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Some results on impulsive boundary value problem for fractional differential inclusions

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Abstract
This paper deals with impulsive fractional differential inclusions with a fractional order multi-point boundary condition and with fractional order impulses. By use of multi-valued analysis and topological fixed point theory, we present some existence results under both convexity and nonconvexity conditions on the multi-valued right-hand side. The compactness of the solutions set and continuous version of Filippov's theorem are also investigated.
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1 Introduction

In recent years, the theory of fractional differential equations has been an object of increasing interest because of its wide applicability in biology, in medicine and in more and more fields, see for instance [4, 14, 15, 16, 17, 26, 28, 34, 35, 36] and references therein. In particular, Tian [35] studied the existence of solutions for equation

$$
\begin{aligned}
&{}^C\!D^{\alpha}u(t) = f(t, u(t)), \quad a.e. t \in J,\\
&\Delta u(t)|_{t=t_k} = I_k(u(t_k)), k = 1, 2, \cdots, m,\\
&\Delta u'(t)|_{t=t_k} = \bar{I}_k(u(t_k)), k = 1, 2, \cdots, m,\\
&u(0) + u'(0) = 0, u(1) + u'(\xi) = 0.
\end{aligned}
$$

By use of Banach's fixed point theorem and Schauder's fixed point theorem, the authors obtained some existence results.

On the other hand, realistic problems, arising from economics, optimal control, etc., can be modeled as differential inclusions. So, differential inclusions have been widely investigated by many authors, see, for instance [1, 6, 12, 18, 19, 20, 21, 22, 23, 25, 32, 33] and references therein.

To the best of our knowledge, there are few papers concerning fractional-order impulsive differential inclusions with a multi-point boundary condition. Motivated by works mentioned above, we consider the following problem:

$$ {}^C\!D^{\alpha}u(t) \in F(t, u(t)), \quad a.e. t \in J, \tag{1.1} $$

$$ \Delta u(t)|_{t=t_k} = I_k(u(t_k)), k = 1, 2, \cdots, m, \tag{1.2} $$

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$$\Delta {}^C\!D^\beta u(t)|_{t=t_k} = \bar{I}_k(u(t_k)), k = 1, 2, \cdots, m, \tag{1.3}$$

$$u(0) + {}^C\!D^\beta u(0) = A, u(1) + {}^C\!D^\beta u(\xi) = B, \tag{1.4}$$

where ${}^C\!D^\alpha$ is the Caputo fractional derivative, and $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multi-valued map with compact values ($\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$). $1 < \alpha \leq 2, 0 < \beta < \alpha - 1$, A, B are real numbers. $J = [0, 1]$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$, $0 < \xi < t_m$, $\xi \neq t_k (k = 1, 2, \cdots, m)$. $I_k, \bar{I}_k \in C[\mathbb{R}, \mathbb{R}]$, $\Delta u(t)|_{t=t_k} = u(t_k^+) - u(t_k^-), \Delta {}^C\!D^\beta u(t)|_{t=t_k} = {}^C\!D^\beta u(t_k^+) - {}^C\!D^\beta u(t_k^-)$, $u(t_k^+)$ and $u(t_k^-)$ represent the right-hand limit and the left-hand limit of the function $u(t)$ at $t = t_k$.

Our goal in this paper is to give some existence results and continuous version of Filippov's theorem for fractional differential inclusion (1.1)-(1.4), where the right-hand side is either convexity or nonconvexity. Furthermore, we prove that the set of solutions is compact under suitable conditions in Theorem 3.1. Our work complement and extend some results of [35].

The remainder of this paper is organized as follows. In Section 2, we introduce some notations, definitions, preliminary facts about the fractional calculus and an auxiliary lemma, which are used in the next two sections. In Section 3, we give the existence of solutions under both convexity and nonconvexity conditions on the multi-valued right-hand side. The compactness of the solutions set is also established. Finally, we give a continuous version of Filippov's theorem in Section 4.

2 Preliminaries

We now introduce some notations, definitions, preliminary facts about the fractional calculus and an auxiliary lemma, which will be used later.

Let $AC^1(J, \mathbb{R})$ be the space of functions $u : J \to \mathbb{R}$, differentiable and whose derivative, u', is absolutely continuous. And let $C(J, \mathbb{R})$ be the Banach space of all continuous functions from J into $\mathbb{R}$ with the usual norm

$$\|y\|_\infty = \sup\{|y(t)| : t \in J\}.$$

$L^1[J, \mathbb{R}]$ denote the Banach space of measurable functions $y : J \to \mathbb{R}$ which are Lebesgue integrable; it is normed by

$$\|y\|_{L^1} = \int_0^1 |y(s)| ds.$$

Definition 2.1. *The fractional (arbitrary) order integral of the function $v(t) \in L^1([0, \infty), \mathbb{R})$ of $\mu \in \mathbb{R}^+$ is defined by*

$$I^\mu v(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t - s)^{\mu - 1} v(s) ds, \ t > 0.$$

Definition 2.2. *The Riemann-Liouville fractional derivative of order $\mu > 0$ for a function $v(t)$ given in the interval $[0, \infty)$ is defined by*

$$D^\mu v(t) = \frac{1}{\Gamma(n - \mu)} (\frac{d}{dt})^n \int_0^t (t - s)^{n - \mu - 1} v(s) ds$$

provided that the right hand side is point-wise defined. Here $n = [\mu] + 1$ and $[\mu]$ means the integral part of the number μ, and Γ is the Euler gamma function.

We observe an alternative definition of fractional derivative, originally introduced by Caputo [7, 8] in the late 1960's and adopted by Caputo and Mainardi [9] in the framework of the theory of Linear Viscoelasticity (see a review in [30]).

Definition 2.3. *The Caputo fractional derivative of order $\mu > 0$ for a function $v(t)$ given in the interval $[0, \infty)$ is defined by*

$$ {}^{C}\!\mathcal{D}^{\mu} v(t) = \frac{1}{\Gamma(n-\mu)} \int_0^t (t-s)^{n-\mu-1} v^{(n)}(s) ds $$

provided that the right hand side is point-wise defined. Here $n = [\mu] + 1$ and $[\mu]$ means the integral part of the number μ, and Γ is the Euler gamma function.

This definition is of course more restrictive than the Riemann-Liouville definition, in that it requires the absolute integrability of the derivative of order n. Whenever we use the operator ${}^{C}\!\mathcal{D}^{\mu}$ we (tacitly) assume that this condition is met. The following properties of the fractional calculus theory are well known, see, e.g., [27, 34, 36].

(i) ${}^{C}\!\mathcal{D}^{\beta} I^{\beta} v(t) = v(t)$ for a.e. $t \in J$, where $v(t) \in L^1[0, 1]$, $\beta > 0$.

(ii) $I^{\beta}\, {}^{C}\!\mathcal{D}^{\beta} v(t) = v(t) - \sum_{j=0}^{n-1} c_j t^j$ for a.e. $t \in J$, where $v(t) \in L^1[0, 1]$, $\beta > 0$, $c_j (j = 0, 1, \cdots n-1)$ are some constants, $n = [\beta] + 1$.

(iii) $I^{\beta} : C[0, 1] \to C[0, 1]$, $I^{\beta} : L^1[0, 1] \to L^1[0, 1]$, $\beta > 0$.

(iv) ${}^{C}\!\mathcal{D}^{\beta} I^{\alpha} = I^{\alpha-\beta}$ and ${}^{C}\!\mathcal{D}^{\beta} 1 = 0$ for $t \in J$, $\alpha - \beta > 0$.

More details on fractional derivatives and their properties can be found in [27, 34].

Let $(X, \|\cdot\|)$ be a separable Banach space, and denote

$\mathcal{P}(X) = \{Y \subset X : Y \neq \varnothing\}$,
$\mathcal{P}_{cv}(X) = \{Y \in \mathcal{P}(X) : Y \text{ convex}\}$,
$\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\}$,
$\mathcal{P}_{b}(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}$,
$\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact}\}$,
$\mathcal{P}_{cv,cp}(X) = \mathcal{P}_{cv}(X) \cap \mathcal{P}_{cp}(X)$.

A multi-valued map $G : X \to \mathcal{P}(X)$ has convex (closed) values if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B) = \cup_{x \in B} G(x)$ is bounded in X for any bounded set B of X (i.e. $\sup_{x \in B}\{\sup\{|u| : u \in G(x)\}\} < +\infty$).

The map G is upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$ the set $G(x_0)$ is a nonempty, closed subset of X, and if, for each open set N of X containing $G(x_0)$, there exists an open neighborhood M of x_0 such that $G(M) \subseteq N$.

Likewise, G is lower semi-continuous (l.s.c.) if $G : X \to \mathcal{P}(X)$ be a multi-valued operator with nonempty closed values, and if, the set $\{x \in X : G(x) \cap B \neq \varnothing\}$ is open for any open set B in X.

G is completely continuous if $G(B)$ is relatively compact for every bounded subset $B \subseteq X$.

If the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e. $x_n \to x_*, y_n \to y_*, y_n \in G(x_n)$ imply $y_* \in G(x_*)$).

We say that G has a fixed point if there exists $x \in X$ such that $x \in G(x)$.

A multi-valued map $G : J \to \mathcal{P}_{cl}(X)$ is said to be measurable if for each $x \in X$ the function $Y : J \to \mathbb{R}^+$ defined by $Y(t) = d(x, G(t)) = \inf\{\|x - z\| : z \in G(t)\}$ is measurable.

Let $\mathbb{K}$ be a subset of $[0, 1] \times \mathbb{R}$. $\mathbb{K}$ is $\mathcal{L} \otimes \mathcal{B}$ measurable if $\mathbb{K}$ belongs to the σ-algebra generated by all sets of the form $\mathcal{J} \times D$ where $\mathcal{J}$ is Lebesgue measurable in $[0, 1]$ and D is Borel measurable in $\mathbb{R}$. A subset $\mathbb{K}$ of $L^1([0, 1], \mathbb{R})$ is decomposable if for all $u, v \in \mathbb{K}$ and $\mathcal{J} \subset [0, 1]$ measurable, $u\mathcal{X}_{\mathcal{J}} + v\mathcal{X}_{[0,1]-\mathcal{J}} \in \mathbb{K}$, where $\mathcal{X}$ stands for the characteristic function.

Definition 2.4. *The multi-valued map $F : J \times X \to \mathcal{P}(X)$ is L^1-Carathéodory if*
(i) $t \mapsto F(t, u)$ is measurable for each $u \in X$;
(ii) $u \mapsto F(t, u)$ is upper semi-continuous for almost all $t \in J$;
(iii) For each $q > 0$, there exists $\phi_q \in L^1(J, \mathbb{R}^+)$ such that

$\|F(t, u)\|_{\mathcal{P}} = \sup\{\|v\| : v \in F(t, u)\} \leq \phi_q(t)$,
for all $\|u\| \leq q$ and for almost all $t \in J$.

For any $u \in C([0,1],\mathbb{R})$, we define the set

$$S_{F,u} = \{v \in L^1(J,\mathbb{R}) : v(t) \in F(t,u(t)) \text{ for a.e. } t \in J\}.$$

This is known as the set of selection functions.

For the sake of convenience, we introduce the following notations.

Let $J_0 = [0,t_1], J_k = (t_k, t_{k+1}], k = 1,2,\cdots,m$ and and let u_k be the restriction of a function u to J_k.

$PC(J) = \{u : [0,1] \to \mathbb{R} | u \in C(J_k,\mathbb{R}),\ u(t_k^+)$ and $u(t_k^-)$ exist, and $u(t_k^-) = u(t_k), k = 0,1,\cdots,m\}$. $PC(J_k) = \{u : u \in C(J_k,\mathbb{R})$ and $u(t_k^+)$ exists$\}$, $k = 0,1,\cdots,m$.

Obviously, $PC(J)$ is a Banach space with the norm $\|u\|_{PC} = \max\{\|u\|_{PC_k} : k = 1,2,\cdots,m\}$, where $\|u\|_{PC_k} = \sup\{|y(t)| : t \in [t_k, t_{k+1}]\}$.

