be singular at $t = 0, 1$ and $f(t, u)$ may be singular at $u = 0$. BVP (1.3) is the spacial case of BVP (1.1), when $p \equiv 1$ and $q \equiv 0$. In [15], [1] and [2], Liu, Jiang and co-author used the same condition to control the singularity of f at $u = 0$ for those BVPs (see (H2) in [1] and [15], (H3) in [2]). In this paper, our condition is less restrictive than that one (see (3.4)), and the conditions of the existence of solutions is simpler than the one in [15] when $\beta_1, \beta_2 > 0$.

In [16], by using some results from the mixed monotone operator theory, Kong concerned with positive solutions of the second order singular BVP

$$\begin{cases} u''(t) + \lambda h(t) f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = \int_0^1 u(s)\mathrm{d}\xi(s), \\ u(1) = \int_0^1 u(s)\mathrm{d}\eta(s), \end{cases} \tag{1.4}$$

where $f(t, u)$ may be singular at $t = 0, 1$ and $u = 0$. When β_1, $\beta_2 = 0$, (1.1) becomes BVP (1.4). Kong [16] studied the existence and uniqueness of positive solutions of (1.4). In this paper, we use different methods from [16] to control the singularity of f at $u = 0$. We improve and extend the results in [16] (see Remark 3.5).

Our results extend some known results from the nonsingular case in [3], [9], [10] (when $H(x) = x$) and [12] to the singular cases, and improve and extend some results from the singular cases in [1], [2], [4], [15] and [16].

The rest of this paper is organized as follows. In section 2, we present some lemmas that are used to prove our main results. In section 3, the existence of positive solutions for BVP (1.1) is stated and proved when β_1, $\beta_2 > 0$ and $\beta_1 = 0$ or $\beta_2 = 0$, respectively.

2 Preliminaries

Lemma 2.1 (*See* [3]) *Suppose ϕ and ψ be the solutions of the linear problems*

$$\begin{cases} -(p(t)\phi'(t))' + q(t)\phi(t) = 0,\ 0 < t < 1, \\ \phi(0) = \beta_1, \phi'(0) = \alpha_1, \end{cases}$$

and

$$\begin{cases} -(p(t)\psi'(t))' + q(t)\psi(t) = 0,\ 0 < t < 1, \\ \psi(1) = \beta_2, \psi'(1) = -\alpha_2, \end{cases}$$

respectively. Then

(i) ϕ is strictly increasing on $[0,1]$, and $\phi(t) > 0$ on $(0,1]$;

(ii) ψ is strictly decreasing on $[0,1]$, and $\psi(t) > 0$ on $[0,1)$;

(iii) $w = p(t)(\phi'(t)\psi(t) - \phi(t)\psi'(t))$ is a constant and $w > 0, \phi$ and ψ are linearly independent. ■

Let

$$G(t,s) = \frac{1}{w}\begin{cases} \phi(t)\psi(s), 0 \le t \le s \le 1, \\ \phi(s)\psi(t), 0 \le s \le t \le 1. \end{cases}$$

Lemma 2.2 (See [3]) *For any $y \in L[0,1]$, u is the solution of the boundary value problem*

$$\begin{cases} -(p(t)u'(t))' + q(t)u(t) = y(t),\ 0 < t < 1, \\ \alpha_1 u(0) - \beta_1 u'(0) = 0, \\ \alpha_2 u(1) + \beta_2 u'(1) = 0 \end{cases}$$

if and only if u can be expressed by

$$u(t) = \int_0^1 G(t,s)y(s)\mathrm{d}s.$$

■

Let

$$a(t) = \frac{\psi(t)}{\alpha_1\psi(0) - \beta_1\psi'(0)} = \frac{p(0)\psi(t)}{w}, \text{ and } b(t) = \frac{\phi(t)}{\alpha_2\phi(1) + \beta_2\phi'(1)} = \frac{p(1)\phi(t)}{w}.$$

Then $a(t)$ and $b(t)$ are solutions of

$$\begin{cases} -(p(t)a'(t))' + q(t)a(t) = 0,\ 0 < t < 1, \\ \alpha_1 a(0) - \beta_1 a'(0) = 1, \\ \alpha_2 a(1) + \beta_2 a'(1) = 0 \end{cases}$$

and

$$\begin{cases} -(p(t)b'(t))' + q(t)b(t) = 0, \ 0 < t < 1, \\ \alpha_1 b(0) - \beta_1 b'(0) = 0, \\ \alpha_2 b(1) + \beta_2 b'(1) = 1, \end{cases}$$

respectively.

