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Periodic boundary value problems for nonlinear impulsive fractional differential equation

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Abstract
In this paper, we investigate the existence and uniqueness of solution of the periodic boundary value problem for nonlinear impulsive fractional differential equation involving Riemann-Liouville fractional derivative by using Banach contraction principle.

Keywords: Riemann-Liouville fractional derivative; Periodic boundary value problem; Impulsive

1. Introduction

This paper deals with the existence of solutions for nonlinear impulsive fractional differential equation with periodic boundary conditions

$$D^\alpha u(t) - \lambda u(t) = f(t, u(t)), \quad t \in (0,1] \setminus \{t_1\}, \quad 0 < \alpha \le 1, \tag{1.1}$$

$$\lim_{t\to 0^+} t^{1-\alpha} u(t) = u(1), \tag{1.2}$$

$$\lim_{t\to t_1^+} (t - t_1)^{1-\alpha}(u(t) - u(t_1)) = I(u(t_1)), \tag{1.3}$$

where D^α is the standard Riemann-Liouville fractional derivative, $\lambda \in R$, $0 < t_1 < 1$, $I \in C(R, R)$, f is continuous at every point $(t, u) \in [0,1] \times R$.

For clarity and brevity, we restrict our attention to BVPs with one impulse, the difference between the theory of one or an arbitrary number of impulses is quite minimal.

In [1], the author investigated the existence and uniqueness of solution to initial value problems for a class of fractional differential equations

$$D^\alpha u(t) = f(t, u(t)), \quad t \in (0, T] \quad 0 < \alpha < 1, \tag{1.4}$$

$$t^{1-\alpha} u(t)|_{t=0} = u_0, \tag{1.5}$$

by using the method of upper and lower solutions and its associated monotone iterative.

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In [2], the existence and uniqueness of solution of the following fractional differential equation with periodic boundary value condition

$$D^{\delta}u(t) - \lambda u(t) = f(t, u(t)), \quad t \in (0, 1] \quad 0 < \delta < 1, \tag{1.6}$$

$$\lim_{t \to 0^+} t^{1-\delta}u(t) = u(1), \tag{1.7}$$

was discussed by using the fixed point theorem of Schaeffer and the Banach contraction principle.

Differential equation with fractional order have recently proved valuable tools in the modeling of many phenomena in various fields of science and engineering [3-7]. There has also been a significant theoretical development in fractional differential equations in recent years, see for examples [8-19]. Recently, many researchers paid attention to existence result of solution of the initial value problem and boundary value problem for fractional differential equations, such as [20-25].

Impulsive differential equations are now recognized as an excellent source of models to simulate process and phenomena observed in control theory, physics, chemistry, population dynamics, biotechnology, industrial robotic, optimal control, etc. [26,27]. Periodic boundary value for impulsive differential equation has drawn much attention, see [28-31]. Anti-periodic problems constitute an important class of boundary value problems and have recently received considerable attention. The recent results on anti-periodic BVPs or impulsive anti-periodic BVPs of fractional differential equations can be found in [32-35]. But till now, the theory of boundary value problems for nonlinear fractional differential equations is still in the initial stages. For some recent work on impulsive fractional differential equations, see [36-41] and the references therein.

To the best of the authors knowledge, no one has studied the existence of solutions for BVP (1.1)-(1.3). The purpose of this paper is to study the existence and uniqueness of solution of the periodic boundary value problem for nonlinear impulsive fractional differential equation involving Riemann-Liouville fractional derivative by using Banach contraction principle.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. Let $C(a, b]$ ($C[a, b]$) be the Banach space of all continuous real functions defined on $(a, b]$ ($[a, b]$).

In order to define the solution of (1.1)-(1.3) we shall consider the space

$$PC_r[0,1] = \{x \;:\; t^r x|_{[0,t_1]} \in C[0,t_1],\ (t-t_1)^r x|_{(t_1,1]} \in C(t_1,1],$$
$$\text{there exist } \lim_{t\to t_1^+}(t-t_1)^r x(t) \text{ and } x(t_1^-) \text{ with } x(t_1^-) = x(t_1)\},$$

where constant $0 \le r < 1$. It is easy to check that the space $PC_r[0,1]$ is a Banach space with norm

$$\|x\|_r = \max\{\sup\{t^r|x(t)| : t \in [0,t_1]\},\ \sup\{(t-t_1)^r|x(t)| : t \in (t_1,1]\}\}.$$

Remark 2.1. If $r = 0$, then the definition of $PC_r[0,1]$ reduces to the following

$$PC[0,1] = \{x \;:\; x|_{[0,t_1]} \in C[0,t_1],\ x|_{(t_1,1]} \in C(t_1,1],$$
$$\text{there exist } x(t_1^-) \text{ and } x(t_1^+) \text{ with } x(t_1^-) = x(t_1)\}.$$