Definition 2.5. *A function $u \in PC(J) \cap \bigcup_{k=0}^{m} AC^1(J_k,\mathbb{R})$ is said to be a solution of (1.1)-(1.4) if there exists $v \in L^1(J,\mathbb{R})$ with $v(t) \in F(t,u(t))$ for a.e. $t \in J$ such that u satisfies the fractional differential equation ${}^C\!D^\alpha u(t) = v(t)$ a.e. on J, and the condition (1.2), (1.3) and (1.4).*

Lemma 2.1. *Let $v \in L^1(J,\mathbb{R})$. $\xi \in (t_l, t_{l+1}), 1 \le l \le m-1$, and l is a nonnegative integer. $1 < \alpha \le 2, 0 < \beta < \alpha - 1$, A, B are real numbers. Then u is the unique solution of the boundary value problem*

$$\begin{cases} {}^C\!D^\alpha u(t) = v(t), \quad a.e. t \in J, t \ne t_k, k = 1,2,\cdots,m, \\ \Delta u(t)|_{t=t_k} = I_k(u(t_k)), \\ \Delta {}^C\!D^\beta u(t)|_{t=t_k} = \bar{I}_k(u(t_k)), k = 1,2,\cdots,m, \\ u(0) + {}^C D^\beta u(0) = A, u(1) + {}^C D^\beta u(\xi) = B, \end{cases} \tag{2.1}$$

if and only if

$$\begin{aligned} u(t) = &\tfrac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} v(s) ds + \tfrac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} v(s) ds \\ &-\Gamma(2-\beta) \sum_{i=1}^{k} \tfrac{(t_i - t_{i-1})^\beta}{\Gamma(\alpha-\beta)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-\beta-1} v(s) ds \\ &- \tfrac{\Gamma(2-\beta)}{(t_{k+1}-t_k)^{1-\beta}} \tfrac{(t-t_k)}{\Gamma(\alpha-\beta)} \int_{t_k}^{t_{k+1}} (t_{k+1} - s)^{\alpha-\beta-1} v(s) ds \\ &+ \mathbb{I}_{k,A,B}(u), t \in J_k, k = 0,1,\cdots,m-1, \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} u(t) = &\tfrac{1}{\Gamma(\alpha)} \int_{t_m}^{t} (t-s)^{\alpha-1} v(s) ds \\ &+ \tfrac{1}{\Gamma(\alpha)} \tfrac{1-t}{1-t_m} \{ \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} v(s) ds \\ &-\Gamma(\alpha)\Gamma(2-\beta) \sum_{i=1}^{m} \tfrac{(t_i - t_{i-1})^\beta}{\Gamma(\alpha-\beta)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-\beta-1} v(s) ds \} \\ &- \tfrac{t-t_m}{1-t_m} \tfrac{1}{\Gamma(\alpha)} \{ \int_{t_m}^{1} (1-s)^{\alpha-1} v(s) ds + \tfrac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} \int_{t_l}^{\xi} (\xi - s)^{\alpha-\beta-1} v(s) ds \\ &- \tfrac{(\xi-t_l)^{1-\beta}}{(t_{l+1}-t_l)^{1-\beta}} \tfrac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} \int_{t_l}^{t_{l+1}} (t_{l+1} - s)^{\alpha-\beta-1} v(s) ds \} \\ &+ \mathbb{I}_{m,A,B}(u), t \in J_m, \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} \mathbb{I}_{k,A,B}(u) = &-\Gamma(2-\beta) \sum_{i=1}^{k} (t_i - t_{i-1})^\beta \bar{I}_i(u(t_i)) + A + \sum_{i=1}^{k} I_i(u(t_i)) \\ &- \tfrac{\Gamma(2-\beta)(t-t_k)}{(t_{k+1}-t_k)^{1-\beta}} \bar{I}_{k+1}(u(t_{k+1})), k = 0,1,\cdots,m-1, \\ \mathbb{I}_{m,A,B}(u) = &-\tfrac{1-t}{1-t_m} \Gamma(2-\beta) \sum_{i=1}^{m} (t_i - t_{i-1})^\beta \bar{I}_i(u(t_i)) \\ &+ \tfrac{A(1-t)}{1-t_m} + \tfrac{B(t-t_m)}{1-t_m} + \tfrac{1-t}{1-t_m} \sum_{i=1}^{m} I_i(u(t_i)) \\ &+ \tfrac{t-t_m}{1-t_m} \tfrac{(\xi-t_l)^{1-\beta}}{(t_{l+1}-t_l)^{1-\beta}} \bar{I}_{l+1}(u(t_{l+1})). \end{aligned} \tag{2.4}$$

Proof. Suppose that u is a solution of (2.1), we have

$$u(t) = I^{\alpha}v(t) - c_0 - d_0 t = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v(s)ds - c_0 - d_0 t, t \in J_0, \tag{2.5}$$

for some $c_0, d_0 \in \mathbb{R}$. Then

$$^{C}\!D^{\beta}u(t) = \frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}v(s)ds - d_0\frac{t^{1-\beta}}{\Gamma(2-\beta)}, t \in J_0. \tag{2.6}$$

If $t \in J_1$, we have

$$\begin{aligned} &u(t) = \tfrac{1}{\Gamma(\alpha)}\textstyle\int_{t_1}^t (t-s)^{\alpha-1}v(s)ds - c_1 - d_1(t-t_1),\\ &^{C}\!D^{\beta}u(t) = \tfrac{1}{\Gamma(\alpha-\beta)}\textstyle\int_{t_1}^t (t-s)^{\alpha-\beta-1}v(s)ds - d_1\tfrac{(t-t_1)^{1-\beta}}{\Gamma(2-\beta)}, \end{aligned}$$

for some $c_1, d_1 \in \mathbb{R}$. Thus

$$\begin{aligned} &u(t_1^-) = \tfrac{1}{\Gamma(\alpha)}\textstyle\int_0^{t_1} (t_1-s)^{\alpha-1}v(s)ds - c_0 - d_0 t_1,\\ &u(t_1^+) = -c_1,\\ &^{C}\!D^{\beta}u(t_1^-) = \tfrac{1}{\Gamma(\alpha-\beta)}\textstyle\int_0^{t_1} (t_1-s)^{\alpha-\beta-1}v(s)ds - d_0\tfrac{t_1^{1-\beta}}{\Gamma(2-\beta)},\\ &^{C}\!D^{\beta}u(t_1^+) = 0. \end{aligned}$$

In view of $\Delta u(t)|_{t=t_1} = I_1(u(t_1)), \Delta {}^{C}\!D^{\beta}u(t)|_{t=t_1} = \bar{I}_1(u(t_1))$, we have

$$\begin{aligned} &-c_1 = \tfrac{1}{\Gamma(\alpha)}\textstyle\int_0^{t_1} (t_1-s)^{\alpha-1}v(s)ds - c_0 - d_0 t_1 + I_1(u(t_1)),\\ &-d_0 = -\tfrac{\Gamma(2-\beta)}{t_1^{1-\beta}}\tfrac{1}{\Gamma(\alpha-\beta)}\textstyle\int_0^{t_1} (t_1-s)^{\alpha-\beta-1}v(s)ds - \tfrac{\Gamma(2-\beta)}{t_1^{1-\beta}}\bar{I}_1(u(t_1)). \end{aligned}$$

Hence, we obtain

$$\begin{aligned} u(t) &= \tfrac{1}{\Gamma(\alpha)}\textstyle\int_{t_1}^t (t-s)^{\alpha-1}v(s)ds + \tfrac{1}{\Gamma(\alpha)}\int_0^{t_1} (t_1-s)^{\alpha-1}v(s)ds\\ &\quad -c_0 - d_0 t_1 + I_1(u(t_1)) - d_1(t-t_1), t \in J_1,\\ -d_0 &= -\tfrac{\Gamma(2-\beta)}{t_1^{1-\beta}}\tfrac{1}{\Gamma(\alpha-\beta)}\textstyle\int_0^{t_1} (t_1-s)^{\alpha-\beta-1}v(s)ds - \tfrac{\Gamma(2-\beta)}{t_1^{1-\beta}}\bar{I}_1(u(t_1)). \end{aligned}$$

Repeating the process in this way, one has

$$\begin{aligned} u(t) &= \tfrac{1}{\Gamma(\alpha)}\textstyle\int_{t_k}^t (t-s)^{\alpha-1}v(s)ds + \tfrac{1}{\Gamma(\alpha)}\sum\limits_{i=1}^{k}\int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-1}v(s)ds\\ &\quad -c_0 - \textstyle\sum\limits_{i=1}^{k} d_{i-1}(t_i - t_{i-1}) + \sum\limits_{i=1}^{k} I_i(u(t_i)) - d_k(t-t_k),\\ &\quad t \in J_k, k = 1, 2, \cdots, m,\\ -d_{i-1} &= -\tfrac{\Gamma(2-\beta)}{(t_i-t_{i-1})^{1-\beta}}\tfrac{1}{\Gamma(\alpha-\beta)}\textstyle\int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-\beta-1}v(s)ds\\ &\quad -\tfrac{\Gamma(2-\beta)}{(t_i-t_{i-1})^{1-\beta}}\bar{I}_i(u(t_i)), i = 1, 2, \cdots, k. \end{aligned} \tag{2.7}$$

By (2.5), (2.6) and the boundary condition $u(0) + {}^{C}D^{\beta}u(0) = A$, we can obtain $-c_0 = A$. On the other hand, by (2.7), we have

$$\begin{aligned} u(1) &= \begin{cases} \tfrac{1}{\Gamma(\alpha)}\textstyle\int_{t_m}^1 (1-s)^{\alpha-1}v(s)ds\\ +\tfrac{1}{\Gamma(\alpha)}\textstyle\sum\limits_{i=1}^{m}\int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-1}v(s)ds\\ +A - \textstyle\sum\limits_{i=1}^{m} d_{i-1}(t_i - t_{i-1}) + \sum\limits_{i=1}^{m} I_i(u(t_i))\\ -d_m(1-t_m), \end{cases}\\ -d_{i-1} &= \begin{cases} -\tfrac{\Gamma(2-\beta)}{(t_i-t_{i-1})^{1-\beta}}\tfrac{1}{\Gamma(\alpha-\beta)}\textstyle\int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-\beta-1}v(s)ds\\ -\tfrac{\Gamma(2-\beta)}{(t_i-t_{i-1})^{1-\beta}}\bar{I}_i(u(t_i)), i = 1, 2, \cdots, m, \end{cases} \end{aligned} \tag{2.8}$$

and

$$ {}^{C}\!D^{\beta}u(\xi) = \frac{1}{\Gamma(\alpha-\beta)}\int_{t_l}^{\xi}(\xi-s)^{\alpha-\beta-1}v(s)ds - d_l\frac{(\xi-t_l)^{1-\beta}}{\Gamma(2-\beta)}. \tag{2.9} $$

By (2.8), (2.9) and the boundary condition $u(1)+ {}^{C}D^{\beta}u(\xi)=B$, we get

$$ -d_m = \frac{1}{1-t_m}\begin{cases} B - A - \frac{1}{\Gamma(\alpha)}\int_{t_m}^{1}(1-s)^{\alpha-1}v(s)ds \\ -\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}v(s)ds \\ +\Gamma(2-\beta)\sum_{i=1}^{m}\{\frac{(t_i-t_{i-1})^{\beta}}{\Gamma(\alpha-\beta)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-\beta-1}v(s)ds \\ +(t_i-t_{i-1})^{\beta}\bar{I}_i(u(t_i))\} - \sum_{i=1}^{m}I_i(u(t_i)) \\ -\frac{1}{\Gamma(\alpha-\beta)}\int_{t_l}^{\xi}(\xi-s)^{\alpha-\beta-1}v(s)ds \\ +\frac{(\xi-t_l)^{1-\beta}}{(t_{l+1}-t_l)^{1-\beta}}\frac{1}{\Gamma(\alpha-\beta)}\int_{t_l}^{t_{l+1}}(t_{l+1}-s)^{\alpha-\beta-1}v(s)ds \\ +\frac{(\xi-t_l)^{1-\beta}}{(t_{l+1}-t_l)^{1-\beta}}\bar{I}_{l+1}(u(t_{l+1})). \end{cases} \tag{2.10} $$

In sum, we get (2.2) and (2.3).

Conversely, we assume that u is a solution of the integral equation (2.2) and (2.3). In view of the relations ${}^{C}\!D^{\beta}I^{\beta}v(t)=v(t)$ for $\beta>0$, we get

$$ {}^{C}\!D^{\alpha}u(t) = v(t), \quad a.e. t\in J, t\neq t_k, k=1,2,\cdots,m. $$

Moreover, it can easily be shown that

$$ \Delta u(t)|_{t=t_k} = I_k(u(t_k)), \Delta\, {}^{C}\!D^{\beta}u(t)|_{t=t_k} = \bar{I}_k(u(t_k)), k=1,2,\cdots,m $$

and $u(0)+{}^{C}D^{\beta}u(0)=A, u(1)+{}^{C}D^{\beta}u(\xi)=B$. The proof is completed. □

3 Existence results

3.1 Convex case

In this subsection, by means of Bohnenblust-Karlin's fixed point theorem, we present a existence result for the problem (1.1)-(1.4) with convex-valued right-hand side. For this, we give some useful lemmas.

Lemma 3.1. *(see [29]) Let X be a Banach space. Let $F: J\times X\to\mathcal{P}_{cp,cv}(X)$ be an L^1-Carathédory multi-valued map, and let Θ be a linear continuous mapping from $L^1(J,X)$ to $C(J,X)$. Then the operator*

$$ \Theta\circ S_F : C(J,X)\to\mathcal{P}_{cp,cv}(J,X), \quad v\mapsto(\Theta\circ S_F)(v) := \Theta(S_{F,v}) $$

is a closed graph operator in $C(J,X)\times C(J,X)$.

Lemma 3.2. *(see [2, Bohnenblust-Karlin]) Let X be a Banach space, D a nonempty subset of X, which is bounded, closed, and convex. Suppose $G: D\to\mathcal{P}(X)$ is u.s.c. with closed, convex values, and such that $G(D)\subset D$ and $\bar{G(D)}$ compact. Then G has a fixed point.*

Lemma 3.3. *(Mazur's Lemma, [31, Theorem 21.4]). Let E be a normed space and $\{x_k\}_{k\in N}\subset E$ a sequence weakly converging to a limit $x\in E$. Then there exists a sequence of convex combinations $u_m=\sum_{k=1}^{m}a_{mk}x_k$, where $a_{mk}>0$ for $k=1,2,\cdots,m$, and $\sum_{k=1}^{m}a_{mk}=1$, which converges strongly to x.*

Then our main contribution of this subsection is the following.

