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A FOUR-POINT NONLOCAL INTEGRAL BOUNDARY VALUE PROBLEM FOR FRACTIONAL DIFFERENTIAL EQUATIONS OF ARBITRARY ORDER

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Abstract

This paper studies a nonlinear fractional differential equation of an arbitrary order with four-point nonlocal integral boundary conditions. Some existence results are obtained by applying standard fixed point theorems and Leray-Schauder degree theory. The involvement of nonlocal parameters in four-point integral boundary conditions of the problem makes the present work distinguished from the available literature on four-point integral boundary value problems which mainly deals with the four-point boundary conditions restrictions on the solution or gradient of the solution of the problem. These integral conditions may be regarded as strip conditions involving segments of arbitrary length of the given interval. Some illustrative examples are presented.

Key words and phrases: Fractional differential equations; four-point integral boundary conditions; existence; Fixed point theorem; Leray-Schauder degree.
AMS (MOS) Subject Classifications: 26A33, 34A12, 34A40.

1 Introduction

Boundary value problems for nonlinear fractional differential equations have recently been studied by several researchers. Fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. These characteristics of the fractional derivatives make the fractional-order models more realistic and practical than the classical integer-order models. As a matter of fact, fractional differential equations arise in many engineering and scientific disciplines such as physics, chemistry, biology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, aerodynamics, fitting of experimental data, etc. [17, 18, 19, 20]. Some recent work on boundary value problems of fractional order can be found in [1, 2, 3, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16, 22] and the references therein.

In this paper, we consider a boundary value problem of nonlinear fractional differential equations of an arbitrary order with four-point integral boundary conditions given by

$$\begin{cases} {}^cD^q x(t) = f(t, x(t)), \quad 0 < t < 1, \quad m-1 < q \le m, \\ x(0) = \alpha \displaystyle\int_0^\xi x(s)ds, \; x'(0) = 0, \; x''(0) = 0, \dots, x^{(m-2)}(0) = 0, \\ x(1) = \beta \displaystyle\int_0^\eta x(s)ds, \quad 0 < \xi, \eta < 1, \end{cases} \tag{1.1}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q, $f : [0,1] \times X \to X$ is continuous and $\alpha, \beta \in \mathbb{R}$. Here, $(X, \|\cdot\|)$ is a Banach space and $\mathcal{C} = \mathrm{C}([0,1], X)$ denotes the Banach space of all continuous functions from $[0,1] \to X$ endowed with a topology of uniform convergence with the norm denoted by $\|\cdot\|$.

Integral boundary conditions have various applications in applied fields such as blood flow problems, chemical engineering, thermoelasticity, underground water flow, population dynamics, etc. For a detailed description of the integral boundary conditions, we refer the reader to the papers [4, 5] and references therein. It has been observed that the limits of integration in the integral part of the boundary conditions are usually taken to be fixed, for instance, from 0 to 1 in case the independent variable belongs to the interval $[0,1]$. In the present study, we have introduced a nonlocal type of integral boundary conditions with limits of integration involving the parameters $0 < \xi,\ \eta < 1$. It is imperative to note that the available literature on nonlocal boundary conditions is confined to the nonlocal parameters involvement in the solution or gradient of the solution of the problem. The present work is motivated by a recent article [10], in which some existence results were obtained for nonlinear fractional differential equations with three-point nonlocal integral boundary conditions.

2 Preliminaries

Let us recall some basic definitions of fractional calculus [17, 18, 20].

Definition 2.1 *For a function $g : [0,\infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as*

$${}^cD^q g(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} g^{(n)}(s)ds, \quad n-1 < q < n, n = [q]+1,$$

where $[q]$ denotes the integer part of the real number q.

Definition 2.2 *The Riemann-Liouville fractional integral of order q is defined as*

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{g(s)}{(t-s)^{1-q}} ds, \quad q > 0,$$

provided the integral exists.

Lemma 2.1 *([17]) For $q > 0$, the general solution of the fractional differential equation ${}^cD^q x(t) = 0$ is given by*

$$x(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R},\ i = 0, 1, 2, \ldots, n-1$ $(n = [q] + 1)$.