And the space $PC[0,1]$ is a Banach space with norm

$$\|x\|_{PC} = \sup\{|x(t)| : t \in [0,1]\}.$$

Lemma 2.1. *The linear impulsive boundary value problem*

$$D^\alpha u(t) - \lambda u(t) = \sigma(t), \quad t \in (0,1] \setminus \{t_1\}, \tag{2.1}$$

$$\lim_{t\to 0^+} t^{1-\alpha}u(t) = u(1), \tag{2.2}$$

$$\lim_{t\to t_1^+}(t-t_1)^{1-\alpha}(u(t)-u(t_1)) = a_1, \tag{2.3}$$

where $\lambda, a_1 \in R$ *are constants and* $\sigma \in C[0,1]$, *has a unique solution* $u \in PC_{1-\alpha}[0,1]$ *given by*

$$u(t) = \int_0^1 G_{\lambda,\alpha}(t,s)\sigma(s)ds + \Gamma(\alpha)G_{\lambda,\alpha}(t,t_1)a_1, \tag{2.4}$$

where

$$G_{\lambda,\alpha}(t,s) = \begin{cases} \frac{\Gamma(\alpha)E_{\alpha,\alpha}(\lambda t^\alpha)E_{\alpha,\alpha}(\lambda(1-s)^\alpha)t^{\alpha-1}(1-s)^{\alpha-1}}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)} + (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha), & 0 \le s \le t \le 1, \\ \frac{\Gamma(\alpha)E_{\alpha,\alpha}(\lambda t^\alpha)E_{\alpha,\alpha}(\lambda(1-s)^\alpha)t^{\alpha-1}(1-s)^{\alpha-1}}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}, & 0 \le t < s \le 1, \end{cases}$$

and $E_{\alpha,\alpha}(\lambda) = \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma((k+1)\alpha)}$ *is Mittag-Leffler function (see [6]).*

Proof. By Theorem 3.2 in [2], we have that

$$\bar{u}(t) = \int_0^1 G_{\lambda,\alpha}(t,s)\sigma(s)ds, \quad t \in (0,1] \setminus \{t_1\} \tag{2.5}$$

is a unique solution of the linear problem (2.1)-(2.2), and $\bar{u} \in PC_{1-\alpha}[0,1]$. Set

$$w(t) = G_{\lambda,\alpha}(t,t_1)\Gamma(\alpha)a_1, \quad t \in [0,1].$$

For each $t < t_1$, we have

$$\begin{aligned} D^\alpha w(t) &= \frac{\Gamma(\alpha)^2 E_{\alpha,\alpha}(\lambda(1-t_1)^\alpha)(1-t_1)^{\alpha-1}a_1}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)} D^\alpha(t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)) \\ &= \frac{\Gamma(\alpha)^2 E_{\alpha,\alpha}(\lambda(1-t_1)^\alpha)(1-t_1)^{\alpha-1}a_1}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)} D^\alpha\left(\sum_{i=1}^{\infty}\frac{\lambda^{i-1}}{\Gamma(\alpha i)}t^{\alpha i-1}\right). \end{aligned}$$

Using the identities

$$D^\alpha t^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu+1-\alpha)}t^{\mu-\alpha}\ (\mu > -1), \qquad D^\alpha t^{\alpha-1} = 0,$$

we get

$$\begin{aligned} D^\alpha w(t) &= \frac{\Gamma(\alpha)^2 E_{\alpha,\alpha}(\lambda(1-t_1)^\alpha)(1-t_1)^{\alpha-1}a_1}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)} \sum_{i=2}^{\infty}\frac{\lambda^{i-1}}{\Gamma(\alpha(i-1))}t^{\alpha(i-1)-1} \\ &= \lambda\frac{\Gamma(\alpha)^2 E_{\alpha,\alpha}(\lambda(1-t_1)^\alpha)(1-t_1)^{\alpha-1}a_1}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)} \sum_{i=1}^{\infty}\frac{(-\lambda)^{i-1}}{\Gamma(\alpha i)}t^{\alpha i-1} \\ &= \lambda\frac{\Gamma(\alpha)^2 E_{\alpha,\alpha}(\lambda(1-t_1)^\alpha)(1-t_1)^{\alpha-1}a_1}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)} t^{\alpha-1}\sum_{i=0}^{\infty}\frac{(\lambda t^\alpha)^i}{\Gamma(\alpha i+\alpha)} \\ &= \lambda\frac{\Gamma(\alpha) E_{\alpha,\alpha}(\lambda(1-t_1)^\alpha)(1-t_1)^{\alpha-1}t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\Gamma(\alpha)a_1 \\ &= \lambda G_{\alpha,\alpha}(t,t_1)\Gamma(\alpha)a_1 = \lambda w(t), \quad t < t_1. \end{aligned}$$

Similarly, we can obtain that

$$D^\alpha w(t) - \lambda w(t) = 0, \quad t > t_1.$$

Thus, we have $D^\alpha w(t) - \lambda w(t) = 0$ for $t \in (0,1] \setminus \{t_1\}$. Moreover, we have

$$\begin{aligned} &\lim_{t\to 0^+} t^{1-\alpha}w(t) - w(1) \\ &= \lim_{t\to 0^+} t^{1-\alpha}\frac{\Gamma(\alpha)^2 E_{\alpha,\alpha}(\lambda(1-t_1)^\alpha)(1-t_1)^{\alpha-1}a_1}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha) \\ &- \left(\frac{\Gamma(\alpha)E_{\alpha,\alpha}(\lambda(1-t_1)^\alpha)(1-t_1)^{\alpha-1}E_{\alpha,\alpha}(\lambda)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)} + (1-t_1)^{\alpha-1}E_{\alpha,\alpha}(\lambda(1-t_1)^\alpha)\right)\Gamma(\alpha)a_1 \\ &= \frac{\Gamma(\alpha)E_{\alpha,\alpha}(\lambda(1-t_1)^\alpha)(1-t_1)^{\alpha-1}a_1}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)} - \frac{(1-t_1)^{\alpha-1}E_{\alpha,\alpha}(\lambda(1-t_1)^\alpha)\Gamma(\alpha)a_1}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)} \end{aligned}$$

$$= 0,$$

and

$$\lim_{t\to t_1^+}(t-t_1)^{1-\alpha}(w(t)-w(t_1))$$
$$= \lim_{t\to t_1^+}(t-t_1)^{1-\alpha}\cdot(t-t_1)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-t_1)^\alpha)\Gamma(\alpha)a_1$$
$$= \lim_{t\to t_1^+}E_{\alpha,\alpha}(\lambda(t-t_1)^\alpha)\Gamma(\alpha)a_1 = \lim_{t\to t_1^+}\sum_{i=0}^{\infty}\frac{(\lambda(t-t_1)^\alpha)^i}{\Gamma(\alpha i+\alpha)}\Gamma(\alpha)a_1$$
$$= \tfrac{1}{\Gamma(\alpha)}\Gamma(\alpha)a_1 = a_1.$$