Theorem 3.1. *Suppose the following hold:*
($\mathcal{A}1$) The function $F : J \times \mathbb{R} \to \mathcal{P}_{cp,cv}(\mathbb{R})$ is L^1-Carathéodory,
($\mathcal{A}2$) There exist a function $p \in L^1(J,\mathbb{R})$ and a continuous nondecreasing function $\Psi : [0,\infty) \to [0,\infty)$ such that

$$\|F(t,z)\|_{\mathcal{P}} \leq p(t)\Psi(|z|) \text{ for a.e. } t \in J, \text{ and each } z \in \mathbb{R},$$

with $\lim\limits_{x\to+\infty} \frac{\Psi(x)}{x} = 0,$
($\mathcal{A}3$) There exist constants $L_1, L_2 > 0$ such that

$$|I_k(u)| \leq L_1, |\bar{I}_k(u)| \leq L_2, \text{ for all } u \in \mathbb{R}, k = 1, 2, \cdots, m.$$

Then the set of solutions for Problem (1.1)-(1.4) is nonempty and compact.

Proof. We can transform the problem into a fixed point problem. The proof consists of three parts, with the first part involving multiple steps.
Part 1. Define

$$N_0(u) := \left\{ \begin{array}{l} h \in PC(J_0) : h(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v(s)ds \\ -\frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)}\frac{t}{t_1^{1-\beta}}\int_0^{t_1}(t_1-s)^{\alpha-\beta-1}v(s)ds \\ +A - \frac{\Gamma(2-\beta)t}{t_1^{1-\beta}}\bar{I}_1(u(t_1)), t \in J_0 \end{array} \right\}, \tag{3.1}$$

where $v \in S_{F,u} = \{v \in L^1(J,\mathbb{R}) : v(t) \in F(t,u(t)) \text{ for a.e. } t \in J_0\}$. Next we shall show that N_0 satisfies all the assumptions of Lemma 3.2, and thus N_0 has a fixed point. For the sake of convenience, we subdivide this part into several steps.

Step 1. $N_0(u)$ is convex for each $u \in PC(J_0)$.
Indeed, if $h_1, h_2 \in N_0(u)$, then there exist $v_1, v_2 \in S_{F,u}$ such that for $j = 1, 2$

$$\begin{aligned} h_j(t) = &\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v_j(s)ds - \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)}\frac{t}{t_1^{1-\beta}}\int_0^{t_1}(t_1-s)^{\alpha-\beta-1}v_j(s)ds \\ &+A - \frac{\Gamma(2-\beta)t}{t_1^{1-\beta}}\bar{I}_1(u(t_1)). \end{aligned}$$

Let $0 \leq \lambda \leq 1$. Then for each $t \in J_0$, we have

$$\begin{aligned} [\lambda h_1 + (1-\lambda)h_2](t) = &\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[\lambda v_1 + (1-\lambda)v_2](s)ds \\ &-\frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)}\frac{t}{t_1^{1-\beta}}\int_0^{t_1}(t_1-s)^{\alpha-\beta-1}[\lambda v_1 + (1-\lambda)v_2](s)ds \\ &+A - \frac{\Gamma(2-\beta)t}{t_1^{1-\beta}}\bar{I}_1(u(t_1)). \end{aligned}$$

Since $S_{F,u}$ is convex (because F has convex values), we obtain $[\lambda h_1 + (1-\lambda)h_2] \in N_0(u)$.

Step 2. For each constant $r > 0$, let $B_r = \{u \in PC(J_0) : \|u\|_{PC_0} \leq r\}$. Then B_r is a bounded closed convex set in $PC(J_0)$. We claim that there exists a positive number R_0 such that $N_0(B_{R_0}) \subseteq B_{R_0}$.
Let $u \in PC(J_0)$ and $h \in N_0(u)$. Thus there exists $v \in S_{F,u}$ such that

$$\begin{aligned} h(t) = &\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v(s)ds - \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)}\frac{t}{t_1^{1-\beta}}\int_0^{t_1}(t_1-s)^{\alpha-\beta-1}v(s)ds \\ &+A - \frac{\Gamma(2-\beta)t}{t_1^{1-\beta}}\bar{I}_1(u(t_1)). \end{aligned}$$

And so

$$\begin{aligned} |h(t)| \leq &\Psi(\|u\|_{PC_0})\{(\frac{1}{\Gamma(\alpha)} + \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)})t_1^{\alpha}\int_0^{t_1} p(s)ds\} \\ &+|A| + \Gamma(2-\beta)t_1^{\beta}L_2. \end{aligned}$$

Immediately,

$$\begin{aligned} \|h(t)\| \leq &\Psi(\|u\|_{PC_0})\{(\frac{1}{\Gamma(\alpha)} + \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)})t_1^{\alpha}\int_0^{t_1} p(s)ds\} \\ &+|A| + \Gamma(2-\beta)t_1^{\beta}L_2. \end{aligned} \tag{3.2}$$

Since $\lim\limits_{x\to+\infty}\frac{\Psi(x)}{x}=0$ by $(\mathcal{A}2)$, there exists a sufficiently large number $R_0>0$, such that

$$\begin{aligned}R_0 > \Psi(R_0)\{(\tfrac{1}{\Gamma(\alpha)}+\tfrac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)})t_1^{\alpha}\textstyle\int_0^{t_1}p(s)ds\}\\ +|A|+\Gamma(2-\beta)t_1^{\beta}L_2.\end{aligned}$$

which together with (3.2) imply that

$$\|N_0(u)\|_{\mathcal{P}}<R_0, \text{ when } \|u\|_{PC_0}\le R_0.$$

We have shown that $N_0(B_{R_0})\subseteq B_{R_0}$.

Step 3. $N_0(B_{R_0})$ is equi-continuous on J_0.

Let $u\in B_{R_0}$ and $h\in N_0(u)$. Thus there exists $v\in S_{F,u}$, $|v(s)|\le p(s)\Psi(R_0)$ such that,

$$\begin{aligned}h'(t)=\tfrac{1}{\Gamma(\alpha-1)}\textstyle\int_0^t(t-s)^{\alpha-2}v(s)ds-\tfrac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)}\tfrac{1}{t_1^{1-\beta}}\int_0^{t_1}(t_1-s)^{\alpha-\beta-1}v(s)ds\\ -\tfrac{\Gamma(2-\beta)}{t_1^{1-\beta}}\bar{I}_1(u(t_1)).\end{aligned}$$

So

$$\begin{aligned}|h'(t)|\le\Psi(R_0)\{\tfrac{t_1^{\alpha-1}}{\Gamma(\alpha)}\textstyle\int_0^{t_1}p(s)ds+\tfrac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)}t_1^{\alpha-1}\int_0^{t_1}p(s)ds\}+\tfrac{\Gamma(2-\beta)}{t_1^{1-\beta}}L_2\\ \le M_{R_0} \text{ (a constant)}.\end{aligned}$$

As a consequence of Steps 1-3 together with the Ascoli-Arzela theorem, we can conclude that N_0 is compact valued map.

Step 4. N_0 has closed graph.

Let $h_n\in N_0(u_n)$, and $h_n\to h_*, u_n\to u_*$ as $n\to\infty$. We will prove that $h_*\in N_0(u_*)$. $h_n\in N_0(u_n)$ implies that there exists $v_n\in S_{F,u_n}$ such that for $t\in J_0$

$$\begin{aligned}h_n(t)=\tfrac{1}{\Gamma(\alpha)}\textstyle\int_0^t(t-s)^{\alpha-1}v_n(s)ds-\tfrac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)}\tfrac{t}{t_1^{1-\beta}}\int_0^{t_1}(t_1-s)^{\alpha-\beta-1}v_n(s)ds\\ +A-\tfrac{\Gamma(2-\beta)t}{t_1^{1-\beta}}\bar{I}_1(u_n(t_1)).\end{aligned}$$

We must show that there exists $v_*\in S_{F,u_*}$ such that for each $t\in J_0$,

$$\begin{aligned}h_*(t)=\tfrac{1}{\Gamma(\alpha)}\textstyle\int_0^t(t-s)^{\alpha-1}v_*(s)ds-\tfrac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)}\tfrac{t}{t_1^{1-\beta}}\int_0^{t_1}(t_1-s)^{\alpha-\beta-1}v_*(s)ds\\ +A-\tfrac{\Gamma(2-\beta)t}{t_1^{1-\beta}}\bar{I}_1(u_*(t_1)).\end{aligned}\tag{3.3}$$

Consider the continuous linear operator

$$\begin{aligned}&\Theta: L^1(J_0,\mathbb{R})\to PC(J_0,\mathbb{R}),\ v\mapsto\Theta(v)(t),\\ &\Theta(v)(t)=\tfrac{1}{\Gamma(\alpha)}\textstyle\int_0^t(t-s)^{\alpha-1}v(s)ds-\tfrac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)}\tfrac{t}{t_1^{1-\beta}}\int_0^{t_1}(t_1-s)^{\alpha-\beta-1}v(s)ds\end{aligned}$$

Clearly, by the continuity of $I_k,\bar{I}_k(k-1,2,\cdots,m)$, we have

$$\begin{aligned}&\|h_n(t)-A+\tfrac{\Gamma(2-\beta)t}{t_1^{1-\beta}}\bar{I}_1(u_n(t_1))-\{h_*(t)-A+\tfrac{\Gamma(2-\beta)t}{t_1^{1-\beta}}\bar{I}_1(u_*(t_1))\}\|_{PC_0}\to 0\\ &\text{as } n\to\infty.\end{aligned}$$

From Lemma 3.1 it follows that $\Theta\circ S_{F,u}$ is a closed graph operator. Moreover, we have

$$h_n(t)-\mathbb{I}_{k,A,B}(u_n)\in\Theta(S_{F,u_n}).$$

Since $u_n\to u_*$, Lemma 3.1 implies that (3.3) hold for some $v_*\in S_{F,u_*}$.

Therefore, N_0 are compact multi-valued map, u.s.c., with convex closed values. As a consequence of Lemma 3.2, we deduce that N_0 has a fixed point $u_{*,0}$.

Part 2. Define

$$N_1(u) := \left\{ \begin{array}{l} h \in PC(J_1) : h(t) = \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t-s)^{\alpha-1} v(s) ds \\ -\frac{\Gamma(2-\beta)}{(t_2-t_1)^{1-\beta}} \frac{(t-t_1)}{\Gamma(\alpha-\beta)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-\beta-1} v(s) ds \\ -\frac{\Gamma(2-\beta) t_1^{\beta}}{\Gamma(\alpha-\beta)} \int_0^{t_1} (t_1-s)^{\alpha-\beta-1} v_0(s) ds \\ +\frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1} v_0(s) ds + \tilde{\mathbb{I}}_{1,A,B}(u), t \in J_1, \end{array} \right\} \tag{3.4}$$

where

$$\begin{array}{rl} \tilde{\mathbb{I}}_{1,A,B}(u) = & -\Gamma(2-\beta) t_1^{\beta} \bar{I}_1(u_{*,0}(t_1)) + A + I_1(u_{*,0}(t_1)) \\ & -\frac{\Gamma(2-\beta)(t-t_1)}{(t_2-t_1)^{1-\beta}} \bar{I}_2(u(t_2)), \end{array} \tag{3.5}$$

and $v \in \{v \in L^1(J_1, \mathbb{R}) : v(t) \in F(t, u(t))$ for a.e. $t \in J_1\}$, $v_0 \in \{v \in L^1(J_1, \mathbb{R}) : v(t) \in F(t, u_{*,0}(t))$ for a.e. $t \in J_1\}$. Clearly, N_1 is convex valued, u.s.c..

We claim that there exists a positive number R_1 such that $N_1(B_{R_1}) \subseteq B_{R_1}$, where $B_{R_1} = \{u \in PC(J_1) : \|u\|_{PC_1} \leq R_1\}$.

Let $u \in PC(J_1)$ and $h \in N_1(u)$. Thus there exists $v \in S_{F,u}$ such that

$$\begin{array}{rl} h(t) = & \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t-s)^{\alpha-1} v(s) ds \\ & -\frac{\Gamma(2-\beta)}{(t_2-t_1)^{1-\beta}} \frac{(t-t_1)}{\Gamma(\alpha-\beta)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-\beta-1} v(s) ds \\ & -\frac{\Gamma(2-\beta) t_1^{\beta}}{\Gamma(\alpha-\beta)} \int_0^{t_1} (t_1-s)^{\alpha-\beta-1} v_0(s) ds \\ & +\frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1} v_0(s) ds + \mathbb{I}_{1,A,B}(u). \end{array}$$

Noticing $\|u_{*,0}\| \leq R_0$, we have

$$\begin{array}{rl} \|h(t)\| \leq & \Psi(\|u\|_{PC_1}) \{ (\frac{(t_2-t_1)^{\alpha}}{\Gamma(\alpha)} + \frac{\Gamma(2-\beta)(t_2-t_1)^{\alpha}}{\Gamma(\alpha-\beta+1)}) \int_{t_1}^{t_2} p(s) ds \} \\ & +\Psi(R_0) \{ (\frac{t_1^{\alpha}}{\Gamma(\alpha)} + \frac{\Gamma(2-\beta) t_1^{\alpha}}{\Gamma(\alpha-\beta+1)}) \int_0^{t_1} p(s) ds \} \\ & +\Gamma(2-\beta) t_1^{\beta} L_2 + |A| + L_1 + \Gamma(2-\beta)(t_2-t_1)^{\beta} L_2. \end{array} \tag{3.6}$$

Since $\lim\limits_{x \to +\infty} \frac{\Psi(x)}{x} = 0$ by $(\mathcal{A}2)$, there exists a sufficiently large number $R_1 > 0$, such that

$$\begin{array}{rl} R_1 > & \Psi(R_1) \{ (\frac{(t_2-t_1)^{\alpha}}{\Gamma(\alpha)} + \frac{\Gamma(2-\beta)(t_2-t_1)^{\alpha}}{\Gamma(\alpha-\beta+1)}) \int_{t_1}^{t_2} p(s) ds \} \\ & +\Psi(R_0) \{ (\frac{t_1^{\alpha}}{\Gamma(\alpha)} + \frac{\Gamma(2-\beta) t_1^{\alpha}}{\Gamma(\alpha-\beta+1)}) \int_0^{t_1} p(s) ds \} \\ & +\Gamma(2-\beta) t_1^{\beta} L_2 + |A| + L_1 + \Gamma(2-\beta)(t_2-t_1)^{\beta} L_2. \end{array}$$

which together with (3.6) imply that

$$\|N_1(u)\|_{\mathcal{P}} < R_1, \text{ when } \|u\|_{PC_1} \leq R_1.$$

Similarly, we conclude that $N_1(B_{R_1})$ is equi-continuous on J_1. As a consequence of Lemma 3.2, we deduce that N_1 has a fixed point $u_{*,1}$.