In view of Lemma 2.1, it follows that

$$I^q\ {}^cD^q x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1}, \tag{2.1}$$

for some $c_i \in \mathbb{R},\ i = 0, 1, 2, ..., n-1$ $(n = [q] + 1)$.

Lemma 2.2 *For a given $\sigma \in C[0, 1]$, the unique solution of the boundary value problem*

$$\begin{cases} {}^cD^q x(t) = \sigma(t), \quad 0 < t < 1, \quad m-1 < q \le m, \\ x(0) = \alpha \displaystyle\int_0^\xi x(s)ds,\ x'(0) = 0,\ x''(0) = 0, \ldots, x^{(m-2)}(0) = 0, \\ x(1) = \beta \displaystyle\int_0^\eta x(s)ds, \quad 0 < \xi, \eta < 1, \end{cases} \tag{2.2}$$

is given by

$$\begin{aligned} x(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \sigma(s)ds + \frac{\alpha}{m\Delta}\Big[(\beta\eta^m - m) \int_0^\xi \Big(\int_0^s \frac{(s-k)^{q-1}}{\Gamma(q)} \sigma(k)dk\Big)ds \\ &- \beta\xi^m \int_0^\eta \Big(\int_0^s \frac{(s-k)^{q-1}}{\Gamma(q)} \sigma(k)dk\Big)ds + \xi^m \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} \sigma(s)ds\Big] \\ &- \frac{t^{m-1}}{\Delta}\Big[\alpha(\beta\eta - 1) \int_0^\xi \Big(\int_0^s \frac{(s-k)^{q-1}}{\Gamma(q)} \sigma(k)dk\Big)ds \\ &- \beta(\alpha\xi - 1) \int_0^\eta \Big(\int_0^s \frac{(s-k)^{q-1}}{\Gamma(q)} \sigma(k)dk\Big)ds + (\alpha\xi - 1) \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} \sigma(s)ds\Big]. \end{aligned}$$

where

$$\Delta = \frac{\alpha\xi^m(\beta\eta - 1) - (\alpha\xi - 1)(\beta\eta^m - m)}{m} \neq 0. \tag{2.3}$$

Proof. It is well known [17] that the general solution of the fractional differential equation in (2.1) can be written as

$$x(t)=\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\sigma(s)ds-c_0-c_1t-c_2t^2-\ldots-c_{m-1}t^{m-1}, \tag{2.4}$$

where $c_0,c_1,c_2,\ldots,c_{m-1}$ are arbitrary constants. Applying the boundary conditions for the problem (2.2), we find that $c_1=0,...,c_{m-2}=0$,

$$\begin{aligned} c_0 &= -\frac{\alpha}{m\Delta}\Big[(\beta\eta^m-m)\int_0^\xi\Big(\int_0^s \frac{(s-k)^{q-1}}{\Gamma(q)}\sigma(k)dk\Big)ds \\ &- \beta\xi^m\int_0^\eta\Big(\int_0^s \frac{(s-k)^{q-1}}{\Gamma(q)}\sigma(k)dk\Big)ds+\xi^m\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}\sigma(s)ds\Big]\end{aligned}$$

and

$$\begin{aligned} c_{m-1} &= \frac{1}{\Delta}\Big[\alpha(\beta\eta-1)\int_0^\xi\Big(\int_0^s \frac{(s-k)^{q-1}}{\Gamma(q)}\sigma(k)dk\Big)ds \\ &- \beta(\alpha\xi-1)\int_0^\eta\Big(\int_0^s \frac{(s-k)^{q-1}}{\Gamma(q)}\sigma(k)dk\Big)ds+(\alpha\xi-1)\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}\sigma(s)ds\Big],\end{aligned}$$

where Δ is given by (2.3). Substituting the values of $c_0,\ c_1,\ldots,c_{m-1}$ in (2.4), we obtain