In consequence,

$$u(t) = \bar{u}(t) + w(t) = \int_0^1 G_{\lambda,\alpha}(t,s)\sigma(s)ds + \Gamma(\alpha)G_{\lambda,\alpha}(t,t_1)a_1$$

is the solution of problem (2.1)-(2.3).

Next we prove that the solution of BVP (2.1)-(2.3) is unique. Suppose that $u_1, u_2 \in PC_{1-\alpha}[0,1]$ are two solutions of BVP (2.1)-(2.3). Let $v = u_1 - u_2$, then we have

$$D^\alpha v(t) - \lambda v(t) = 0, \quad t \in (0,1]\setminus\{t_1\}, \tag{2.6}$$

$$\lim_{t\to 0^+} t^{1-\alpha}v(t) - v(1) = 0, \tag{2.7}$$

$$\lim_{t\to t_1^+}(t-t_1)^{1-\alpha}(v(t)-v(t_1)) = 0. \tag{2.8}$$

By (2.5) and (2.6)-(2.7), we get that $v(t) \equiv 0$ for any $t \in (0,1]\setminus\{t_1\}$. Since $v \in PC_{1-\alpha}[0,1]$, we have $\lim_{t\to t_1^-} t^{1-\alpha}v(t) = t_1^{1-\alpha}v(t_1)$. On the other hand, $\lim_{t\to t_1^-} t^{1-\alpha}v(t) = 0$. Thus, we obtain that $v(t_1) = 0$. Hence, $u_1(t) = u_2(t)$ for each $t \in (0,1]$. Moreover, by (2.7), we have $\lim_{t\to 0^+} t^{1-\alpha}v(t) = v(1) = 0$, which implies that $\lim_{t\to 0^+} t^{1-\alpha}u_1(t) = \lim_{t\to 0^+} t^{1-\alpha}u_2(t)$. Therefore, $u_1 = u_2$.

Remark 2.2. For $\alpha = 1$, Lemma 2.1 reduces to the one for a first order linear impulsive boundary value problem

3. Main results

Lemma 3.1. *Suppose that*

(H_1) *there exist positive constants M and m such that*

$$|f(t,u)| \leq M, \quad |I(u)| \leq m, \quad \forall t \in [0,1],\ u \in R, \tag{3.1}$$

holds, then the operator $A : PC_{1-\alpha}[0,1] \to PC_{1-\alpha}[0,1]$ *is well defined.*

Proof. Define the operator A as follows :

$$(Ax)(t) := \int_0^1 G_{\lambda,\alpha}(t,s)f(s,x(s))ds + \Gamma(\alpha)G_{\lambda,\alpha}(t,t_1)I(x(t_1)). \tag{3.2}$$

By Lemma 2.1, it is easy to see that a function x is a solution to (1.1)-(1.3) if and only if x is a fixed point of A. Using (H$_1$), we check that $t^{1-\alpha}(Au)(t)|_{[0,t_1]} \in C[0,t_1]$ and $(t-t_1)^{1-\alpha}(Au)(t)|_{(t_1,1]} \in C(t_1,1]$. Here, we only prove that $(t-t_1)^{1-\alpha}(Au)(t)|_{(t_1,1]} \in C(t_1,1]$. Similarly, we can prove that $t^{1-\alpha}(Au)(t)|_{[0,t_1]} \in C[0,t_1]$. For every $u \in PC_{1-\alpha}[0,1]$, for any $\tau_1, \tau_2 \in (t_1,1]$ with $\tau_1 < \tau_2$, we have