Denote

$$v_1 = 1 - \int_0^1 a(\tau)\mathrm{d}\alpha(\tau), \ v_2 = 1 - \int_0^1 b(\tau)\mathrm{d}\beta(\tau), \ v_3 = \int_0^1 a(\tau)\mathrm{d}\beta(\tau), \ v_4 = \int_0^1 b(\tau)\mathrm{d}\alpha(\tau),$$

$$A(s) = \frac{v_2 \int_0^1 G(\tau,s)\mathrm{d}\alpha(\tau) + v_4 \int_0^1 G(\tau,s)\mathrm{d}\beta(\tau)}{v_1v_2 - v_3v_4},$$

and

$$B(s) = \frac{v_1 \int_0^1 G(\tau,s)\mathrm{d}\beta(\tau) + v_3 \int_0^1 G(\tau,s)\mathrm{d}\alpha(\tau)}{v_1v_2 - v_3v_4}.$$

We will use the following hypothesis:

(**H1**) $v_1 > 0, \ v_1v_2 - v_3v_4 > 0.$

Obviously, $v_3, \ v_4 \geq 0$. And $v_2 > 0$ if (H1) holds.

Lemma 2.3 *Suppose* (H1) *holds. For any $y \in L[0,1]$, u is the solution of the nonlinear BVP*

$$\begin{cases} -(p(t)u'(t))' + q(t)u(t) = y(t), \ 0 < t < 1, \\ \alpha_1 u(0) - \beta_1 u'(0) = \int_0^1 u(\tau)\mathrm{d}\alpha(\tau), \\ \alpha_2 u(1) + \beta_2 u'(1) = \int_0^1 u(\tau)\mathrm{d}\beta(\tau) \end{cases} \tag{2.1}$$

if and only if u can be expressed by

$$u(t) = \int_0^1 \big(G(t,s) + a(t)A(s) + b(t)B(s)\big)y(s)\mathrm{d}s. \tag{2.2}$$

■

The equation (2.2) is proved in [12] using the methods of [13] with a different notation from the one here.

Let $M = \max\limits_{0\leq t\leq 1}\{\|\phi\|, \|\psi\|\}$. For any $0 < \theta < \frac{1}{2}$, we denote

$$\gamma = \min\left\{\frac{\phi(\theta)}{\phi(1)}, \frac{\psi(1-\theta)}{\psi(0)}\right\}$$

and

$$\gamma_0 = \frac{1}{M}\min\{\beta_1, \beta_2\}.$$

It follows Lemma 2.4 and Lemma 2.5 from Lemma 2.1.

Lemma 2.4 *(1)* $G(t,s)=G(s,t)\leq G(s,s)\leq \frac{M^2}{w}$ *for all* $(t,s)\in[0,1]\times[0,1]$;

(2) $0<\gamma G(s,s)\leq G(t,s)$, *for* $t\in[\theta,1-\theta]$ *and* $s\in[0,1]$;

(3) $0<\gamma_0 G(s,s)\leq G(t,s)$, *for* $t,s\in[0,1]$, *if* $\beta_1,\beta_2>0$. ■

Lemma 2.5 *Suppose* (H1) *holds. Then*

(1) $A(s)$ *and* $B(s)$ *are nonnegative and bounded on* $[0,1]$;

(2) $a(t)$ *is strictly decreasing on* $[0,1]$, *and* $a(t)>0$ *on* $[0,1)$;

(3) $b(t)$ *is strictly increasing on* $[0,1]$, *and* $b(t)>0$ *on* $(0,1]$. ■

Let

$$c(t)=\min\left\{\frac{\phi(t)}{\phi(1)},\frac{\psi(t)}{\psi(0)}\right\}$$

and

$$\Phi(s)=G(s,s)+a(0)A(s)+b(1)B(s),$$

we can easily obtain the following Lemma 2.6 from Lemma 2.4 and Lemma 2.5.