$$\begin{aligned} x(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\sigma(s)ds+\frac{\alpha}{m\Delta}\Big[(\beta\eta^m-m)\int_0^\xi\Big(\int_0^s \frac{(s-k)^{q-1}}{\Gamma(q)}\sigma(k)dk\Big)ds \\ &- \beta\xi^m\int_0^\eta\Big(\int_0^s \frac{(s-k)^{q-1}}{\Gamma(q)}\sigma(k)dk\Big)ds+\xi^m\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}\sigma(s)ds\Big] \\ &- \frac{t^{m-1}}{\Delta}\Big[\alpha(\beta\eta-1)\int_0^\xi\Big(\int_0^s \frac{(s-k)^{q-1}}{\Gamma(q)}\sigma(k)dk\Big)ds \\ &- \beta(\alpha\xi-1)\int_0^\eta\Big(\int_0^s \frac{(s-k)^{q-1}}{\Gamma(q)}\sigma(k)dk\Big)ds+(\alpha\xi-1)\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}\sigma(s)ds\Big].\end{aligned}$$

This completes the proof. □

In view of Lemma 2.2, we define an operator $\mathbf{F}:\mathcal{C}\to\mathcal{C}$ by

$$
\begin{aligned}
(\mathbf{F}x)(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds \\
&+\frac{\alpha}{m\Delta}\Big[(\beta\eta^m - m)\int_0^{\xi}\Big(\int_0^s \frac{(s-k)^{q-1}}{\Gamma(q)} f(k,x(k))dk\Big)ds \\
&-\beta\xi^m \int_0^{\eta}\Big(\int_0^s \frac{(s-k)^{q-1}}{\Gamma(q)} f(k,x(k))dk\Big)ds \\
&+\xi^m \textstyle\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds\Big] \qquad (2.5)\\
&-\frac{t^{m-1}}{\Delta}\Big[\alpha(\beta\eta-1)\int_0^{\xi}\Big(\int_0^s \frac{(s-k)^{q-1}}{\Gamma(q)} f(k,x(k))dk\Big)ds \\
&-\beta(\alpha\xi-1)\int_0^{\eta}\Big(\int_0^s \frac{(s-k)^{q-1}}{\Gamma(q)} f(k,x(k))dk\Big)ds \\
&+(\alpha\xi-1)\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds\Big], \quad t\in[0,1].
\end{aligned}
$$

For convenience, let us set

$$
\begin{aligned}
\vartheta &= \max_{t\in[0,1]}\Big\{\frac{t^q}{\Gamma(q+1)} + \frac{|\alpha|}{m|\Delta|}\Big[\frac{|\beta\eta^m - m|\xi^{q+1} + (|\beta|\eta^{q+1} + (q+1))\xi^m}{\Gamma(q+2)}\Big] \\
&+ \frac{t^{m-1}}{|\Delta|}\Big[\frac{|\alpha(\beta\eta-1)|\xi^{q+1} + |\alpha\xi-1|(|\beta|\eta^{q+1} + (q+1))}{\Gamma(q+2)}\Big]\Big\} \\
&= \frac{1}{\Gamma(q+1)}\Big(1+\frac{\vartheta_1+\vartheta_2}{m|\Delta|(q+1)}\Big), \qquad (2.6)
\end{aligned}
$$

where

$$\vartheta_1 = |\alpha|(|\beta\eta^m - m| + m|\beta\eta-1|)\xi^{q+1}, \quad \vartheta_2 = (|\alpha|\xi^m + m|\alpha\xi-1|)(|\beta|\eta^{q+1} + (q+1)).$$

Observe that the problem (1.1) has solutions if the operator equation $\mathbf{F}x = x$ has fixed points.

For the forthcoming analysis, we need the following assumption:

(H) Assume that $f:[0,1]\times X\to X$ is a jointly continuous function and maps bounded subsets of $[0,1]\times X$ into relatively compact subsets of X.

Furthermore, we need the following fixed point theorem to prove the existence of solutions for the problem at hand.

Theorem 2.1 [21] *Let X be a Banach space. Assume that Ω is an open bounded subset of X with $\theta\in\Omega$ and let $T:\overline{\Omega}\to X$ be a completely continuous operator such that*

$$\|Tu\| \le \|u\|, \qquad \forall u\in\partial\Omega.$$

Then T has a fixed point in $\overline{\Omega}$.