$$\begin{aligned}
&\Big|(\tau_1-t_1)^{1-\alpha}(Au)(\tau_1) - (\tau_2-t_1)^{1-\alpha}(Au)(\tau_2)\Big| \\
&\leq \Big|(\tau_1-t_1)^{1-\alpha}\int_0^1 G_{\lambda,\alpha}(\tau_1,s)f(s,u(s))ds - (\tau_2-t_1)^{1-\alpha}\int_0^1 G_{\lambda,\alpha}(\tau_2,s)f(s,u(s))ds\Big| \\
&\quad + \Big|(\tau_1-t_1)^{1-\alpha}\Gamma(\alpha)G_{\lambda,\alpha}(\tau_1,t_1)I(u(t_1)) - (\tau_2-t_1)^{1-\alpha}\Gamma(\alpha)G_{\lambda,\alpha}(\tau_2,t_1)I(u(t_1))\Big| \\
&= \Big|\int_0^1 \frac{\Gamma(\alpha)E_{\alpha,\alpha}(\lambda(1-s)^\alpha)(1-s)^{\alpha-1}}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)} \\
&\qquad \times[(\tau_1-t_1)^{1-\alpha}E_{\alpha,\alpha}(\lambda\tau_1^\alpha)\tau_1^{\alpha-1} - (\tau_2-t_1)^{1-\alpha}E_{\alpha,\alpha}(\lambda\tau_2^\alpha)\tau_2^{\alpha-1}]f(s,u(s))ds \\
&\quad + \int_0^{\tau_1}[(\tau_1-t_1)^{1-\alpha}(\tau_1-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(\tau_1-s)^\alpha) \\
&\qquad -(\tau_2-t_1)^{1-\alpha}(\tau_2-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(\tau_2-s)^\alpha)]f(s,u(s))ds \\
&\quad - \int_{\tau_1}^{\tau_2}(\tau_2-t_1)^{1-\alpha}(\tau_2-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(\tau_2-s)^\alpha f(s,u(s))ds\Big| \\
&\quad + \Big|[(\tau_1-t_1)^{1-\alpha}G_{\lambda,\alpha}(\tau_1,t_1) - (\tau_2-t_1)^{1-\alpha}G_{\lambda,\alpha}(\tau_2,t_1)]\Gamma(\alpha)I(u(t_1))\Big| \\
&\leq \frac{\Gamma(\alpha)M}{|1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)|}|(\tau_1-t_1)^{1-\alpha}E_{\alpha,\alpha}(\lambda\tau_1^\alpha)\tau_1^{\alpha-1} - (\tau_2-t_1)^{1-\alpha}E_{\alpha,\alpha}(\lambda\tau_2^\alpha)\tau_2^{\alpha-1}| \\
&\qquad \times\int_0^1 E_{\alpha,\alpha}(\lambda(1-s)^\alpha)(1-s)^{\alpha-1}ds \\
&\quad +M((\tau_2-t_1)^{1-\alpha} - (\tau_1-t_1)^{1-\alpha})\int_0^{\tau_1}(\tau_1-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(\tau_1-s)^\alpha)ds \\
&\quad +M(\tau_2-t_1)^{1-\alpha}\int_0^{\tau_1}|(\tau_1-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(\tau_1-s)^\alpha) - (\tau_2-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(\tau_2-s)^\alpha)|ds
\end{aligned}$$

$$
\begin{aligned}
&+M(\tau_2-t_1)^{1-\alpha}\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(\tau_2-s)^{\alpha})ds\\
&+\frac{\Gamma^2(\alpha)mE_{\alpha,\alpha}(\lambda(1-t_1)^{\alpha})(1-t_1)^{1-\alpha}}{|1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)|}\\
&\quad\times|(\tau_1-t_1)^{1-\alpha}\tau_1^{\alpha-1}E_{\alpha,\alpha}(\lambda\tau_1^{\alpha})-(\tau_2-t_1)^{1-\alpha}\tau_2^{\alpha-1}E_{\alpha,\alpha}(\lambda\tau_2^{\alpha})|\\
&+|E_{\alpha,\alpha}(\lambda(\tau_1-t_1)^{\alpha})-E_{\alpha,\alpha}(\lambda(\tau_2-t_1)^{\alpha})|\\
\to 0,&\quad \text{as } \tau_1\to\tau_2,
\end{aligned}
$$

since

$$
\begin{aligned}
&\int_0^1 E_{\alpha,\alpha}(\lambda(1-s)^{\alpha})(1-s)^{\alpha-1}ds=E_{\alpha,\alpha+1}(\lambda)\le E_{\alpha,\alpha+1}(|\lambda|),\\
&\int_0^{\tau_1} E_{\alpha,\alpha}(\lambda(\tau_1-s)^{\alpha})(\tau_1-s)^{\alpha-1}ds=\tau_1^{\alpha}E_{\alpha,\alpha+1}(\lambda\tau_1^{\alpha})\le\tau_1^{\alpha}E_{\alpha,\alpha+1}(|\lambda|\tau_1^{\alpha}),\\
&\int_{\tau_1}^{\tau_2} E_{\alpha,\alpha}(\lambda(\tau_2-s)^{\alpha})(\tau_2-s)^{\alpha-1}ds=(\tau_2-\tau_1)^{\alpha}E_{\alpha,\alpha+1}(\lambda(\tau_2-\tau_1)^{\alpha}),
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^{\tau_1}|(\tau_1-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(\tau_1-s)^{\alpha})-(\tau_2-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(\tau_2-s)^{\alpha})|ds\\
&\quad\le\sum_{i=0}^{\infty}\frac{|\lambda|^i}{\Gamma(\alpha i+\alpha)}\int_0^{\tau_1}\left|(\tau_1-s)^{\alpha i+\alpha-1}-(\tau_2-s)^{\alpha i+\alpha-1}\right|ds
\end{aligned}
$$

tend to zero as $\tau_1\to\tau_2$ by (4.13) and (4.14) in [2].

So, $(t-t_1)^{1-\alpha}(Au)(t)|_{(t_1,1]}\in C(t_1,1]$. On the other hand, we have $\lim_{t\to t_1^-}(Au)(t)$ and

$$
\begin{aligned}
&\lim_{t\to t_1^+}(t-t_1)^{1-\alpha}(Au)(t)\\
&\quad=\lim_{t\to t_1^+}(t-t_1)^{1-\alpha}\left[\int_0^1\frac{\Gamma(\alpha)E_{\alpha,\alpha}(\lambda t^{\alpha})E_{\alpha,\alpha}(\lambda(1-s)^{\alpha})t^{\alpha-1}(1-s)^{\alpha-1}}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}f(s,u(s))ds\right.\\
&\qquad+\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^{\alpha})f(s,u(s))ds\\
&\qquad+\Gamma(\alpha)\left(\frac{\Gamma(\alpha)E_{\alpha,\alpha}(\lambda t^{\alpha})E_{\alpha,\alpha}(\lambda(1-t_1)^{\alpha})t^{\alpha}(1-t_1)^{\alpha-1}}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\right.\\
&\qquad\left.\left.+(t-t_1)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-t_1)^{\alpha})\right)I(u(t_1))\right]\\
&\quad=\lim_{t\to t_1^+}(t-t_1)^{1-\alpha}\cdot\Gamma(\alpha)(t-t_1)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-t_1)^{\alpha})I(u(t_1))
\end{aligned}
$$

$$= I(u(t_1))$$

exist. Thus, $A : PC_{1-\alpha}[0,1] \to PC_{1-\alpha}[0,1]$.