Lemma 2.6 *Suppose* (H1) *holds. Then*

$$c(t)\Phi(s)\leq G(t,s)+a(t)A(s)+b(t)B(s)\leq\Phi(s),\ t,s\in[0,1].$$

■

Remark 2.7 *Denote* $Q_1=\sup\limits_{0\leq s\leq 1}A(s)$ *and* $Q_2=\sup\limits_{0\leq s\leq 1}B(s)$. *Then* Q_1, $Q_2\geq 0$ *if* (H1) *holds.*

For convenience, let us list the following hypothesis:

(**H2**) There exist functions $h\in C((0,1),[0,+\infty))$ and $g\in C((0,+\infty),[0,+\infty))$ such that

$$f(t,u)\leq h(t)g(u),t\in(0,1),u\in(0,+\infty),$$

and

$$0<\int_0^1 h(s)\mathrm{d}s<+\infty.$$

Let $E=C[0,1]$ with $\|u\|=\max\limits_{0\leq t\leq 1}|u(t)|$ for any $u\in C[0,1]$. Then E is a Banach space with the norm $\|\cdot\|$. Let $P=\{u\in E\ :\ u(t)\geq 0,t\in[0,1]\}$. Clearly, P is a cone in E.

We define $T:P\to P$ by

$$(Tu)(t)=\int_0^1\big(G(t,s)+a(t)A(s)+b(t)B(s)\big)u(s)\mathrm{d}s,\ t\in[0,1].$$

Let $f_n(t,u)=f(t,\max\{\frac{1}{n},u\})$ for $n\in\mathbb{N}^+$ and $t\in(0,1)$. Define $A_n:P\to P$ by

$$(A_nu)(t)=\int_0^1\big(G(t,s)+a(t)A(s)+b(t)B(s)\big)f_n(s,u(s))\mathrm{d}s,\ t\in[0,1].\tag{2.3}$$

For any $u\in P$, if (H2) holds, we have $f_n(s,u(s))\leq h(s)g\left(\max\{\frac{1}{n},u(s)\}\right)$ and A_n is well defined.

Define

$$K = \{u \in P \ : \ u(t) \geq \gamma_0 \|u\|, \ t \in [0,1]\} \text{ if } \beta_1, \beta_2 > 0,$$

and

$$K = \{u \in P \ : \ \min_{t\in[\theta,1-\theta]} u(t) \geq \gamma \|u\|, \text{ and } u(t) \geq c(t)\|u\|, \ t \in [0,1]\} \text{ if } \beta_1 = 0 \text{ or } \beta_2 = 0.$$

Clearly, $K \subset P$ is a cone.

Noticing Lemma 2.4, 2.5 and Lemma 2.6, we can easily to get the following Lemma 2.8.

Lemma 2.8 *Suppose that (H1) and (H2) hold. Then $A_n(K) \subseteq K$ is a completely continuous operator for any fixed positive integer n.* ■

Let E be a Banach space, $K \subset E$ a cone. K is said to be reproducing if $E = K - K$, and is a total cone if $E = \overline{K - K}$. (See [18] and [19]).

Lemma 2.9 (*See* [*18*] *Page 219 Proposition 19.1*) *Let E be a Banach space and $K \subset E$ a cone. Then we can get that $\mathring{K} \neq \emptyset \Rightarrow K$ is reproducing. The converse fails.* ■

We take $u^*(t) \equiv 1$ for $t \in [0,1]$, obviously, $u^* \in \mathring{K}$. It follows K we define is reproducing from Lemma 2.9.

Lemma 2.10 *Suppose that (H1) holds. Then $T : K \to K$ is a completely continuous, positive, linear operator and the spectral radius $r(T) > 0$.*

Proof Since (H1) holds, by Lemma 2.4, 2.5 and Lemma 2.6, it is easy to show $T : K \to K$ is a completely continuous, positive, linear operator.

Noticing Lemma 2.6, we can get the spectral radius $r(T) > 0$ from Lemma 2.5 in [17]. ■

Lemma 2.11 (*Krein-Rutman theorem. See* [*18*] *Page 226 Theorem 19.2*) *Let E be a Banach space, $K \subset E$ a total cone and $T \in L(E)$ a compact, linear, operator with $T(K) \subset K$ (positive) and spectral radius $r(T) > 0$. Then $r(T)$ is an eigenvalue with an eigenvector in K.* ■