Part 3. Continue this process. We can define, for $k = 2, 3, \cdots, m-1$,

$$N_k(u) := \left\{ \begin{array}{l} h \in PC(J_k) : h(t) = \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} v(s) ds \\ -\frac{\Gamma(2-\beta)}{(t_{k+1}-t_k)^{1-\beta}} \frac{(t-t_k)}{\Gamma(\alpha-\beta)} \int_{t_k}^{t_{k+1}} (t_{k+1}-s)^{\alpha-\beta-1} v(s) ds \\ +\frac{1}{\Gamma(\alpha)} \sum\limits_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-1} v_{i-1}(s) ds \\ -\Gamma(2-\beta) \sum\limits_{i=1}^{k} \frac{(t_i-t_{i-1})^{\beta}}{\Gamma(\alpha-\beta)} \int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-\beta-1} v_{i-1}(s) ds \\ +\tilde{\mathbb{I}}_{k,A,B}(u), t \in J_k, \end{array} \right\} \tag{3.7}$$

and

$$
N_m(u) := \left\{ \begin{array}{l} h \in PC(J_m) : h(t) = \frac{1}{\Gamma(\alpha)} \int_{t_m}^{t} (t-s)^{\alpha-1} v(s) ds \\ -\frac{t-t_m}{1-t_m} \frac{1}{\Gamma(\alpha)} \int_{t_m}^{1} (1-s)^{\alpha-1} v(s) ds \\ +\frac{1}{\Gamma(\alpha)} \frac{1-t}{1-t_m} \{ \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-1} v_{i-1}(s) ds \\ -\Gamma(\alpha)\Gamma(2-\beta) \sum_{i=1}^{m} \frac{(t_i-t_{i-1})^{\beta}}{\Gamma(\alpha-\beta)} \int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-\beta-1} v_{i-1}(s) ds \} \\ -\frac{t-t_m}{1-t_m} \frac{1}{\Gamma(\alpha)} \{ \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} \int_{t_l}^{\xi} (\xi-s)^{\alpha-\beta-1} v_l(s) ds \\ -\frac{(\xi-t_l)^{1-\beta}}{(t_{l+1}-t_l)^{1-\beta}} \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} \int_{t_l}^{t_{l+1}} (t_{l+1}-s)^{\alpha-\beta-1} v_l(s) ds \} \\ +\tilde{\mathbb{I}}_{m,A,B}(u), t \in J_m, \end{array} \right\} \tag{3.8}
$$

where

$$
\begin{aligned} \tilde{\mathbb{I}}_{k,A,B}(u) = & -\Gamma(2-\beta) \sum_{i=1}^{k} (t_i-t_{i-1})^{\beta} \bar{I}_i(u_{*,i-1}(t_i)) \\ & +A + \sum_{i=1}^{k} I_i(u_{*,i-1}(t_i)) - \frac{\Gamma(2-\beta)(t-t_k)}{(t_{k+1}-t_k)^{1-\beta}} \bar{I}_{k+1}(u(t_{k+1})), \\ \tilde{\mathbb{I}}_{m,A,B}(u) = & -\frac{1-t}{1-t_m} \Gamma(2-\beta) \sum_{i=1}^{m} (t_i-t_{i-1})^{\beta} \bar{I}_i(u_{*,i-1}(t_i)) \\ & +\frac{A(1-t)}{1-t_m} + \frac{B(t-t_m)}{1-t_m} + \frac{1-t}{1-t_m} \sum_{i=1}^{m} I_i(u_{*,i-1}(t_i)) \\ & +\frac{t-t_m}{1-t_m} \frac{(\xi-t_l)^{1-\beta}}{(t_{l+1}-t_l)^{1-\beta}} \bar{I}_{l+1}(u_{*,l}(t_{l+1})), \end{aligned} \tag{3.9}
$$

and $v \in S_{F,u}$, $v_i \in S_{F,u_{*,i}}, i = 1, 2, \cdots, m-1$. Then we can similarly prove that $N_k (k = 2, 3, \cdots, m)$ possesses also a fixed point $u_{*,k}$. The solution u of Problem (1.1)-(1.4) can be then defined by

$$
u(t) = \begin{cases} u_{*,0}(t), & t \in [0, t_1], \\ u_{*,1}(t), & t \in (t_1, t_2], \\ \vdots & \\ u_{*,m}(t), & t \in (t_m, 1]. \end{cases}
$$

Using the fact that $F(\cdot, \cdot) \in \mathcal{P}_{cv,cp}(\mathbb{R})$, $F(t, \cdot)$ is u.s.c. and Mazur's lemma, by Ascoli's theorem, we can prove that the solution set of Problem (1.1)-(1.4) is compact. □

3.2 Nonconvex case

In this subsection we present two existence results for the problem (1.1)-(1.4) with nonconvex-valued right-hand side. Let (X, d) be a metric space induced from the normed space $(X, \|\cdot\|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}^+ \cup \{\infty\}$, given by

$$
H_d(\mathbb{A}, \mathbb{B}) = \max\{\sup_{a \in \mathbb{A}} d(a, \mathbb{B}), \sup_{b \in \mathbb{B}} d(\mathbb{A}, b)\},
$$

where $d(a, \mathbb{B}) = \inf_{b \in \mathbb{B}} d(a, b)$, $d(\mathbb{A}, b) = \inf_{a \in \mathbb{A}} d(a, b)$.

Then $(\mathcal{P}_{b,cl}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized (complete) metric space (see [32]).

Definition 3.6. *A multi-valued operator $G : X \to \mathcal{P}_{cl}(X)$ is called*
(a) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$
H_d(G(x), G(y)) \le \gamma d(x, y), \text{ for each } x, y \in X,
$$

(b) a contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

Our considerations are based on the following fixed point theorem for contractive multi-valued operators given by Covitz and Nadler [10] (see also Deimling [13, Theorem 11.1]).

Lemma 3.4. *Let (X, d) be a complete metric space. If $G : X \to \mathcal{P}_{cl}(X)$ is a contraction, then Fix $G \neq \varnothing$.*

Let us introduce the following hypotheses:
($\mathcal{B}1$) $F : J \times \mathbb{R} \to \mathcal{P}_{cl}(\mathbb{R})$ satisfies
(a) $t \mapsto F(t, u)$ is measurable for each $u \in \mathbb{R}$;
(b) the map $t \mapsto H_d(0, F(t, 0))$ is integrable bounded.
($\mathcal{B}2$) There exist a function $p \in L^1(J, \mathbb{R})$ such that for a.e. $t \in J$ and all $u, v \in \mathbb{R}$,

$$H_d(F(t, u), F(t, v)) \leq p(t)|u - v|,$$

($\mathcal{B}3$) There exist constants $\varsigma_k, \bar{\varsigma}_k \geq 0$ such that

$$|I_k(u_1) - I_k(u_2)| \leq \varsigma_k|u_1 - u_2|, |\bar{I}_k(u_1) - \bar{I}_k(u_2)| \leq \bar{\varsigma}_k|u_1 - u_2|,$$
for each $u_1, u_2 \in \mathbb{R}$,

($\mathcal{B}4$) $\Lambda = (\frac{4}{\Gamma(\alpha)} + \frac{\Gamma(2-\beta)+2}{\Gamma(\alpha-\beta+1)})\|p\|_{L^1} + \Gamma(2 - \beta) \sum_{i=1}^{m} \bar{\varsigma}_i + \sum_{i=1}^{m} \varsigma_i + \bar{\varsigma}_{l+1} < 1.$

Remark 3.1. *The hypotheses ($\mathcal{B}4$) can be improved by a weaker substitute in Theorem 3.2, and Theorem 4.1, respectively.*

Theorem 3.2. *Suppose that hypotheses ($\mathcal{B}1$)-($\mathcal{B}4$) are satisfied. Then the problem (1.1)-(1.4) has at least one solution.*

Proof. The proof will be given in several steps.

Step 1. Let N_0 be defined as (3.1) in the proof of Theorem 3.1. We show that N_0 satisfies the assumptions of Lemma 3.4.

Firstly, $N_0(u) \in \mathcal{P}_{cl}(PC(J_0))$ for each $u \in PC(J_0)$. In fact, let $\{u_n\}_{n\geq 1} \subset N_0(u)$ such that $u_n \to \tilde{u}$. Then there exists $x_n \in S_{F,u}$ such that

$$\begin{aligned} u_n(t) = {} & \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} x_n(s) ds - \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)} \frac{t}{t_1^{1-\beta}} \int_0^{t_1} (t_1 - s)^{\alpha-\beta-1} x_n(s) ds \\ & + A - \frac{\Gamma(2-\beta)t}{t_1^{1-\beta}} \bar{I}_1(u(t_1)), t \in J_0. \end{aligned}$$

Then $\{x_n\}$ is integrably bounded. Since $F(\cdot, \cdot)$ has closed values, let $\omega(\cdot) \in F(\cdot, 0)$ be such that $|\omega(t)| = H_d(0, F(t, 0))$.

From ($\mathcal{B}1$) we infer that for a.e. $t \in J_0$,

$$\begin{aligned} |x_n(t)| & \leq |x_n(t) - \omega(t)| + |\omega(t)| \\ & \leq p(t)\|x(t)\|_{PC_k} + H_d(0, F(t, 0)) := M_*(t), \forall n \in \mathbb{N}, \end{aligned}$$

that is,

$$x_n(t) \in M_*(t)B(0, 1), \text{ a.e. } t \in J_0.$$

Since $B(0, 1)$ is compact in $\mathbb{R}$, there exists a subsequence still denoted $\{x_n\}$ which converges to x.

Then the Lebesgue dominated convergence theorem implies that, as $n \to \infty$,

$$\|x_n - x\|_{L^1} \to 0 \text{ and thus}$$
$$\begin{aligned} \tilde{u}(t) = {} & \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} x(s) ds - \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)} \frac{t}{t_1^{1-\beta}} \int_0^{t_1} (t_1 - s)^{\alpha-\beta-1} x(s) ds \\ & + A - \frac{\Gamma(2-\beta)t}{t_1^{1-\beta}} \bar{I}_1(u(t_1)), t \in J_0, \end{aligned}$$

proving that $\tilde{u} \in N_0(u)$.

Secondly, there exists $\gamma < 1$ such that $H_d(N_0(u), N_0(\bar{u})) \leq \gamma \|u - \bar{u}\|_{PC_0}$, for each $u, \bar{u} \in PC(J_0)$. Let $u, \bar{u} \in PC(J_0)$ and $h \in N_0(u)$. Then there exists $x(t) \in F(t, u(t))$, such that for each $t \in J_0$

$$\begin{aligned} h(t) = & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds - \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)} \frac{t}{t_1^{1-\beta}} \int_0^{t_1} (t_1 - s)^{\alpha-\beta-1} x(s) ds \\ & + A - \frac{\Gamma(2-\beta) t}{t_1^{1-\beta}} \bar{I}_1(u(t_1)), t \in J_0. \end{aligned}$$

From $(\mathcal{B}1)$ it follows that

$$H_d(F(t, u(t)), F(t, \bar{u}(t))) \leq p(t)|u(t) - \bar{u}(t)|.$$

Hence there is $y \in F(t, u)$, such that

$$|x(t) - y| \leq p(t)|u(t) - \bar{u}(t)|, t \in J_0.$$

Consider $U_0 : PC(J_0) \to \mathcal{P}(\mathbb{R})$, given by

$$U_0(t) = \{y \in \mathbb{R} : |x(t) - y| \leq p(t)|u(t) - \bar{u}(t)|\}.$$

Since the multi-valued operator $\overline{U}_0(t) = U_0(t) \cap F(t, \bar{u}(t))$ is measurable (see [11, Proposition III.4]), there exists a function $\bar{x}(t)$, which is a measurable selection for $\overline{U}_0$. So $\bar{x}(t) \in F(t, \bar{u}(t))$ and

$$|x(t) - \bar{x}(t)| \leq p(t)|u(t) - \bar{u}(t)|, t \in J_0.$$

We can define for each $t \in J_0$

$$\begin{aligned} \bar{h}(t) = & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \bar{x}(s) ds - \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)} \frac{t}{t_1^{1-\beta}} \int_0^{t_1} (t_1 - s)^{\alpha-\beta-1} \bar{x}(s) ds \\ & + A - \frac{\Gamma(2-\beta) t}{t_1^{1-\beta}} \bar{I}_1(u(t_1)), t \in J_0. \end{aligned}$$

Therefore,

$$\begin{aligned} |h(t) - \bar{h}(t)| & \leq \left(\frac{t_1^\alpha}{\Gamma(\alpha)} + \frac{\Gamma(2-\beta) t_1^\alpha}{\Gamma(\alpha-\beta+1)}\right) \int_0^{t_1} p(s)|u(s) - \bar{u}(s)| ds \\ & \quad + \Gamma(2-\beta) t_1^\beta \varsigma_1 |u(s) - \bar{u}(s)| \\ & \leq \Lambda \|u - \bar{u}\|_{PC_0}. \end{aligned}$$

Then

$$\|h(t) - \bar{h}(t)\|_{PC_0} \leq \Lambda \|u - \bar{u}\|_{PC_0}.$$

By an analogous relation, obtained by interchanging the roles of u and $\bar{u}$, it follows that

$$H_d(N_0(u), N_0(\bar{u})) \leq \Lambda \|u - \bar{u}\|_{PC_0}.$$

So, N_0 is a contraction. By Lemma 3.4, N_0 has a fixed point $u_{*,0}$.