For convenience, set

$$M_1 := \frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|t_1^\alpha)E_{\alpha,\alpha}(|\lambda|)(1-t_1)^{\alpha-1}t_1^\alpha}{|1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)|\alpha} + t_1^\alpha E_{\alpha,\alpha}(|\lambda|t_1^\alpha)\frac{(\Gamma(\alpha))^2}{\Gamma(2\alpha)}, \tag{3.3}$$

$$M_2 := \frac{(\Gamma(\alpha))^3 E_{\alpha,\alpha}(|\lambda|t_1^\alpha)E_{\alpha,\alpha}(|\lambda|(1-t_1)^\alpha)(1-t_1)^{2\alpha-1}}{|1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)|\Gamma(2\alpha)}, \tag{3.4}$$

$$M_3 := \begin{cases} \frac{\Gamma(\alpha)(E_{\alpha,\alpha}(|\lambda|))^2 t_1^{2\alpha-1}}{|1-\Gamma(\alpha)(E_{\alpha,\alpha}(\lambda)|\alpha} + t_1^{2\alpha-1}(1-t_1)^{1-\alpha}E_{\alpha,\alpha}(|\lambda|)\frac{(\Gamma(\alpha))^2}{\Gamma(2\alpha)}, & \text{if } 0 < \alpha \le \frac{1}{2}, \\ \frac{\Gamma(\alpha)(E_{\alpha,\alpha}(|\lambda|))^2 t_1^{2\alpha-1}}{|1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)|\alpha} + (1-t_1)^{1-\alpha}E_{\alpha,\alpha}(|\lambda|)\frac{(\Gamma(\alpha))^2}{\Gamma(2\alpha)}, & \text{if } \frac{1}{2} < \alpha \le 1, \end{cases} \tag{3.5}$$

$$M_4 := \frac{(\Gamma(\alpha))^3 E_{\alpha,\alpha}(|\lambda|)E_{\alpha,\alpha}(|\lambda|(1-t_1)^\alpha)t_1^{\alpha-1}(1-t_1)^\alpha}{|1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)|\Gamma(2\alpha)}, \tag{3.6}$$

$$N_1 := \frac{(\Gamma(\alpha))^2 E_{\alpha,\alpha}(|\lambda|t_1^\alpha)E_{\alpha,\alpha}(|\lambda|(1-t_1)^\alpha)t_1^{\alpha-1}(1-t_1)^{\alpha-1}}{|1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)|}, \tag{3.7}$$

$$N_2 := \frac{(\Gamma(\alpha))^2 E_{\alpha,\alpha}(|\lambda|)E_{\alpha,\alpha}(|\lambda|(1-t_1)^\alpha)t_1^{2\alpha-2}}{|1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)|} + \Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|(1-t_1)^\alpha)t_1^{\alpha-1}. \tag{3.8}$$

Theorem 3.2. *Suppose that* (H_1) *and the following condition hold:*

(H_2) *there exist positive constant k, and l such that*

$$|f(t,u) - f(t,v)| \le k|u-v|, \quad |I(u) - I(v)| \le l|u-v|, \quad \forall t \in [0,1],\ u,v \in R. \tag{3.9}$$

Then the problem (1.1)-(1.3) has a unique solution in $PC_{1-\alpha}[0,1]$, provided that

$$\max\{k(M_1+M_2) + lN_1,\ k(M_3+M_4) + lN_2\} < 1. \tag{3.10}$$

Proof. By (H_1) and Lemma 3.1, we have $A : PC_{1-\alpha}[0,1] \to PC_{1-\alpha}[0,1]$. For any $x, y \in PC_{1-\alpha}[0,1]$, and each $t \in [0,t_1]$, we obtain by (H_2) that

$$t^{1-\alpha}|(Ax)(t) - (Ay)(t)| \le \int_0^1 t^{1-\alpha}|G_{\lambda,\alpha}(t,s)||f(s,x(s)) - f(s,y(s))|ds$$

$$+t^{1-\alpha}\Gamma(\alpha)|G_{\lambda,\alpha}(t,t_1)||I(x(t_1)) - I(y(t_1))|$$

$$\le \int_0^1 t^{1-\alpha}|G_{\lambda,\alpha}(t,s)|k|x(s) - y(s)|ds + t^{1-\alpha}\Gamma(\alpha)|G_{\lambda,\alpha}(t,t_1)|l|x(t_1) - y(t_1)|$$

$$= k\int_0^{t_1} t^{1-\alpha}|G_{\lambda,\alpha}(t,s)|s^{\alpha-1}s^{1-\alpha}|x(s) - y(s)|ds$$

$$+k\int_{t_1}^{1} t^{1-\alpha}|G_{\lambda,\alpha}(t,s)|(s-t_1)^{\alpha-1}(s-t_1)^{1-\alpha}|x(s)-y(s)|ds$$

$$+l\Gamma(\alpha)t^{1-\alpha}|G_{\lambda,\alpha}(t,t_1)|t_1^{\alpha-1}t_1^{1-\alpha}|x(t_1)-y(t_1)|$$

$$\leq \|x-y\|_{1-\alpha}\left[k\int_0^{t_1} t^{1-\alpha}|G_{\lambda,\alpha}(t,s)|s^{\alpha-1}ds\right.$$

$$\left.+k\int_{t_1}^{1} t^{1-\alpha}|G_{\lambda,\alpha}(t,s)|(s-t_1)^{\alpha-1}ds + l\Gamma(\alpha)t^{1-\alpha}|G_{\lambda,\alpha}(t,t_1)|t_1^{\alpha-1}\right]. \tag{3.11}$$