According to Lemmas above, we can let u_0 denote the eigenfunction in K corresponding to its eigenvalue $r(T)$ such that $r(T)u_0 = Tu_0$. We write

$$\lambda = (r(T))^{-1}. \tag{2.4}$$

Lemma 2.12 (See [*20*]) *Let K be a cone of a real Banach space E, Ω be a bounded open set in E. Suppose $A : K \cap \Omega \to K$ is a completely continuous operator. If there exists $u_0 \in K\backslash\{\theta\}$ such that $u - Au \neq \rho u_0$ for all $u \in \partial\Omega \cap K$ and all $\rho \geq 0$, then $i(A, \Omega \cap K, K) = 0$.* ■

Lemma 2.13 (See [*20*]) *Let K be a cone of a real Banach space E, Ω be a bounded open set in E, with $0 \in \Omega$. Suppose $A : K \cap \Omega \to K$ be a completely continuous operator. If $Au \neq \rho u$ for all $u \in \partial\Omega \cap K$ and all $\rho \geq 1$, then $i(A, \Omega \cap K, K) = 1$.* ■

3 The main results

We will also use the following hypotheses on the nonlinear term f.

(A1) $\underline{\lim}_{u\to 0^+} \inf_{t\in(0,1)} \dfrac{f(t,u)}{u} > \lambda$;

(A2) $\overline{\lim}_{u\to +\infty} \sup_{t\in(0,1)} \dfrac{f(t,u)}{u} < \lambda$,

where λ is defined by (2.4).

Let $0 < \varepsilon_0 < \lambda$, $r_0 > 0$ and $R_0 > \max\{1, r_0\}$ be such that

$$f(t,u) > (\lambda+\varepsilon_0)u, \text{ for } 0 < u \le r_0, \text{ and } f(t,u) < (\lambda-\varepsilon_0)u, \text{ for } u \ge R_0.$$

When $\beta_1 > 0$ and $\beta_1 > 0$ the singularity of $f(t,u)$ at $u = 0$ is easily dealt with as nonzero solutions in the cone are strictly positive.

Theorem 3.1 *Suppose (H1), (H2), (A1) and (A2) hold, $\beta_1, \beta_2 > 0$. Let*

$$K = \{u \in P \ : \ u(t) \ge \gamma_0\|u\|, \ t \in [0,1]\}.$$

Then the BVP (1.1) has at least one positive solution $u \in K$ with $r_0 \le \|u\|$.

Proof. Take $n_0 \in \mathbb{N}^+$ and $n_0 > [\frac{1}{r_0}]$, then $\frac{1}{n} < r_0$ for $n > n_0$. Hence, if $n > n_0$ we have

$$f_n(t,u) = f(t, \max\{\frac{1}{n}, u\}) > \lambda \max\{\frac{1}{n}, u\} \ge \lambda u, \ 0 < u \le r_0, \ t \in (0,1). \tag{3.1}$$

By Theorem 3.4 in [17], $i(A_n, B_{r_0} \cap K, K) = 0$ for $n > 1/r_0$, where $B_{r_0} = \{u \in C[0,1] : \|u\| < r_0\}$.

On the other hand, for each $n \in \mathbb{N}^+$,

$$f_n(t,u) = f(t, \max\{\frac{1}{n}, u\}) \le (\lambda-\varepsilon_0) \max\{\frac{1}{n}, u\} = (\lambda-\varepsilon_0)u, \ u \ge R_0, \ t \in (0,1). \tag{3.2}$$

Therefore, by Theorem 3.3 in [17], there exists a constant $R_n > R_0$ such that $i(A_n, B_{R_n} \cap K, K) = 1$. By the additivity property of fixed point index, A_n has a fixed point $u_n \in K$ with $r_0 \le \|u_n\| \le R_n$. For $n_1 > 1/(\gamma_0 r_0)$ and $t \in (0,1)$, it follows that $u_{n_1}(t) \ge \gamma_0\|u_{n_1}\| \ge \gamma_0 r_0 > 1/n_1$. We have $f_{n_1}(t, u_{n_1}(t)) = f(t, u_{n_1}(t))$. Hence, u_{n_1} is a positive solution of the BVP (1.1) and $u_{n_1} \in K$ with $r_0 \le \|u_{n_1}\|$. ■

Now we consider the case when $\beta_1 = 0$ or $\beta_2 = 0$. We will use the cone

$$K = \{u \in P \ : \ \min_{t\in[\theta,1-\theta]} u(t) \ge \gamma\|u\|, \text{ and } u(t) \ge c(t)\|u\|, \ t \in [0,1]\}.$$