3 Existence results in Banach space

Lemma 3.1 *The operator* $\boldsymbol{F}:\mathcal{C}\to\mathcal{C}$ *is completely continuous.*

Proof. Clearly, continuity of the operator $\mathbf{F}$ follows from the continuity of f. Let $\Omega\subset\mathcal{C}$ be bounded. Then, $\forall x\in\Omega$, by the assumption (H), there exists $L_1>0$ such that $|f(t,x)|\le L_1$. Thus, we have

$$\begin{aligned}
|(\mathbf{F})(t)| &\le \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds \\
&\quad+\frac{|\alpha|}{m|\Delta|}\Big[|\beta\eta^m-m|\int_0^\xi\Big(\int_0^s\frac{(s-k)^{q-1}}{\Gamma(q)}|f(k,x(k))|dk\Big)ds \\
&\quad+|\beta|\xi^m\int_0^\eta\Big(\int_0^s\frac{(s-k)^{q-1}}{\Gamma(q)}|f(k,x(k))|dk\Big)ds \\
&\quad+\xi^m\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds\Big] \\
&\quad+\frac{|t^{m-1}|}{|\Delta|}\Big[|\alpha(\beta\eta-1)|\int_0^\xi\Big(\int_0^s\frac{(s-k)^{q-1}}{\Gamma(q)}|f(k,x(k))|dk\Big)ds \\
&\quad+|\beta(\alpha\xi-1)|\int_0^\eta\Big(\int_0^s\frac{(s-k)^{q-1}}{\Gamma(q)}|f(k,x(k))|dk\Big)ds \qquad (3.1)\\
&\quad+|\alpha\xi-1|\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds\Big] \\
&\le L_1\Big\{\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}ds+\frac{|\alpha|}{m|\Delta|}\Big[|\beta\eta^m-m|\int_0^\xi\Big(\int_0^s\frac{(s-k)^{q-1}}{\Gamma(q)}dk\Big)ds \\
&\quad+|\beta|\xi^m\int_0^\eta\Big(\int_0^s\frac{(s-k)^{q-1}}{\Gamma(q)}dk\Big)ds+\xi^m\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}ds\Big] \\
&\quad+\frac{|t^{m-1}|}{|\Delta|}\Big[|\alpha(\beta\eta-1)|\int_0^\xi\Big(\int_0^s\frac{(s-k)^{q-1}}{\Gamma(q)}dk\Big)ds \\
&\quad+|\beta(\alpha\xi-1)|\int_0^\eta\Big(\int_0^s\frac{(s-k)^{q-1}}{\Gamma(q)}dk\Big)ds+|\alpha\xi-1|\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}ds\Big]\Big\} \\
&\le \frac{L_1}{\Gamma(q+1)}\Big(1+\frac{\vartheta_1+\vartheta_2}{m|\Delta|(q+1)}\Big)=L_2,
\end{aligned}$$

which implies that $\|(\mathbf{F}x)\|\le L_2$. Furthermore,

$$\begin{aligned}
|(\mathbf{F}x)'(t)| &= \int_0^t\frac{(t-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s))|ds \\
&\quad+\frac{|(m-1)t^{m-2}|}{|\Delta|}\Big[|\alpha(\beta\eta-1)|\int_0^\xi\Big(\int_0^s\frac{(s-k)^{q-1}}{\Gamma(q)}|f(k,x(k))|dk\Big)ds
\end{aligned}$$