From the expression of $G_{\lambda,\alpha}(t,s)$, and (4.21) in [2], we have

$$\int_0^{t_1} t^{1-\alpha}|G_{\lambda,\alpha}(t,s)|s^{\alpha-1}ds \leq \int_0^{t_1} t^{1-\alpha}\left|\frac{\Gamma(\alpha)E_{\alpha,\alpha}(\lambda t^{\alpha})E_{\alpha,\alpha}(\lambda(1-s)^{\alpha})t^{\alpha-1}(1-s)^{\alpha-1}}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\right|s^{\alpha-1}ds$$

$$+\int_0^{t} t^{1-\alpha}(t-s)^{\alpha-1}|E_{\alpha,\alpha}(\lambda(t-s)^{\alpha})|s^{\alpha-1}ds$$

$$\leq \frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|t_1^{\alpha})E_{\alpha,\alpha}(|\lambda|)(1-t_1)^{\alpha-1}}{|1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)|}\int_0^{t_1} s^{\alpha-1}ds$$

$$+t^{1-\alpha}E_{\alpha,\alpha}(|\lambda|t_1^{\alpha})\int_0^{t}(t-s)^{\alpha-1}s^{\alpha-1}ds$$

$$= \frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|t_1^{\alpha})E_{\alpha,\alpha}(|\lambda|)(1-t_1)^{\alpha-1}t_1^{\alpha}}{|1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)|\alpha} + t^{\alpha}E_{\alpha,\alpha}(|\lambda|t_1^{\alpha})\frac{(\Gamma(\alpha))^2}{\Gamma(2\alpha)}$$

$$\leq \frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|t_1^{\alpha})E_{\alpha,\alpha}(|\lambda|)(1-t_1)^{\alpha-1}t_1^{\alpha}}{|1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)|\alpha} + t_1^{\alpha}E_{\alpha,\alpha}(|\lambda|t_1^{\alpha})\frac{(\Gamma(\alpha))^2}{\Gamma(2\alpha)} = M_1. \tag{3.12}$$

And

$$\int_{t_1}^{1} t^{1-\alpha}|G_{\lambda,\alpha}(t,s)|(s-t_1)^{\alpha-1}ds$$

$$\leq \int_{t_1}^{1} t^{1-\alpha}\left|\frac{\Gamma(\alpha)E_{\alpha,\alpha}(\lambda t^{\alpha})E_{\alpha,\alpha}(\lambda(1-s)^{\alpha})t^{\alpha-1}(1-s)^{\alpha-1}}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\right|(s-t_1)^{\alpha-1}ds$$

$$\leq \frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|t_1^{\alpha})E_{\alpha,\alpha}(|\lambda|(1-t_1)^{\alpha})}{|1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)|}\int_{t_1}^{1}(1-s)^{\alpha-1}(s-t_1)^{\alpha-1}ds$$

$$= \frac{(\Gamma(\alpha))^3E_{\alpha,\alpha}(|\lambda|t_1^{\alpha})E_{\alpha,\alpha}(|\lambda|(1-t_1)^{\alpha})(1-t_1)^{2\alpha-1}}{|1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)|\Gamma(2\alpha)} = M_2, \tag{3.13}$$

since

$$\int_{t_1}^{1}(1-s)^{\alpha-1}(s-t_1)^{\alpha-1}ds = \int_0^{1-t_1} s^{\alpha-1}(1-t_1-s)^{\alpha-1}ds = (1-t_1)^{2\alpha-1}\frac{(\Gamma(\alpha))^2}{\Gamma(2\alpha)}. \tag{3.14}$$

Moreover, we have

$$\begin{aligned}&\Gamma(\alpha)t^{1-\alpha}|G_{\lambda,\alpha}(t,t_1)|t_1^{\alpha-1}\\&\quad\le\frac{(\Gamma(\alpha))^2E_{\alpha,\alpha}(|\lambda|t_1^{\alpha})E_{\alpha,\alpha}(|\lambda|(1-t_1)^{\alpha})t_1^{\alpha-1}(1-t_1)^{\alpha-1}}{|1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)|}=N_1.\end{aligned}\tag{3.15}$$

Substituting (3.12), (3.13) and (3.15) into (3.11), we get

$$\sup_{t\in[0,t_1]}t^{1-\alpha}|(Ax)(t)-(Ay)(t)|\le(k(M_1+M_2)+N_1)\|x-y\|_{1-\alpha}.\tag{3.16}$$