Lemma 3.2 *Suppose (H1), (H2), (A1) and (A2) hold, and $\beta_1 = 0$ or $\beta_2 = 0$. Then for $n > \frac{1}{\gamma r_0}$ and $n \in \mathbb{N}^+$ we have*

$$i(A_n, B_{r_0} \cap K, K) = 0. \tag{3.3}$$

Proof. This is the same as the first part of Theorem 3.1. ■

Since $r(\lambda-\varepsilon_0)T) = (\lambda-\varepsilon_0)r(T) < 1$ and $T : P \to P$ is a completely continuous, linear operator, it follows $\big(I - (\lambda - \varepsilon_0)T\big)^{-1}$ is a bounded linear operator and maps P into P.

Theorem 3.3 *Suppose (H1), (H2), (A1) and (A2) hold, and $\beta_1 = 0$ or $\beta_2 = 0$, there exist constants $M_0 > 0$, $0 < m_0 < \frac{1}{2}$ such that*

$$\sup_{\{u\in K\,:\,\gamma r_0\le\|u\|\le R_0\}} \int_{E(m_0)} h(s)g(u(s))\mathrm{d}s \le M_0, \tag{3.4}$$

where $E(m_0) = [0, m_0] \cup [1 - m_0, 1]$ and

$$K = \{u \in P : \min_{t\in[m_0,1-m_0]} u(t) \ge \gamma\|u\|, \text{ and } u(t) \ge c(t)\|u\|,\ t \in [0, 1]\}.$$

Then the BVP (1.1) has at least one positive solution $u \in K$ with $r_0 \le \|u\|$.

Proof. We denote $\widetilde{M} = \Big(\frac{M^2}{w} + a(0)Q_1 + b(1)Q_2\Big) M_0$ and take

$$R > \max\{\frac{R_0}{\gamma}, \|\big(I - (\lambda - \varepsilon_0)T\big)^{-1}\|\widetilde{M}\}.$$

Let $B_R = \{u \in C[0, 1] : \|u\| < R\}$. We can prove

$$A_n u \ne \rho u, \text{ for all } u \in \partial B_R \cap K, \rho \ge 1 \text{ and } n > \frac{1}{\gamma R_0}. \tag{3.5}$$

If (3.5) is not true, there exist $u^* \in \partial B_R \cap K$ and $\rho_1 \ge 1$ such that

$$A_n u^* = \rho_1 u^*, \text{ for some } n > \frac{1}{\gamma R_0}. \tag{3.6}$$

We have

$$\|u^*\| = R,\ u^*(t) \ge \gamma\|u^*\| = \gamma R > R_0 > r_0 > \gamma r_0, \text{ for } t \in [m_0, 1 - m_0]. \tag{3.7}$$

Let $E_1 = \{s \in [0, 1] : 0 \le u^*(s) \le R_0\}$, it is easy to see that $E_1 \subset E(m_0)$.

Denote $\overline{u}^*(s) = \max\{\frac{1}{n}, u^*(s)\}$, then $\overline{u}^* \in K$,

$$\|\overline{u}^*\| = R,\ \overline{u}^*(s) \ge \gamma\|\overline{u}^*\| = \gamma R > R_0 > \gamma r_0, \text{ for } s \in [m_0, 1 - m_0]. \tag{3.8}$$

Let

$$\overline{u}^{**}(t) = \begin{cases} \overline{u}^*(t),\ t \in E_1, \\ R_0,\ t \in [0, 1]\backslash E_1. \end{cases}$$

We have $\gamma r_0 \le \|\overline{u}^{**}\| \le R_0$.