Step 2. Define $N_1(u)$ as (3.4) in the proof of Theorem 3.1. We can prove that $N_1(u)$ satisfies also the assumptions of Lemma 3.4. So N_1 has a fixed point $u_{*,1}$.

Continue this process. We can define $N_k(u)(k = 2, 3, \cdots, m)$ as (3.7) (3.8) and similarly prove that $N_k(k = 2, 3, \cdots, m)$ possesses also a fixed point $u_{*,k}$. Then Problem (1.1)-(1.4) has a solution u defined by

$$u(t) = \begin{cases} u_{*,0}(t), & t \in [0, t_1], \\ u_{*,1}(t), & t \in (t_1, t_2], \\ \vdots & \\ u_{*,m}(t), & t \in (t_m, 1]. \end{cases}$$

□

For our another result in this subsection, we give some definitions and preliminary facts.

Definition 3.7. *Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0,1],\mathbb{R}))$ be a multi-valued operator. We say N has property (BC) if*
(1) N is l.s.c.;
(2) N has nonempty closed and decomposable values.

Let $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multi-valued map with nonempty compact values. Assign to F the multi-valued operator $\mathcal{F} : C([0,1],\mathbb{R}) \to \mathcal{P}(L^1([0,1],\mathbb{R}))$ defined by $\mathcal{F}(u) = S_{F,u}$. The operator $\mathcal{F}$ is called the Niemytzki operator associated with F.

Definition 3.8. *Let $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multi-valued map with nonempty compact values. We say F is of lower semi-continuous type (l.s.c. type) if its associated Niemytzki operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values.*

Next, we introduce a selection theorem due to Bressan and Colombo and a crucial lemma.

Theorem 3.3 ([5])**.** *Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0,1],\mathbb{R}))$ be a multi-valued operator which has property (BC). Then N has a continuous selection, i.e. there exists a continuous function (single-valued) $\tilde{g} : Y \to L^1([0,1],\mathbb{R})$ such that $\tilde{g}(y) \in N(y)$ for every $y \in Y$.*

Lemma 3.5 ([13, 19])**.** *Let $F : J \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ be an integrably bounded multi-valued function satisfying, in addition to ($\mathcal{A}2$),*
($\mathcal{B}5$) The function $F : J \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ is such that

$$\begin{cases} (a)\ (t,u) \mapsto F(t,u) \text{ is } \mathcal{L} \otimes \mathcal{B} \text{ measurable} \\ (b)\ u \mapsto F(t,u) \text{ is lower semi-continuous for a.e. } t \in J. \end{cases}$$

Then F is of lower semi-continuous type.

Now, we present another existence result for the problem (1.1)-(1.4) in the spirit of the nonlinear alternative of Leray-Schauder type [24] for single-valued maps, combined with a selection theorem due to Bressan and Colombo [5] for lower semi-continuous multi-valued maps with decomposable values.

Theorem 3.4. *Assume that ($\mathcal{A}2$), ($\mathcal{A}3$) and ($\mathcal{B}5$) hold, Then Problem (1.1)-(1.4) has at least one solution.*

Proof. From Lemma 3.5 and Theorem 3.3, there exists a continuous function $f : PC(J) \to L^1(J,\mathbb{R})$, such that $f(u)(t) \in \mathcal{F}(u)$, for each $u \in PC(J)$ and a.e. $t \in J$.

Consider the following impulsive fractional differential equation

$$\begin{array}{l} {}^{C}\!D^{\alpha}u(t) = f(u)(t), \quad a.e. t \in J, \\ \Delta x(t)|_{t=t_k} = I_k(x(t_k)), k = 1,2,\cdots,m, \\ \Delta {}^{C}\!D^{\beta}x(t)|_{t=t_k} = \bar{I}_k(x(t_k)), k = 1,2,\cdots,m, \\ u(0) + {}^{C}D^{\beta}u(0) = A, u(1) + {}^{C}D^{\beta}u(\xi) = B. \end{array} \tag{3.10}$$

Clearly, if $u \in PC(J)$ is a solution of Problem (3.10), then u is a solution to Problem (1.1)-(1.4).

Problem (3.10) is then reformulated by Lemma 2.1 as a fixed point problem for some single-valued operators. The remainder of the proof will be given in several steps.

Step 1. Define the single-valued operator $\dot{N}_0 : PC(J_0) \to PC(J_0)$ by

$$\begin{array}{rl} \dot{N}_0(u) = & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(u)(s)ds \\ & -\frac{\Gamma(2-\beta)}{t_1^{1-\beta}} \frac{t}{\Gamma(\alpha-\beta)} \int_0^{t_1} (t_1-s)^{\alpha-\beta-1} f(u)(s)ds \\ & +A - \frac{\Gamma(2-\beta)t}{t_1^{1-\beta}} \bar{I}_1(u(t_1)), t \in J_0. \end{array}$$

Next, we will show that all $\dot{N}_0$ have a fixed point $u_{*,0}$.

Firstly, $\dot{N}_0$ is completely continuous. Indeed, from the continuity of $f, \bar{I}_1$, the Lebesgue dominated convergence theorem and $\alpha - 1 > 0, \alpha - \beta - 1 \geq 0$, we have $\dot{N}_0$ is continuous. As in Theorem 3.1, we can similarly show that $\dot{N}_0$ is completely continuous on any bounded subset.

Secondly, we show that there exists an open set $S_0 \subset PC(J_0)$ with no $u = \lambda \dot{N}_0(u)$ for any $\lambda \in (0,1)$ and $u \in \partial S_0$.

Let $u \in PC(J_0)$ and $u = \lambda \dot{N}_0(u)$ for some $0 < \lambda < 1$. Thus for each $t \in J_0$, we have

$$\begin{aligned} \|u\|_{PC_0} \leq & [\tfrac{t_1^\alpha}{\Gamma(\alpha)} + \tfrac{\Gamma(2-\beta)t_1^\alpha}{\Gamma(\alpha-\beta+1)}] \int_0^{t_1} p(s)ds\Psi(\|u\|_{PC_0}) \\ & + |A| + \Gamma(2-\beta)t_1^\beta L_2. \end{aligned} \tag{3.11}$$

Since $\lim\limits_{x\to+\infty} \frac{\Psi(x)}{x} = 0$ by $(\mathcal{A}2)$, there exists a sufficiently large number $R_0 > 0$, such that

$$\begin{aligned} R_0 - 1 > & \Psi(R_0)[\tfrac{t_1^\alpha}{\Gamma(\alpha)} + \tfrac{\Gamma(2-\beta)t_1^\alpha}{\Gamma(\alpha-\beta+1)}] \int_0^{t_1} p(s)ds \\ & + |A| + \Gamma(2-\beta)t_1^\beta L_2. \end{aligned}$$

which together with (3.11) imply that

$$\|N_0(u)\|_{PC_0} < R_0 - 1, \text{ when } \|u\|_{PC_0} \leq R_0.$$

Set

$$S_0 = \{u \in C([0,t_1], \mathbb{R}) : \|u\|_{PC_0} < R_0\}.$$

From the choice of S_0, there is no $u \in \partial S_0$ such that $u = \lambda \dot{N}_0(u)$ for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type [24], we deduce that $\dot{N}_0$ has a fixed point $u_{*,0}$ in S_0.

Step 2. Define the single-valued operator $\dot{N}_1 : PC(J_1) \to PC(J_1)$ by

$$\begin{aligned} \dot{N}_1(u) = & \tfrac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} f(u)(s)ds \\ & - \tfrac{\Gamma(2-\beta)}{(t_2-t_1)^{1-\beta}} \tfrac{(t-t_1)}{\Gamma(\alpha-\beta)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-\beta-1} f(u)(s)ds \\ & + \tfrac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1} f(u_{*,0})(s)ds \\ & - \Gamma(2-\beta) \tfrac{t_1^\beta}{\Gamma(\alpha-\beta)} \int_0^{t_1} (t_1-s)^{\alpha-\beta-1} f(u_{*,0})(s)ds \\ & + \tilde{\mathbb{I}}_{1,A,B}(u), t \in J_1, \end{aligned}$$

where

$$\begin{aligned} \tilde{\mathbb{I}}_{1,A,B}(u) = & -\Gamma(2-\beta)t_1^\beta \bar{I}_1(u_{*,0}(t_1)) + A + I_1(u_{*,0}(t_1)) \\ & - \tfrac{\Gamma(2-\beta)(t-t_1)}{(t_2-t_1)^{1-\beta}} \bar{I}_2(u(t_2)). \end{aligned}$$

Similarly, $\dot{N}_1$ is completely continuous on any bounded subset. Let $u \in PC(J_1)$ and $u = \lambda \dot{N}_1(u)$ for some $0 < \lambda < 1$. Thus, we have

$$\begin{aligned} \|u\|_{PC_1} \leq & \Psi(\|u\|_{PC_1})\{(\tfrac{(t_2-t_1)^\alpha}{\Gamma(\alpha)} + \tfrac{\Gamma(2-\beta)(t_2-t_1)^\alpha}{\Gamma(\alpha-\beta+1)}) \int_{t_1}^{t_2} p(s)ds\} \\ & + \Psi(R_0)\{(\tfrac{t_1^\alpha}{\Gamma(\alpha)} + \tfrac{\Gamma(2-\beta)t_1^\alpha}{\Gamma(\alpha-\beta+1)}) \int_0^{t_1} p(s)ds\} \\ & + \Gamma(2-\beta)t_1^\beta L_2 + |A| + L_1 + \Gamma(2-\beta)(t_2-t_1)^\beta L_2. \end{aligned} \tag{3.12}$$

As in Step 1, there exists a sufficiently large number $R_1 > 0$, such that $\|N_1(u)\|_{PC_1} < R_1 - 1$, when $\|u\|_{PC_1} \leq R_1$. Set

$$S_1 = \{u \in PC(J_1) : \|u\|_{PC_1} < R_1\}.$$

From the choice of S_1, there is no $u \in \partial S_1$ such that $u = \lambda \dot{N}_1(u)$ for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type [24], we deduce that $\dot{N}_1$ has a fixed point $u_{*,1}$ in S_1.