On the other hand, for each $x,y\in PC_{1-\alpha}[0,1]$ and $t\in(t_1,1]$, we obtain that

$$\begin{aligned}&(t-t_1)^{1-\alpha}|(Ax)(t)-(Ay)(t)|\\&\quad\le\int_0^1(t-t_1)^{1-\alpha}|G_{\lambda,\alpha}(t,s)||f(s,x(s))-f(s,y(s))|ds\\&\qquad+(t-t_1)^{1-\alpha}\Gamma(\alpha)|G_{\lambda,\alpha}(t,t_1)||I(x(t_1))-I(y(t_1))|\\&\quad\le\int_0^1(t-t_1)^{1-\alpha}|G_{\lambda,\alpha}(t,s)|k|x(s)-y(s)|ds\\&\qquad+(t-t_1)^{1-\alpha}\Gamma(\alpha)|G_{\lambda,\alpha}(t,t_1)|l|x(t_1)-y(t_1)|\\&\quad=k\int_0^{t_1}(t-t_1)^{1-\alpha}|G_{\lambda,\alpha}(t,s)|s^{\alpha-1}s^{1-\alpha}|x(s)-y(s)|ds\\&\qquad+k\int_{t_1}^1(t-t_1)^{1-\alpha}|G_{\lambda,\alpha}(t,s)|(s-t_1)^{\alpha-1}(s-t_1)^{1-\alpha}|x(s)-y(s)|ds\\&\qquad+(t-t_1)^{1-\alpha}\Gamma(\alpha)l|G_{\lambda,\alpha}(t,t_1)|t_1^{\alpha-1}t_1^{1-\alpha}|x(t_1)-y(t_1)|\\&\quad\le\|x-y\|_{1-\alpha}\Bigg[k\int_0^{t_1}(t-t_1)^{1-\alpha}|G_{\lambda,\alpha}(t,s)|s^{\alpha-1}ds\\&\qquad+k\int_{t_1}^1(t-t_1)^{1-\alpha}|G_{\lambda,\alpha}(t,s)|(s-t_1)^{\alpha-1}ds+l\Gamma(\alpha)|G_{\lambda,\alpha}(t,t_1)|(t-t_1)^{1-\alpha}t_1^{\alpha-1}\Bigg].\end{aligned}\tag{3.17}$$

By (4.21) in [2], we have

$$\begin{aligned}&\int_0^{t_1}(t-t_1)^{1-\alpha}|G_{\lambda,\alpha}(t,s)|s^{\alpha-1}ds\\&\quad\le\int_0^{t_1}(t-t_1)^{1-\alpha}\left|\frac{\Gamma(\alpha)E_{\alpha,\alpha}(\lambda t^{\alpha})E_{\alpha,\alpha}(\lambda(1-s)^{\alpha})t^{\alpha-1}(1-s)^{\alpha-1}}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\right|s^{\alpha-1}ds\\&\quad+\int_0^t(t-t_1)^{1-\alpha}(t-s)^{\alpha-1}|E_{\alpha,\alpha}(\lambda(t-s)^{\alpha})|s^{\alpha-1}ds\end{aligned}$$

$$\leq (1-t_1)^{1-\alpha}\frac{\Gamma(\alpha)(E_{\alpha,\alpha}(|\lambda|))^2 t_1^{\alpha-1}(1-t_1)^{\alpha-1}}{|1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)|}\int_0^{t_1} s^{\alpha-1}ds$$

$$+(1-t_1)^{1-\alpha}E_{\alpha,\alpha}(|\lambda|)\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}ds$$

$$= \frac{\Gamma(\alpha)(E_{\alpha,\alpha}(|\lambda|))^2 t_1^{2\alpha-1}}{|1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)|\alpha} + (1-t_1)^{1-\alpha}E_{\alpha,\alpha}(|\lambda|)t^{2\alpha-1}\frac{(\Gamma(\alpha))^2}{\Gamma(2\alpha)}$$

$$\leq \frac{\Gamma(\alpha)(E_{\alpha,\alpha}(|\lambda|))^2 t_1^{2\alpha-1}}{|1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)|\alpha}$$

$$+\begin{cases} t_1^{2\alpha-1}(1-t_1)^{1-\alpha}E_{\alpha,\alpha}(|\lambda|)\frac{(\Gamma(\alpha))^2}{\Gamma(2\alpha)}, & if\ 0<\alpha\leq\frac{1}{2}, \\ (1-t_1)^{1-\alpha}E_{\alpha,\alpha}(|\lambda|)\frac{(\Gamma(\alpha))^2}{\Gamma(2\alpha)}, & if\ \frac{1}{2}<\alpha\leq 1, \end{cases}$$

$$= M_3. \tag{3.18}$$

From (3.14), we get

$$\int_{t_1}^1 (t-t_1)^{1-\alpha}|G_{\lambda,\alpha}(t,s)|(s-t_1)^{\alpha-1}ds$$

$$= \int_{t_1}^1 (t-t_1)^{1-\alpha}\left|\frac{\Gamma(\alpha)E_{\alpha,\alpha}(\lambda t^\alpha)E_{\alpha,\alpha}(\lambda(1-s)^\alpha)t^{\alpha-1}(1-s)^{\alpha-1}}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\right|(s-t_1)^{\alpha-1}ds$$

$$\leq (1-t_1)^{1-\alpha}\frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)E_{\alpha,\alpha}(|\lambda|(1-t_1)^\alpha)t_1^{\alpha-1}}{|1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)|}\int_{t_1}^1 (s-t_1)^{\alpha-1}(1-s)^{\alpha-1}ds$$

$$= \frac{(\Gamma(\alpha))^3 E_{\alpha,\alpha}(|\lambda|)E_{\alpha,\alpha}(|\lambda|(1-t_1)^\alpha)t_1^{\alpha-1}(1-t_1)^\alpha}{|1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)|\Gamma(2\alpha)} = M_4. \tag{3.19}$$