Hence, by (3.4), (H2) and Lemma 2.4, for $t \in [0,1]$, we can show

$$\begin{aligned}
&\int_{E_1} \big(G(t,s)+a(t)A(s)+b(t)B(s)\big) f_n(s,u^*(s))\mathrm{d}s \\
=&\int_{E_1} \big(G(t,s)+a(t)A(s)+b(t)B(s)\big) f\left(s,\max\{\frac{1}{n},u^*(s)\}\right)\mathrm{d}s \\
\leq&\int_{E_1} \big(G(t,s)+a(t)A(s)+b(t)B(s)\big) h(s)g\left(\max\{\frac{1}{n},u^*(s)\}\right)\mathrm{d}s \\
=&\int_{E_1} \big(G(t,s)+a(t)A(s)+b(t)B(s)\big) h(s)g\left(\overline{u}^*(s)\right)\mathrm{d}s \\
=&\int_{E_1} \big(G(t,s)+a(t)A(s)+b(t)B(s)\big) h(s)g\left(\overline{u}^{**}(s)\right)\mathrm{d}s \\
\leq&\left(\frac{M^2}{w}+a(0)Q_1+b(1)Q_2\right)\sup_{\{u\in K\,:\,\gamma r_0\leq\|u\|\leq R_0\}}\int_{E(m_0)} h(s)g(u(s))\mathrm{d}s \\
\leq&\widetilde{M}.
\end{aligned} \tag{3.9}$$

Noticing (3.2) and (3.9), for $t\in[0,1]$, we can obtain

$$\begin{aligned}
(A_nu^*)(t)=&\int_0^1 \big(G(t,s)+a(t)A(s)+b(t)B(s)\big) f_n(s,u^*(s))\mathrm{d}s \\
=&\int_{[0,1]\setminus E_1} \big(G(t,s)+a(t)A(s)+b(t)B(s)\big) f_n(s,u^*(s))\mathrm{d}s \\
&+\int_{E_1} \big(G(t,s)+a(t)A(s)+b(t)B(s)\big) f_n(s,u^*(s))\mathrm{d}s \\
\leq&\int_0^1 \big(G(t,s)+a(t)A(s)+b(t)B(s)\big)(\lambda-\varepsilon_0)u^*(s)\mathrm{d}s+\widetilde{M} \\
=&(\lambda-\varepsilon_0)(Tu^*)(t)+\widetilde{M}.
\end{aligned}$$

It is easy to get

$$0\leq u^*\leq \rho_1u^*=A_nu^*\leq(\lambda-\varepsilon_0)(Tu^*)+\widetilde{M}.$$

Then $(I-(\lambda-\varepsilon_0)T)u^*\leq\widetilde{M}$, $u^*\leq(I-(\lambda-\varepsilon_0)T)^{-1}\widetilde{M}$ and

$$\|u^*\|\leq\|(I-(\lambda-\varepsilon_0)T)^{-1}\|\widetilde{M}<R, \tag{3.10}$$

which is a contradiction with the definition of $u^*\in\partial B_R\cap K$.

So (3.5) holds. It follows from Lemma 2.13, we have

$$i(A_n,B_R\cap K,K)=1 \text{ for } n>\frac{1}{\gamma R_0}. \tag{3.11}$$

By (3.3) and (3.11), we obtain

$$i\big(A_n,(B_R\cap K)\backslash(\overline{B}_{r_0}\cap K),K\big)=i(A_n,(B_R\cap K),K)-i(A_n,B_{r_0}\cap K,K)=1,$$

for $n>\frac{1}{\gamma r_0}$.

We can get A_n has a fixed point $u_n \in K$ with $r_0 \leq \| u_n \| \leq R$ when $n > \frac{1}{\gamma r_0}$. Denote $n_0 = [\frac{1}{\gamma r_0}] + 1$. Let

$$\Omega = \{u_n \in K \ : r_0 \leq \| u_n \| \leq R,\ A_n u_n = u_n,\ n > n_0\}.$$

It is easy to see that Ω is uniformly bounded. And we have

$$\gamma r_0 \leq \gamma \|u_n\| \leq u_n(t) \leq R, \text{ for } n > n_0 \text{ and } t \in [m_0, 1 - m_0].$$

Hence,

$$\begin{aligned}\int_0^1 f_n(s, u_n(s))\mathrm{d}s = &\int_{\{s\in[0,1]\,:\,\gamma r_0 < u_n(s) \leq R\}} f(s, \max\{\frac{1}{n}, u_n(s)\})\mathrm{d}s \\ &+ \int_{\{s\in[0,1]\,:\,0 \leq u_n(s) \leq \gamma r_0\}} f(s, \max\{\frac{1}{n}, u_n(s)\})\mathrm{d}s\end{aligned}$$

Let $\overline{u}_n(s) = \max\{\frac{1}{n}, u_n(s)\}$, then for $n > n_0$, $\overline{u}_n \in K$ and

$$R \geq \overline{u}_n(t) \geq \gamma \|\overline{u}_n\| \geq \gamma r_0 \text{ for } n > n_0 \text{ and } t \in [m_0, 1 - m_0],$$