Step 3. We continue this process. Define

$$\begin{aligned}\dot{N}_k(u) = &\frac{1}{\Gamma(\alpha)}\int_{t_k}^t (t-s)^{\alpha-1}f(u)(s)ds\\ &+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}f(u_{*,i-1})(s)ds\\ &-\Gamma(2-\beta)\sum_{i=1}^{k}\frac{(t_i-t_{i-1})^{\beta}}{\Gamma(\alpha-\beta)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-\beta-1}f(u_{*,i-1})(s)ds\\ &-\frac{\Gamma(2-\beta)}{(t_{k+1}-t_k)^{1-\beta}}\frac{(t-t_k)}{\Gamma(\alpha-\beta)}\int_{t_k}^{t_{k+1}}(t_{k+1}-s)^{\alpha-\beta-1}f(u)(s)ds\\ &+\tilde{\mathbb{I}}_{k,A,B}(u), t\in J_k, k=2,3,\cdots,m-1,\end{aligned}$$

and

$$\begin{aligned}\dot{N}_m(u) = &\frac{1}{\Gamma(\alpha)}\int_{t_m}^t (t-s)^{\alpha-1}f(u)(s)ds - \frac{t-t_m}{1-t_m}\frac{1}{\Gamma(\alpha)}\int_{t_m}^1(1-s)^{\alpha-1}f(u)(s)ds\\ &+\frac{1}{\Gamma(\alpha)}\frac{1-t}{1-t_m}\{\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}f(u_{*,i-1})(s)ds\\ &-\Gamma(\alpha)\Gamma(2-\beta)\sum_{i=1}^{m}\frac{(t_i-t_{i-1})^{\beta}}{\Gamma(\alpha-\beta)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-\beta-1}f(u_{*,i-1})(s)ds\}\\ &-\frac{t-t_m}{1-t_m}\frac{1}{\Gamma(\alpha)}\{+\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}\int_{t_l}^{\xi}(\xi-s)^{\alpha-\beta-1}f(u_{*,l})(s)ds\\ &-\frac{(\xi-t_l)^{1-\beta}}{(t_{l+1}-t_l)^{1-\beta}}\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}\int_{t_l}^{t_{l+1}}(t_{l+1}-s)^{\alpha-\beta-1}f(u_{*,l})(s)ds\}\\ &+\tilde{\mathbb{I}}_{m,A,B}(u), t\in J_m,\end{aligned}$$

where

$$\begin{aligned}\tilde{\mathbb{I}}_{k,A,B}(u) = &-\Gamma(2-\beta)\sum_{i=1}^{k}(t_i-t_{i-1})^{\beta}\bar{I}_i(u_{*,i-1}(t_i))\\ &+A+\sum_{i=1}^{k}I_i(u_{*,i-1}(t_i)) - \frac{\Gamma(2-\beta)(t-t_k)}{(t_{k+1}-t_k)^{1-\beta}}\bar{I}_{k+1}(u(t_{k+1})),\\ \tilde{\mathbb{I}}_{m,A,B}(u) = &-\frac{1-t}{1-t_m}\Gamma(2-\beta)\sum_{i=1}^{m}(t_i-t_{i-1})^{\beta}\bar{I}_i(u_{*,i-1}(t_i))\\ &+\frac{A(1-t)}{1-t_m}+\frac{B(t-t_m)}{1-t_m}+\frac{1-t}{1-t_m}\sum_{i=1}^{m}I_i(u_{*,i-1}(t_i))\\ &+\frac{t-t_m}{1-t_m}\frac{(\xi-t_l)^{1-\beta}}{(t_{l+1}-t_l)^{1-\beta}}\bar{I}_{l+1}(u_{*,l}(t_{l+1})).\end{aligned}$$

By a similar argument to the one above, we can prove that all $\dot{N}_k (k=2,3,\cdots,m)$ have a fixed point $u_{*,k}$. Then Problem (1.1)-(1.4) has a solution u defined by

$$u(t) = \begin{cases} u_{*,0}(t), & t\in[0,t_1],\\ u_{*,1}(t), & t\in(t_1,t_2],\\ \vdots & \\ u_{*,m}(t), & t\in(t_m,1].\end{cases}$$

□

4 Filippov's Theorem

In this section, we will present a Filippov's result for the Problem (1.1)-(1.4). For this, we list some lemmas.

Lemma 4.1. *[12] Consider an l.s.c. multi-valued map $G : S \to \mathcal{D}$ and assume that $\phi : S \to L^1(J,\mathbb{R}^n)$ and $\psi : S \to L^1(J,\mathbb{R}^+)$ are continuous maps, and for every $s \in S$, the set*

$$H(s) = \overline{\{u\in G(s) : |u(t)-\phi(s)(t)| < \psi(s)(t)\}}$$

is nonempty. Then the map $H : S \to \mathcal{D}$ is l.s.c., and so it admits a continuous selection.

Lemma 4.2. *[25, Corollary 2.3.] Let* $G : [0,b] \to \mathcal{P}_{cp}(X)$ *be a measurable multifunction and* $g : [0,b] \to X$ *be a measurable function. Then there exists a measurable selection* u *of* G *such that*

$$|u(t) - g(t)| \leq d(g(t), G(t)).$$

Let $\bar{A}, \bar{B} \in \mathbb{R}$, and a continuous mapping $g(\cdot) : PC(J) \to L^1(J, \mathbb{R})$. Let $x \in PC(J)$, be a solution of the impulsive differential problem with fractional order:

$$^{C}D^{\alpha}x(t) = g(x)(t), \quad a.e. t \in J, t \neq t_k, k = 1, 2, \cdots, m, \tag{4.1}$$

$$\Delta x(t)|_{t=t_k} = I_k(x(t_k)), k = 1, 2, \cdots, m, \tag{4.2}$$

$$\Delta {^{C}D^{\beta}}x(t)|_{t=t_k} = \bar{I}_k(x(t_k)), k = 1, 2, \cdots, m, \tag{4.3}$$

$$x(0) + {^{C}D^{\beta}}x(0) = \bar{A}, x(1) + {^{C}D^{\beta}}x(\xi) = \bar{B}. \tag{4.4}$$

By Lemma 2.1, x should satisfy (2.2) and (2.3) with respect to $v(s) = g(x)(s))$ and A, B submitted by $\bar{A}, \bar{B}$.

Our main result in this section is contained in the following theorem.

Theorem 4.1. *Assume that, in addition to (B2), (B3), (B4), (B5), the following also holds:*

(H1) There exist a function $\gamma \in L^1(J, \mathbb{R}^+)$ *such that* $d(g(x)(t), F(t, x(t))) < \gamma(t)$ *a.e.* $t \in J$*, where* $x \in PC(J)$ *is a solution of the impulsive differential problem (4.1)-(4.4).*
Then problem (1.1)-(1.4) has at least one solution u *satisfying the estimates*

$$\begin{aligned} &\|x - u\|_{PC_k} \leq \frac{\|p\|_{L^1}(\frac{1}{\Gamma(\alpha)} + \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)})}{1 - \Gamma(2-\beta)\bar{\varsigma}_{k+1}} H_k + \frac{\varsigma_0}{1 - \Gamma(2-\beta)\bar{\varsigma}_{k+1}}, k = 0, 1, \cdots, m-1, \\ &\|x - u\|_{PC_m} \leq \frac{2\|p\|_{L^1}}{\Gamma(\alpha)} H_m + \varsigma_0, \end{aligned} \tag{4.5}$$

and

$$|{^{C}D^{\alpha}}u(t) - g(x)(t))| \leq p(t)H_k + \gamma(t), \quad t \in J_k, k = 0, 1, \cdots, m, \tag{4.6}$$

where

$$\begin{aligned} H_k &= \frac{2\|\gamma\|_{L^1}(\frac{1}{\Gamma(\alpha)} + \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)}) + 2\varsigma_0}{1 - 2(\frac{1}{\Gamma(\alpha)} + \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)})\|p\|_{L^1} - \Gamma(2-\beta)\bar{\varsigma}_{k+1}}, k = 0, 1, \cdots, m-1, \\ H_m &= \frac{4\|\gamma\|_{L^1} + 2\varsigma_0\Gamma(\alpha)}{\Gamma(\alpha) - 4\|p\|_{L^1}} \text{ and } \varsigma_0 = |A - \bar{A}| + |B - \bar{B}|. \end{aligned}$$

Proof. The proof will be given in four steps.

Step 1. We construct a sequence of functions $\{u_n : n \in \mathbb{N}\}$ which will be shown to converge to some solution $u_{*,0}$, which is a fixed point of multi-valued operator N_0 defined by (3.1).

Let $f_0(u_0)(t) = g(x)(t))$ and

$$\begin{aligned} u_0(t) = &\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_0(u_0)(s)ds \\ &- \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)} \frac{t}{t_1^{1-\beta}} \int_0^{t_1} (t_1 - s)^{\alpha-\beta-1} f_0(u_0)(s)ds \\ &+ \bar{A} - \frac{\Gamma(2-\beta)t}{t_1^{1-\beta}} \bar{I}_1(x(t_1)), t \in J_0. \end{aligned} \tag{4.7}$$

Let $G_1 : PC(J_0) \to \mathcal{P}(L^1(J_0, \mathbb{R}))$ be given by

$$G_1(u) = \{v \in L^1(J_0, \mathbb{R}) : v(t) \in F(t, u(t)), \text{ a.e. } t \in J_0\}.$$

and $\tilde{G}_1 : PC(J_0) \to \mathcal{P}(L^1(J_0, \mathbb{R}))$ be defined by

$$\tilde{G}_1(u) = \overline{\{v \in G_1(u) : |v(t) - g(u_0)(t)| < p(t)|u(t) - u_0(t)| + \gamma(t)\}}.$$

Since $t \to F(t, u(t))$ is measurable multifunction, and from Lemma 4.2, there exists a function v_1 which is a measurable selection of $F(t, u(t))$, such that for a.e. $t \in J_0$,

$$\begin{aligned} |v_1(t) - g(u_0)(t)| &< d(g(u_0)(t), F(t, u(t))) \\ &< p(t)|u(t) - u_0(t)| + \gamma(t). \end{aligned}$$

Then $v_1 \in \tilde{G}_1(u) \neq \varnothing$. By Lemma 3.5, F is of lower semi-continuous type. Then $u \to G_1(u)$ is l.s.c., and has decomposable values. So $u \to \tilde{G}_1(u)$ is l.s.c. with decomposable values from $PC(J_0)$ into $\mathcal{P}(PC(J_0))$.

Then from Lemma 4.1 and Theorem 3.3, there exists a continuous function $f_1 : PC(J_0) \to L^1(J_0, \mathbb{R})$ such that $f_1(u) \in \tilde{G}_1(u)$ for all $u \in PC(J_0)$. From Theorem 3.4, the single-valued operator $\tilde{N}_0$ defined by

$$\begin{aligned} \tilde{N}_0(u) = &\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_1(u)(s) ds \\ &- \frac{\Gamma(2-\beta)}{t_1^{1-\beta}} \frac{t}{\Gamma(\alpha-\beta)} \int_0^{t_1} (t_1 - s)^{\alpha-\beta-1} f_1(u)(s) ds \\ &+ A - \frac{\Gamma(2-\beta)t}{t_1^{1-\beta}} \bar{I}_1(u(t_1)), t \in J_0, \end{aligned}$$

has a fixed point which we denote by u_1. Then

$$\begin{aligned} u_1(t) = &\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_1(u_1)(s) ds \\ &- \frac{\Gamma(2-\beta)}{t_1^{1-\beta}} \frac{t}{\Gamma(\alpha-\beta)} \int_0^{t_1} (t_1 - s)^{\alpha-\beta-1} f_1(u_1)(s) ds \\ &+ A - \frac{\Gamma(2-\beta)t}{t_1^{1-\beta}} \bar{I}_1(u_1(t_1)), t \in J_0. \end{aligned}$$

For every $t \in J_0$, we have

$$\begin{aligned} |u_1(t) - u_0(t)| \leq\ &\frac{t_1^\alpha}{\Gamma(\alpha)} \int_0^t |f_1(u_1)(s) - f_0(u_0)(s)| ds \\ &+ \frac{\Gamma(2-\beta)t_1^\alpha}{\Gamma(\alpha-\beta+1)} \int_0^{t_1} |f_1(u_1)(s) - f_0(u_0)(s)| ds \\ &+ |A - \bar{A}| + \Gamma(2-\beta) t_1^\beta |\bar{I}_1(u_1(t_1)) - \bar{I}_1(u_0(t_1))| \\ \leq\ &\{\frac{1}{\Gamma(\alpha)} + \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)}\} \int_0^{t_1} p(s)|u_1(s) - u_0(s)| ds \\ &+ \varsigma_0 + \{\frac{1}{\Gamma(\alpha)} + \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)}\} \|\gamma\|_{L^1} + \Gamma(2-\beta)\bar{\varsigma}_1 |u_1(t_1) - u_0(t_1)|. \end{aligned}$$

Thus,

$$\|u_1 - u_0\|_{PC_0} \leq \frac{\|\gamma\|_{L^1}(\frac{1}{\Gamma(\alpha)} + \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)}) + \varsigma_0}{1 - (\frac{1}{\Gamma(\alpha)} + \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)})\|p\|_{L^1} - \Gamma(2-\beta)\bar{\varsigma}_1}.$$

Let $G_2 : PC(J_0) \to \mathcal{P}(L^1(J_0, \mathbb{R}))$ be given by

$$G_2(u) = \{v \in L^1(J_0, \mathbb{R}) : v(t) \in F(t, u(t)), \text{ a.e. } t \in J_0\}.$$

and $\tilde{G}_2 : PC(J_0) \to \mathcal{P}(L^1(J_0, \mathbb{R}))$ be defined by

$$\begin{aligned} &\tilde{G}_2(u) \\ = &\overline{\{v \in G_2(u) : |v(t) - f_1(u_1)(t)| < p(t)|u(t) - u_1(t)| + p(t)|u_0(t) - u_1(t)|\}}. \end{aligned}$$

Since $t \to F(t, u(t))$ is measurable multifunction, and from Lemma 4.2, there exists a function v_2 which is a measurable selection of $F(t, u(t))$, such that for a.e. $t \in J_0$,

$$\begin{aligned} |v_2(t) - f_1(u_1)(t)| &< d(f_1(u_1)(t), F(t, u(t))) \\ &< H_d(f_1(u_1)(t), F(t, u(t))) \\ &< p(t)|u(t) - u_1(t)| \\ &< p(t)|u(t) - u_1(t)| + p(t)|u_0(t) - u_1(t)|. \end{aligned}$$

Then $v_2 \in \tilde{G}_2(u) \neq \varnothing$. By Lemma 3.5, F is of lower semi-continuous type. Then $u \to G_2(u)$ is l.s.c., and has decomposable values. So $u \to \tilde{G}_2(u)$ is l.s.c. with decomposable values from $PC(J_0)$ into $\mathcal{P}(PC(J_0))$.