Moreover, we have

$$\Gamma(\alpha)|G_{\lambda,\alpha}(t,t_1)|(t-t_1)^{1-\alpha}t_1^{\alpha-1}$$

$$\leq \frac{(\Gamma(\alpha))^2 E_{\alpha,\alpha}(|\lambda|t^\alpha)E_{\alpha,\alpha}(|\lambda|(1-t_1)^\alpha)t^{\alpha-1}(1-t_1)^{\alpha-1}(t-t_1)^{1-\alpha}t_1^{\alpha-1}}{|1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)|}$$

$$+\Gamma(\alpha)t_1^{\alpha-1}E_{\alpha,\alpha}(|\lambda|(t-t_1)^\alpha)$$

$$\leq \frac{(\Gamma(\alpha))^2 E_{\alpha,\alpha}(|\lambda|)E_{\alpha,\alpha}(|\lambda|(1-t_1)^\alpha)t_1^{2\alpha-2}}{|1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)|}$$

$$+\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|(1-t_1)^\alpha)t_1^{\alpha-1} = N_2. \tag{3.20}$$

Substituting (3.18)-(3.20) into (3.17), we obtain

$$\sup_{t\in(t_1,1]}(t-t_1)^{1-\alpha}|(Ax)(t)-(Ay)(t)| \leq (k(M_3+M_4)+lN)\|x-y\|_{1-\alpha}. \tag{3.21}$$

From (3.11), (3.16) and (3.21), we have

$$\begin{aligned}
&\|Ax - Ay\|_{1-\alpha}\\
&\quad = \max\left\{ \sup_{t\in[0,t_1]} t^{1-\alpha}|(Ax)(t)-(Ay)(t)|,\ \sup_{t\in(t_1,1]} (t-t_1)^{1-\alpha}|(Ax)(t)-(Ay)(t)| \right\}\\
&\quad \le \max\{k(M_1+M_2)+lN_1,\ k(M_3+M_4)+lN_2\}\|x-y\|_{1-\alpha}, \qquad (3.22)
\end{aligned}$$

which implies that A is a contraction (by (3.10)). Therefore it has a unique fixed point.

4. An example

The following illustrative example will demonstrate the effectiveness of our main result.

Example 4.1. Consider the nonlinear impulsive fractional differential equation with periodic boundary conditions as follows:

$$D^{0.9}u(t) - \frac{1}{3}u(t) = \frac{t^2}{t^2+11}\arctan u(t), \quad t\in(0,1]\setminus\{0.5\}, \qquad (4.1)$$

$$\lim_{t\to 0^+} t^{0.1}u(t) = u(1), \qquad (4.2)$$

$$\lim_{t\to 0.5^+} (t-0.5)^{0.1}(u(t)-u(0.5)) = \frac{1}{13}\sin(u(0.5)). \qquad (4.3)$$

Then BVP (4.1)-(4.3) reduces to BVP (1.1)-(1.3) with $\alpha = 0.9$, $\lambda = \frac{1}{3}$, $f(t,u) = \frac{t^2}{t^2+11}\arctan u$, $I(u) = \frac{1}{13}\sin u$ and $t_1 = 0.5$.

Obviously, there exist positive constants $M = \frac{\pi}{24}$, $m = \frac{1}{13}$, $k = \frac{1}{12}$ and $l = \frac{1}{13}$ such that

$$|f(t,u)| \le M, \quad |I(u)| \le m, \quad t\in[0,1],\ u\in R,$$

and

$$|f(t,u)-f(t,v)| \le k|u-v|, \quad |I(u)-I(v)| \le l|u-v|, \quad t\in[0,1],\ u,v\in R.$$

So that the conditions (H_1) and (H_2) hold. Moreover, we have

$$E_{\alpha,\alpha}(\lambda) = E_{0.9,0.9}\left(\frac{1}{3}\right) = \frac{1}{\Gamma(0.9)} + \frac{1}{3\Gamma(1.8)} + \frac{1}{3^2\Gamma(2.7)} + \frac{1}{3^3\Gamma(3.6)} + \frac{1}{3^4\Gamma(4.5)} + \cdots$$

Thus, we obtain

$$E_{0.9,0.9}\left(\frac{1}{3}\right) > \frac{1}{\Gamma(0.9)} + \frac{1}{3\Gamma(1.8)} + \frac{1}{3^2\Gamma(2.7)} = 1.3656, \qquad (4.4)$$

and

$$E_{0.9,0.9}\left(\frac{1}{3}\right) < \frac{1}{\Gamma(0.9)} + \frac{1}{3\Gamma(1.8)} + \frac{1}{3^2\Gamma(2.7)} + \frac{1}{3}\left(\frac{1}{3^3} + \frac{1}{3^4} + \cdots\right)$$
$$= 1.3656 + 0.0185 = 1.3841. \tag{4.5}$$

Similarly, we have

$$E_{\alpha,\alpha}(|\lambda|t_1^\alpha) = E_{\alpha,\alpha}(|\lambda|(1-t_1)^\alpha) = E_{0.9,0.9}\left(\frac{1}{3\cdot 2^{0.9}}\right) < 1.149. \tag{4.6}$$

By (4.4)-(4.6) and simple calculation, we can obtain the estimation of M_1, M_2, M_3, M_4, N_1 and N_2 (are as in (3.3)-(3.8) as follows :

$$M_1 < 3.019, \quad M_2 < 1.9351, \quad M_3 < 4.2091,$$

$$M_4 < 2.4333, \quad N_1 < 3.6153, \quad N_2 < 5.6709.$$

Hence, we get

$$\max\{k(M_1 + M_2) + lN_1,\ k(M_3 + M_4) + lN_2\} = 0.9898 < 1,$$

that is (3.10) holds. So, it follows from Theorem 3.2 that the BVP (4.1)-(4.3) has a unique solution in $PC_{0.1}[0, 1]$.

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