It is similar to the proof above, we can show

$$\int_0^1 f_n(s, u_n(s))\mathrm{d}s \leq \max_{\gamma r_0 \leq u \leq R} g(u) \int_0^1 h(s)\mathrm{d}s + M_0.$$

That is, for each $u_n \in \Omega$, we have

$$\int_0^1 f_n(s, u_n(s))\mathrm{d}s \leq M_1, \tag{3.12}$$

where $M_1 = \max\limits_{\gamma r_0 \leq u \leq R} g(u) \int_0^1 h(s)\mathrm{d}s + M_0$.

In the following, we prove that Ω is equicontinuous.

By the continuity of $G(t, s), \phi(t)$ and $\psi(t)$, for any $\varepsilon > 0$, there exists $\delta_1 \in (0, \frac{1}{2})$ such that

$$|G(t, s) - G(0, s)| < \varepsilon,$$

$$|a(t) - a(0)| < \varepsilon \text{ and } |b(t) - b(0)| < \varepsilon.$$

for $t \in (0, \delta_1)$.

By (2.3), for each $u_n \in \Omega$ and $t \in (0, \delta_1)$, we have

$$\begin{aligned}|u_n(t) - u_n(0)| &= |\int_0^1 \Big((G(t, s) - G(0, s)) + (a(t) - a(0))A(s) + (b(t) - b(0))B(s)\Big) f_n(s, u_n(s))\mathrm{d}s| \\ &\leq \int_0^1 \Big(|G(t, s) - G(0, s)| + |a(t) - a(0)|A(s) + |b(t) - b(0)|B(s)\Big) f_n(s, u_n(s))\mathrm{d}s \\ &\leq \varepsilon M_1(1 + Q_1 + Q_2).\end{aligned}$$

Then we can show that

$$\lim_{t\to 0^+} |u_n(t) - u_n(0)| = 0,\ n > n_0. \tag{3.13}$$

Similarly, we can easily prove

$$\lim_{t\to 1^-} |u_n(t) - u_n(1)| = 0,\ n \geq n_0. \tag{3.14}$$

Since $G(t,s), \phi$ and ψ are uniformly continuous on $t \in [\xi, 1-\xi]$, $\xi \in (0, \frac{1}{2})$ and $s \in [0,1]$. For $\varepsilon > 0$, there exists $\delta_2 > 0$ such that

$$|G(t_1,s) - G(t_2,s)| < \varepsilon,$$

$$|a(t_1) - a(t_2)| < \varepsilon \text{ and } |b(t_1) - b(t_2)| < \varepsilon,$$

whenever $|t_1 - t_2| < \delta_2$, $t_1, t_2 \in [\xi, 1-\xi]$ and $s \in [0,1]$.

Then, for all $n > n_0$, we have

$$\begin{aligned}
&|u_n(t_1) - u_n(t_2)| \\
&= |\int_0^1 \Big((G(t_1,s) - G(t_2,s)) + (a(t_1) - a(t_2))A(s) + (b(t_1) - b(t_2))B(s)\Big) f_n(s, u_n(s))\mathrm{d}s| \\
&\leq \int_0^1 \Big(|G(t_1,s) - G(t_2,s)| + |a(t_1) - a(t_2)|A(s) + |b(t_1) - b(t_2)|B(s)\Big) f_n(s, u_n(s))\mathrm{d}s \\
&\leq \varepsilon M_1(1 + Q_1 + Q_2).
\end{aligned}$$

Thus, Ω is equicontinuous on $[\xi, 1-\xi] \subset (0,1)$.

Noticing (3.13) and (3.14), we can obtain Ω is equicontinuous.

It follows that the $\{u_n\}_{n>n_0}$ has a subsequence which is uniformly convergent on $[0,1]$ from Ascoli-Arzela theorem. Without loss of generality, we can assume that $\{u_n\}_{n>n_0}$ itself converges uniformly to u on $[0,1]$, then $r_0 \leq \|u\| \leq R$ and $u \in K$.