Then from Lemma 4.1 and Theorem 3.3, there exists a continuous function $f_2 : PC(J_0) \to L^1(J_0, \mathbb{R})$ such that $f_2(u) \in \tilde{G}_2(u)$ for all $u \in PC(J_0)$. From Theorem 3.4, the single-valued operator $\tilde{N}_0$ defined by

$$\begin{aligned}\tilde{N}_0(u) = & \tfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f_2(u)(s)ds \\ & -\tfrac{\Gamma(2-\beta)}{t_1^{1-\beta}}\tfrac{t}{\Gamma(\alpha-\beta)}\int_0^{t_1}(t_1-s)^{\alpha-\beta-1} f_2(u)(s)ds \\ & +A - \tfrac{\Gamma(2-\beta)t}{t_1^{1-\beta}}\bar{I}_1(u(t_1)), t\in J_0,\end{aligned}$$

has a fixed point which we denote by u_2. Then

$$\begin{aligned}u_2(t) = & \tfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f_2(u_2)(s)ds \\ & -\tfrac{\Gamma(2-\beta)}{t_1^{1-\beta}}\tfrac{t}{\Gamma(\alpha-\beta)}\int_0^{t_1}(t_1-s)^{\alpha-\beta-1} f_2(u_2)(s)ds \\ & +A - \tfrac{\Gamma(2-\beta)t}{t_1^{1-\beta}}\bar{I}_1(u_2(t_1)), t\in J_0.\end{aligned}$$

For every $t \in J_0$, we have

$$\begin{aligned}|u_2(t)-u_1(t)| \leq & \tfrac{t_1^{\alpha}}{\Gamma(\alpha)}\int_0^t |f_1(u_1)(s)-f_2(u_2)(s)|ds \\ & +\tfrac{\Gamma(2-\beta)t_1^{\alpha}}{\Gamma(\alpha-\beta+1)}\int_0^{t_1}|f_1(u_1)(s)-f_2(u_2)(s)|ds \\ & +\Gamma(2-\beta)t_1^{\beta}|\bar{I}_1(u_1(t_1))-\bar{I}_1(u_2(t_1))| \\ \leq & \{\tfrac{1}{\Gamma(\alpha)}+\tfrac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)}\}\int_0^{t_1} p(s)|u_1(s)-u_2(s)|ds \\ & +\{\tfrac{1}{\Gamma(\alpha)}+\tfrac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)}\}\int_0^{t_1} p(s)|u_1(s)-u_0(s)|ds \\ & +\Gamma(2-\beta)\bar{\varsigma}_1|u_1(t_1)-u_2(t_1)|.\end{aligned}$$

Thus,

$$\begin{aligned}\|u_2-u_1\|_{PC_0} \leq & \tfrac{\|\gamma\|_{L^1}\|p\|_{L^1}(\frac{1}{\Gamma(\alpha)}+\frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)})^2}{[1-(\frac{1}{\Gamma(\alpha)}+\frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)})\|p\|_{L^1}-\Gamma(2-\beta)\bar{\varsigma}_1]^2} \\ & +\tfrac{\varsigma_0[\|p\|_{L^1}(\frac{1}{\Gamma(\alpha)}+\frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)})]}{[1-(\frac{1}{\Gamma(\alpha)}+\frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)})\|p\|_{L^1}-\Gamma(2-\beta)\bar{\varsigma}_1]^2}.\end{aligned}$$

Let

$$G_3(u) = \{v \in L^1(J_0, \mathbb{R}) : v(t) \in F(t, u(t)), \text{ a.e. } t \in J_0\},$$

and

$$\begin{aligned}& \tilde{G}_3(u) \\ & = \{v \in G_3(u) : |v(t)-f_2(u_2)(t)| < p(t)|u(t)-u_2(t)| + p(t)|u_2(t)-u_1(t)|\}.\end{aligned}$$

Arguing as we did for $\tilde{G}_2$, we can show that $\tilde{G}_3$ is an l.s.c. with nonempty decomposable values. So there exists a continuous selection $f_3(u) \in \tilde{G}_3(u)$ for all $u \in PC(J_0)$. From Theorem 3.4, the single-valued operator $\tilde{N}_0$ defined by

$$\begin{aligned}\tilde{N}_0(u) = & \tfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f_3(u)(s)ds \\ & -\tfrac{\Gamma(2-\beta)}{t_1^{1-\beta}}\tfrac{t}{\Gamma(\alpha-\beta)}\int_0^{t_1}(t_1-s)^{\alpha-\beta-1} f_3(u)(s)ds \\ & +A - \tfrac{\Gamma(2-\beta)t}{t_1^{1-\beta}}\bar{I}_1(u(t_1)), t\in J_0,\end{aligned}$$

has a fixed point which we denote by u_3. Then

$$\begin{aligned}u_3(t) = & \tfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f_3(u_3)(s)ds \\ & -\tfrac{\Gamma(2-\beta)}{t_1^{1-\beta}}\tfrac{t}{\Gamma(\alpha-\beta)}\int_0^{t_1}(t_1-s)^{\alpha-\beta-1} f_3(u_3)(s)ds \\ & +A - \tfrac{\Gamma(2-\beta)t}{t_1^{1-\beta}}\bar{I}_1(u_3(t_1)), t\in J_0.\end{aligned}$$

For every $t \in J_0$, we have

$$\begin{aligned}|u_2(t)-u_3(t)| \leq\ & \frac{t_1^{\alpha}}{\Gamma(\alpha)}\int_0^t |f_3(u_3)(s)-f_2(u_2)(s)|ds \\ & +\frac{\Gamma(2-\beta)t_1^{\alpha}}{\Gamma(\alpha-\beta+1)}\int_0^{t_1}|f_3(u_3)(s)-f_2(u_2)(s)|ds \\ & +\Gamma(2-\beta)t_1^{\beta}|\bar{I}_1(u_3(t_1))-\bar{I}_1(u_2(t_1))| \\ \leq\ & \{\frac{1}{\Gamma(\alpha)}+\frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)}\}\int_0^{t_1}p(s)|u_3(s)-u_2(s)|ds \\ & +\{\frac{1}{\Gamma(\alpha)}+\frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)}\}\int_0^{t_1}p(s)|u_1(s)-u_2(s)|ds \\ & +\Gamma(2-\beta)\bar{\varsigma}_1|u_3(t_1)-u_2(t_1)|.\end{aligned}$$

Thus,

$$\begin{aligned}\|u_2-u_3\|_{PC_0} \leq\ & \frac{\|\gamma\|_{L^1}\|p\|_{L^1}^2(\frac{1}{\Gamma(\alpha)}+\frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)})^3}{[1-(\frac{1}{\Gamma(\alpha)}+\frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)})\|p\|_{L^1}-\Gamma(2-\beta)\bar{\varsigma}_1]^3} \\ & +\frac{\varsigma_0[\|p\|_{L^1}(\frac{1}{\Gamma(\alpha)}+\frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)})]^2}{[1-(\frac{1}{\Gamma(\alpha)}+\frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)})\|p\|_{L^1}-\Gamma(2-\beta)\bar{\varsigma}_1]^3}.\end{aligned}$$

Repeating the process, we can arrive at the bound

$$\begin{aligned}\|u_n-u_{n-1}\|_{PC_0} \leq\ & \frac{\|\gamma\|_{L^1}\|p\|_{L^1}^{n-1}(\frac{1}{\Gamma(\alpha)}+\frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)})^n}{[1-(\frac{1}{\Gamma(\alpha)}+\frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)})\|p\|_{L^1}-\Gamma(2-\beta)\bar{\varsigma}_1]^n} \\ & +\frac{\varsigma_0[\|p\|_{L^1}(\frac{1}{\Gamma(\alpha)}+\frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)})]^{n-1}}{[1-(\frac{1}{\Gamma(\alpha)}+\frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)})\|p\|_{L^1}-\Gamma(2-\beta)\bar{\varsigma}_1]^n}.\end{aligned} \tag{4.8}$$

By induction, suppose that (4.8) holds for some n. Let

$$\underline{\tilde{G}_{n+1}(u) = \{v \in G_{n+1}(u) : |v(t)-f_n(u_n)(t)| < p(t)|u(t)-u_n(t)| + p(t)|u_n(t)-u_{n-1}(t)|\}}.$$

Since again $\tilde{G}_{n+1}$ is an l.s.c. type multifunction, there exists a continuous function $f_{n+1}(u) \in \tilde{G}_{n+1}(u)$ which allows us to define

$$\begin{aligned}u_{n+1}(t) =\ & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f_{n+1}(u_{n+1})(s)ds \\ & -\frac{\Gamma(2-\beta)}{t_1^{1-\beta}}\frac{t}{\Gamma(\alpha-\beta)}\int_0^{t_1}(t_1-s)^{\alpha-\beta-1}f_{n+1}(u_{n+1})(s)ds \\ & +A-\frac{\Gamma(2-\beta)t}{t_1^{1-\beta}}\bar{I}_1(u_{n+1}(t_1)), t \in J_0.\end{aligned} \tag{4.9}$$

Therefore, we have

$$\begin{aligned}\|u_{n+1}-u_n\|_{PC_0} \leq\ & \frac{\|\gamma\|_{L^1}\|p\|_{L^1}^{n}(\frac{1}{\Gamma(\alpha)}+\frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)})^{n+1}}{[1-(\frac{1}{\Gamma(\alpha)}+\frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)})\|p\|_{L^1}-\Gamma(2-\beta)\bar{\varsigma}_1]^{n+1}} \\ & +\frac{\varsigma_0[\|p\|_{L^1}(\frac{1}{\Gamma(\alpha)}+\frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)})]^{n}}{[1-(\frac{1}{\Gamma(\alpha)}+\frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)})\|p\|_{L^1}-\Gamma(2-\beta)\bar{\varsigma}_1]^{n+1}}.\end{aligned} \tag{4.10}$$

Hence, (4.8) holds for all $n \in \mathbb{N}$. And so, by ($\mathcal{B}4$), $\{u_n\}$ is a Cauchy sequence in $PC(J_0)$, converging uniformly to a function $u_{*,0} \in PC(J_0)$. Moreover, from the definition of $\tilde{G}_n, n \in \mathbb{N}$, we have, for a.e. $t \in J_0$,

$$|f_{n+1}(u_{n+1})(t)-f_n(u_n)(t)| < p(t)|u_{n+1}(t)-u_n(t)| + p(t)|u_n(t)-u_{n-1}(t)|.$$

Therefore, for almost every $t \in J_0$, $\{f_n(u_n)(t) : n \in \mathbb{N}\}$ is also a Cauchy sequence in $\mathbb{R}$ and converges almost everywhere to some measurable function $f(\cdot)$ in $\mathbb{R}$. Moreover, since $f_0 = g$, we have

$$\begin{aligned} |f_n(u_n)(t)| \leq & \ |f_n(u_n)(t) - f_{n-1}(u_{n-1})(t)| \\ & + |f_{n-1}(u_{n-1})(t) - f_{n-2}(u_{n-2})(t)| \\ & + \cdots + |f_1(u_1)(t) - f_0(u_0)(t)| + |f_0(u_0)(t)| \\ \leq & \ 2\sum_{k=1}^{n} p(t)|u_k(t) - u_{k-1}(t)| + |f_0(u_0)(t)| + \gamma(t) \\ \leq & \ 2p(t)\sum_{k=1}^{\infty} |u_k(t) - u_{k-1}(t)| + |g(x)(t)| + \gamma(t) \\ \leq & \ 2p(t)\frac{\|\gamma\|_{L^1}(\frac{1}{\Gamma(\alpha)} + \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)}) + \varsigma_0}{1 - 2(\frac{1}{\Gamma(\alpha)} + \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)})\|p\|_{L^1} - \Gamma(2-\beta)\bar{\varsigma}_1} + |g(x)(t)| + \gamma(t). \end{aligned} \tag{4.11}$$

From (4.11) and the Lebesgue dominated convergence theorem, we conclude that $f_n(u_n)$ converges to $f(u_{*,0})$ in $L^1(J_0, \mathbb{R})$. Passing to the limit in (4.9), the function

$$\begin{aligned} u_{*,0}(t) = & \ \tfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(u_{*,0})(s)ds \\ & - \tfrac{\Gamma(2-\beta)}{t_1^{1-\beta}}\tfrac{t}{\Gamma(\alpha-\beta)}\int_0^{t_1}(t_1-s)^{\alpha-\beta-1} f(u_{*,0})(s)ds \\ & + A - \tfrac{\Gamma(2-\beta)t}{t_1^{1-\beta}}\bar{I}_1(u_{*,0}(t_1)), t \in J_0 \end{aligned}$$

is a fixed point of multi-valued operator N_0 defined by (3.1).