By (2.3), we can show

$$u_n(t) = \int_0^1 \big(G(t,s) + a(t)A(s) + b(t)B(s)\big) f_n(s, u_n(s))\mathrm{d}s,\ n > n_0. \tag{3.15}$$

Since $f \in C((0,1) \times (0, +\infty))$, we have

$$\lim_{n\to+\infty} f_n(t, u_n(t)) = f(t, u(t)),\ t \in (0,1).$$

Noticing (3.4), (3.12) and (H2), according to the Lebesgue's dominated convergence theorem, we can get that $f(s, u(s)) \in L[0,1]$ and that

$$u(t) = \int_0^1 \big(G(t,s) + a(t)A(s) + b(t)B(s)\big) f(s, u(s))\mathrm{d}s$$

for $t \in [0,1]$ from (3.15).

Hence, it follows the BVP (1.1) has at least one positive solution u from Lemma 2.3, and $u \in K$ with $r_0 \leq \|u\|$. ■

Example 3.4 *We consider the BVP*

$$\begin{cases} -u''(t) = f(t, u(t)), \ 0 < t < 1, \\ u(0) = 0, \\ u(1) = \int_0^1 u(s)\mathrm{d}s, \end{cases} \tag{3.16}$$

where

$$f(t,u) = \left(4 + \sin\frac{1}{t} + \sin\frac{1}{1-t}\right)\left(u^\mu + \frac{1}{u^\nu}\right),$$

and $0 \le \mu, \nu < 1$ *are constants. Then the BVP (3.16) has at least one positive solution.*

Proof. It is easy to see that

$$\underline{\lim}_{u\to 0^+} \inf_{t\in(0,1)} \frac{f(t,u)}{u} = +\infty,$$

and

$$\overline{\lim}_{u\to +\infty} \sup_{t\in(0,1)} \frac{f(t,u)}{u} = 0.$$

We can take $h(t) = 6$, $g(u) = u^\mu + \frac{1}{u^\nu}$, $\phi(t) = t$, $\psi(t) = 1 - t$ and $m_0 = \frac{1}{4}$. Then $c(t) = \min\{t, 1-t\}$ and $\gamma = \frac{1}{4}$. Let

$$K = \{u \in P : \min_{t\in[\frac{1}{4},\frac{3}{4}]} u(t) \ge \frac{1}{4}\|u\|, \text{ and } u(t) \ge c(t)\|u\|, \ t \in [0,1]\}.$$

We can easily verify that (A1), (A2), (H1) and (H2) hold.

For each $u \in K$ and $\frac{1}{4}r_0 \le \|u\| \le R_0$,

$$h(t)g(u) \le 6\left(\|u\|^\mu + \frac{1}{(c(t)\|u\|)^\nu}\right) \le 6\left(R_0^\mu + \frac{4}{r_0^\nu (c(t))^\nu}\right).$$

Since $\int_{E(\frac{1}{4})} 6\left(R_0^\mu + \frac{4}{r_0^\nu(c(t))^\nu}\right)\mathrm{d}t$ is convergent, its value is denoted M_0. Therefore, the condition (3.4) is satisfied.

By means of Theorem 3.3, we can obtain that the BVP (3.16) has at least one positive solution. ■

In fact, about the BVP (1.1), if (H1), (H2), (A1) and (A2) hold, when $h(t)g(u) = h(t)\left(u^\mu + \frac{1}{u^\nu}\right)$, for each $u \in K$ and $\gamma r_0 \le \|u\| \le R_0$,

$$h(t)g(u) \le h(t)\left(\|u\|^\mu + \frac{1}{(c(t)\|u\|)^\nu}\right) \le h(t)\left(R_0^\mu + \frac{1}{(\gamma r_0)^\nu (c(t))^\nu}\right).$$

As long as $\int_{E(m_0)} h(t)\left(R_0^\mu + \frac{1}{(\gamma r_0)^\nu(c(t))^\nu}\right)\mathrm{d}t$ is convergent, we can get (3.4) holds.

Remark 3.5 *In the BVP (1.1), let* $p \equiv 1$, $q \equiv 0$, $\alpha_1 = \alpha_2 = 1$ *and* $\beta_1 = \beta_2 = 0$, *(1.1) becomes the BVP (1.4). Moreover, let* $\alpha(t) \equiv C$ *and* $\beta(t) = t$, *where* C *is a constant, that is* $\xi(t) \equiv C$ *and* $\eta(t) = t$ *in (1.4). Then* $M = 1$ *of (2.2) in [16]. Hence, the assumption (H1) does not hold in [16], and the existence of positive solutions of (1.4) cannot be obtained.*

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