Next, we give estimate for $\|x - u_{*,0}\|_{PC_0}$. We have

$$\begin{aligned} |x(t) - u_{*,0}(t)| \leq & \ \tfrac{1}{\Gamma(\alpha)}\int_0^t |g(x)(s) - f(u_{*,0})(s)|ds \\ & + \tfrac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)}\int_0^{t_1} |g(x)(s) - f(u_{*,0})(s)|ds \\ & + |\bar{A} - A| + \Gamma(2-\beta)|\bar{I}_1(x(t_1)) - \bar{I}_1(u_{*,0}(t_1))| \\ \leq & \ (\tfrac{1}{\Gamma(\alpha)} + \tfrac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)})\int_0^{t_1} |f_n(u_n)(s) - f(u_{*,0})(s)|ds \\ & + (\tfrac{1}{\Gamma(\alpha)} + \tfrac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)})\int_0^{t_1} |f_0(u_0)(s) - f_n(u_n)(s)|ds \\ & + \varsigma_0 + \Gamma(2-\beta)\bar{\varsigma}_1|u_{*,0}(t_1) - x(t_1)|, t \in J_0. \end{aligned}$$

As $n \to \infty$, we arrive at

$$\begin{aligned} \|x - u_{*,0}\|_{PC_0} \leq & \ \frac{2\|\gamma\|_{L^1}\|p\|_{L^1}(\frac{1}{\Gamma(\alpha)} + \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)})^2 + 2\varsigma_0\|p\|_{L^1}(\frac{1}{\Gamma(\alpha)} + \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)})}{(1-\Gamma(2-\beta)\bar{\varsigma}_1)(1 - 2(\frac{1}{\Gamma(\alpha)} + \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)})\|p\|_{L^1} - \Gamma(2-\beta)\bar{\varsigma}_1)} \\ & + \frac{\varsigma_0}{(1-\Gamma(2-\beta)\bar{\varsigma}_1)}. \end{aligned} \tag{4.12}$$

Step 2. Let $f_0(u^0) = g$ and set

$$\begin{aligned} u^0(t) = & \ \tfrac{1}{\Gamma(\alpha)}\int_{t_1}^t (t-s)^{\alpha-1} f_0(u^0)(s)ds \\ & - \tfrac{\Gamma(2-\beta)}{(t_2-t_1)^{1-\beta}}\tfrac{(t-t_1)}{\Gamma(\alpha-\beta)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-\beta-1} f_0(u^0)(s)ds \\ & - \tfrac{\Gamma(2-\beta)t_1^{\beta}}{\Gamma(\alpha-\beta)}\int_0^{t_1}(t_1-s)^{\alpha-\beta-1} v_0(s)ds \\ & + \tfrac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1} v_0(s)ds + \tilde{\mathbb{I}}_{1,\bar{A},\bar{B}}(u^0), t \in J_1, \end{aligned} \tag{4.13}$$

where $v_0(t) \in F(t, u_{*,0}(t))$ for a.e. $t \in J_1$ and

$$\begin{aligned} \tilde{\mathbb{I}}_{1,A,B}(u^0) = & -\Gamma(2-\beta)t_1^{\beta}\bar{I}_1(u_{*,0}(t_1)) + A + I_1(u_{*,0}(t_1)) \\ & - \tfrac{\Gamma(2-\beta)(t-t_1)}{(t_2-t_1)^{1-\beta}}\bar{I}_2(u^0(t_2)). \end{aligned} \tag{4.14}$$

As in Step 1, let the multi-valued map $G_1 : PC(J_1) \to \mathcal{P}(L^1(J_1, \mathbb{R}))$ be given by

$$G_1(u) = \{v \in L^1(J_1, \mathbb{R}) : v(t) \in F(t, u(t)), \text{ a.e. } t \in J_1\}.$$

and $\tilde{G}_1 : PC(J_1) \to \mathcal{P}(L^1(J_1,\mathbb{R}))$ be defined by

$$\tilde{G}_1(u) = \overline{\{v \in G_1(u) : |v(t) - g(u_0)(t)| < p(t)|u(t) - u_0(t)| + \gamma(t)\}}.$$

Then there exists a continuous selection $f_1(u) \in \tilde{G}_1(u)$ for all $u \in PC_1$. Define

$$\begin{aligned} u^1(t) = & \tfrac{1}{\Gamma(\alpha)}\int_{t_1}^t (t-s)^{\alpha-1} f_1(u^1)(s)ds \\ & -\tfrac{\Gamma(2-\beta)}{(t_2-t_1)^{1-\beta}}\tfrac{(t-t_1)}{\Gamma(\alpha-\beta)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-\beta-1} f_1(u^1)(s)ds \\ & -\tfrac{\Gamma(2-\beta)t_1^\beta}{\Gamma(\alpha-\beta)}\int_0^{t_1}(t_1-s)^{\alpha-\beta-1}v_0(s)ds \\ & +\tfrac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}v_0(s)ds + \tilde{\mathbb{I}}_{1,A,B}(u^1), t \in J_1. \end{aligned}$$

Next define $G_2 : PC_1 \to \mathcal{P}(L^1(J_1,\mathbb{R}))$ by

$$G_2(u) = \{v \in L^1(J_1,\mathbb{R}) : v(t) \in F(t,u(t)),\ \text{a.e. } t \in J_1\}.$$

and

$$\begin{aligned} & \tilde{G}_2(u) \\ = & \overline{\{v \in G_2(u) : |v(t) - f_1(u^1)(t)| < p(t)|u(t) - u^1(t)| + p(t)|u^0(t) - u^1(t)|\}}. \end{aligned}$$

It has a continuous selection $f_2(u) \in \tilde{G}_2(u)$ for all $u \in PC(J_1)$. Repeating the process of selection as in Step 1, we can define by induction a sequence of multi-valued maps

$$\begin{aligned} & \tilde{G}_{n+1}(u) = \\ & \overline{\{v \in G_{n+1}(u) : |v(t) - f_n(u^n)(t)| < p(t)|u(t) - u^n(t)| + p(t)|u^n(t) - u^{n-1}(t)|\}}. \end{aligned}$$

Since again $\tilde{G}_{n+1}$ is an l.s.c. type multifunction, there exists a continuous function $f_{n+1}(u) \in \tilde{G}_{n+1}(u)$ which allows us to define

$$\begin{aligned} u^{n+1}(t) = & \tfrac{1}{\Gamma(\alpha)}\int_{t_1}^t (t-s)^{\alpha-1} f_{n+1}(u^{n+1})(s)ds \\ & -\tfrac{\Gamma(2-\beta)}{(t_2-t_1)^{1-\beta}}\tfrac{(t-t_1)}{\Gamma(\alpha-\beta)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-\beta-1} f_{n+1}(u^{n+1})(s)ds \\ & -\tfrac{\Gamma(2-\beta)t_1^\beta}{\Gamma(\alpha-\beta)}\int_0^{t_1}(t_1-s)^{\alpha-\beta-1}v_0(s)ds \\ & +\tfrac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}v_0(s)ds + \tilde{\mathbb{I}}_{1,A,B}(u^{n+1}), t \in J_1. \end{aligned} \tag{4.15}$$

Therefore, we can easily prove that

$$\begin{aligned} \|u^{n+1} - u^n\|_{PC_1} \le & \tfrac{\|\gamma\|_{L^1}\|p\|_{L^1}^n(\frac{1}{\Gamma(\alpha)} + \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)})^{n+1}}{[1-(\frac{1}{\Gamma(\alpha)} + \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)})\|p\|_{L^1} - \Gamma(2-\beta)\bar{\varsigma}_2]^{n+1}} \\ & + \tfrac{\varsigma_0[\|p\|_{L^1}(\frac{1}{\Gamma(\alpha)} + \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)})]^n}{[1-(\frac{1}{\Gamma(\alpha)} + \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)})\|p\|_{L^1} - \Gamma(2-\beta)\bar{\varsigma}_2]^{n+1}}. \end{aligned}$$

Then, we have constructed a sequence of functions $\{u^n : n \in \mathbb{N}\}$. As in Step 1, we can show that $\{u^n\}$ is a Cauchy sequence converging uniformly to some $u_{*,1} \in PC_1$ a fixed point of multi-valued operator N_1 defined by (3.4) and that $f_n(u^n)$ converges to $f(u_{*,1})$ in $L^1(J_1,\mathbb{R})$. Moreover,

$$\begin{aligned} \|x - u_{*,1}\|_{PC_1} \le & \tfrac{2\|\gamma\|_{L^1}\|p\|_{L^1}(\frac{1}{\Gamma(\alpha)} + \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)}) + 2\varsigma_0\|p\|_{L^1}(\frac{1}{\Gamma(\alpha)} + \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)})}{(1-\Gamma(2-\beta)\bar{\varsigma}_2)(1-2(\frac{1}{\Gamma(\alpha)} + \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)})\|p\|_{L^1} - \Gamma(2-\beta)\bar{\varsigma}_2)} \\ & + \tfrac{\varsigma_0}{(1-\Gamma(2-\beta)\bar{\varsigma}_2)}. \end{aligned} \tag{4.16}$$

Step 3. Continuing this process, we can arrive at the functions $f(u_{*,k}) \in L^1(J_k,\mathbb{R})$, $u_{*,k} \in PC(J_k)(k = 2,3,\cdots,m)$. And here $u_{*,k}$ is a fixed point of multi-valued operator N_k defined by (3.7) and (3.8), respectively. Similarly, the following estimates are easily obtained

$$\begin{aligned} \|x - u_{*,k}\|_{PC_k} \le & \tfrac{2\|\gamma\|_{L^1}\|p\|_{L^1}(\frac{1}{\Gamma(\alpha)} + \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)})^2 + 2\varsigma_0\|p\|_{L^1}(\frac{1}{\Gamma(\alpha)} + \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)})}{(1-\Gamma(2-\beta)\bar{\varsigma}_{k+1})(1-2(\frac{1}{\Gamma(\alpha)} + \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)})\|p\|_{L^1} - \Gamma(2-\beta)\bar{\varsigma}_{k+1})} \\ & + \tfrac{\varsigma_0}{(1-\Gamma(2-\beta)\bar{\varsigma}_{k+1})}, k = 2,3,\cdots,m-1, \end{aligned} \tag{4.17}$$

and

$$\|x - u_{*,m}\|_{PC_m} \leq \frac{4\|p\|_{L^1}}{\Gamma(\alpha) - 4\|p\|_{L^1}}(\frac{2\|\gamma\|_{L^1}}{\Gamma(\alpha)} + \varsigma_0) + \varsigma_0. \tag{4.18}$$

Step 4. Summarizing, we deduce that a solution u of Problem (1.1)-(1.4) can be defined as

$$u(t) = \begin{cases} u_{*,0}(t), & t \in [0, t_1], \\ u_{*,1}(t), & t \in (t_1, t_2], \\ \vdots \\ u_{*,m}(t), & t \in (t_m, 1]. \end{cases}$$

From (4.12), (4.16), (4.17) and (4.18), we know that the estimates (4.5) hold. Moreover, the function $u(t)$ defined above is a solution of the following impulsive boundary value problem for fractional order differential equation

$$\begin{array}{l} {}^C\!D^\alpha u(t) = f(u)(t), \quad a.e. t \in J, t \neq t_k, k = 1, 2, \cdots, m, \\ \Delta u(t)|_{t=t_k} = I_k(u(t_k)), k = 1, 2, \cdots, m, \\ \Delta {}^C\!D^\beta u(t)|_{t=t_k} = \bar{I}_k(u(t_k)), k = 1, 2, \cdots, m, \\ u(0) + {}^C\!D^\beta u(0) = A, u(1) + {}^C\!D^\beta u(\xi) = B, \end{array} \tag{4.19}$$

where $f(u)(t) = f(u_{*,k})(t), t \in J_k, k = 0, 1, \cdots, m$. From Step 1 to 3, we have that

$$\begin{aligned} |{}^C\!D^\alpha u(t) - g(x)(t))| \leq\ & |f(u)(t) - f_0(u_0)(t)| \\ \leq\ & |f(u)(t) - f_n(u_n)(t)| + |f_n(u_n)(t) - f_0(u_0)(t)| \\ \leq\ & |f(u)(t) - f_n(u_n)(t)| \\ & + \sum_{k=1}^{n} |f_k(u_k)(t) - f_{k-1}(u_{k-1})(t)| \\ \leq\ & |f(u)(t) - f_n(u_n)(t)| \\ & + 2p(t) \sum_{k=1}^{n} |u_k(t) - u_{k-1}(t)| + \gamma(t). \end{aligned} \tag{4.20}$$

Passing to the limit as $n \to \infty$, we obtain

$$|{}^C\!D^\alpha u(t) - g(x)(t))| \leq 2p(t) \sum_{k=1}^{\infty} |u_k(t) - u_{k-1}(t)| + \gamma(t). \tag{4.21}$$

Using (4.10) and (4.21), we get for $t \in J_0$ that

$$|{}^C\!D^\alpha u(t) - g(x)(t))| \leq \frac{2p(t)(\|\gamma\|_{L^1}(\frac{1}{\Gamma(\alpha)} + \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)}) + \varsigma_0)}{1 - 2(\frac{1}{\Gamma(\alpha)} + \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)})\|p\|_{L^1} - \Gamma(2-\beta)\bar{\varsigma}_1} + \gamma(t). \tag{4.22}$$

Similarly, for $t \in J_k, k = 1, 2, \cdots, m-1$,

$$|{}^C\!D^\alpha u(t) - g(x)(t))| \leq \frac{2p(t)(\|\gamma\|_{L^1}(\frac{1}{\Gamma(\alpha)} + \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)}) + \varsigma_0)}{1 - 2(\frac{1}{\Gamma(\alpha)} + \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)})\|p\|_{L^1} - \Gamma(2-\beta)\bar{\varsigma}_{k+1}} + \gamma(t), \tag{4.23}$$

and for $t \in J_m$,

$$|{}^C\!D^\alpha u(t) - g(x)(t))| \leq \frac{2p(t)(2\|\gamma\|_{L^1} + \Gamma(\alpha)\varsigma_0)}{\Gamma(\alpha) - 4\|p\|_{L^1}} + \gamma(t). \tag{4.24}$$

The inequalities (4.22), (4.23) and (4.24) imply that the estimate (4.6) valid. The proof is completed. □

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