$$
\begin{aligned}
&+|\beta(\alpha\xi-1)|\int_0^\eta\Big(\int_0^s\frac{(s-k)^{q-1}}{\Gamma(q)}|f(k,x(k))|dk\Big)ds\\
&+|\alpha\xi-1|\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds\Big]\\
\le\;& L_1\Big\{\int_0^t\frac{(t-s)^{q-2}}{\Gamma(q-1)}ds \qquad (3.2)\\
&+\frac{|(m-1)t^{m-2}|}{|\Delta|}\Big[|\alpha(\beta\eta-1)|\int_0^\xi\Big(\int_0^s\frac{(s-k)^{q-1}}{\Gamma(q)}dk\Big)ds\\
&+|\beta(\alpha\xi-1)|\int_0^\eta\Big(\int_0^s\frac{(s-k)^{q-1}}{\Gamma(q)}dk\Big)ds+|\alpha\xi-1|\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}ds\Big]\Big\}\\
\le\;& L_1\Big\{\frac{1}{\Gamma(q)}+\frac{|m-1|}{|\Delta|}\Big[\frac{|\alpha(\beta\eta-1)|\xi^{q+1}+|\beta(\alpha\xi-1)|\eta^{q+1}}{\Gamma(q+2)}+\frac{|(\alpha\xi-1)|}{\Gamma(q+1)}\Big]\Big\}\\
=\;& L_3.
\end{aligned}
$$

Hence, for $t_1, t_2 \in [0,1]$, we have

$$|(\mathbf{F}x)(t_2)-(\mathbf{F}x)(t_1)|\le\int_{t_1}^{t_2}|(\mathbf{F}x)'(s)|ds\le L_3(t_2-t_1).$$

This implies that $\mathbf{F}$ is equicontinuous on $[0,1]$. Thus, by the Arzela-Ascoli theorem, we have that $\mathbf{F}(\Omega)(t)$ is relatively compact in X for every t, and so the operator $\mathbf{F}:\mathcal{C}\to\mathcal{C}$ is completely continuous.

Theorem 3.1 *Assume that* (H) *holds and* $\lim_{x\to 0}\frac{f(t,x)}{x}=0$, *where the limit is uniform with respect to* t. *Then the problem* (1.1) *has at least one solution.*

Proof. Since $\lim_{x\to 0}\frac{f(t,x)}{x}=0$, therefore there exists a constant $r>0$ such that $|f(t,x)|\le\delta|x|$ for $0<|x|<r$, where $\delta>0$ is such that

$$\vartheta\delta\le 1, \tag{3.3}$$

where ϑ is given by (2.6). Define $\Omega_1=\{x\in\mathcal{C}\mid \|x\|<r\}$ and take $x\in \mathrm{C}$ such that $\|x\|=r$, that is, $x\in\partial\Omega_1$. As before, it can be shown that $\mathbf{F}$ is completely continuous

and

$$\begin{aligned}|\mathbf{F}x(t)| &\le \max_{t\in[0,1]}\Big\{\frac{t^q}{\Gamma(q+1)}+\frac{|\alpha|}{m|\Delta|}\Big[\frac{|\beta\eta^m-m|\xi^{q+1}+(|\beta|\eta^{q+1}+(q+1))\xi^m}{\Gamma(q+2)}\Big]\\ &+\frac{t^{m-1}}{|\Delta|}\Big[\frac{|\alpha(\beta\eta-1)|\xi^{q+1}+|\alpha\xi-1|(|\beta|\eta^{q+1}+(q+1))}{\Gamma(q+2)}\Big]\Big\}\delta\|x\| \\ &=\vartheta\delta\|x\|,\end{aligned} \tag{3.4}$$

which, in view of (3.3), yields $\|\mathbf{F}x\| \le \|x\|$, $x \in \partial\Omega_1$. Therefore, by Theorem 2.1, the operator $\mathbf{F}$ has at least one fixed point, which in turn implies that the problem (1.1) has at least one solution. □

Theorem 3.2 *Assume that $f : [0,1]\times X \to X$ is a jointly continuous function and satisfies the condition*

$$\|f(t,x)-f(t,y)\| \le L\|x-y\|, \forall t\in[0,1],\ L>0,\ x,y\in X,$$

with $L < 1/\vartheta$, where ϑ is given by (2.6). Then the boundary value problem (1.1) has a unique solution.

Proof. Setting $\sup_{t\in[0,1]}|f(t,0)| = M$ and choosing $r \ge \dfrac{\vartheta M}{1-L\vartheta}$, we show that $\mathbf{F}B_r \subset B_r$, where $B_r = \{x\in\mathcal{C} : \|x\|\le r\}$. For $x\in B_r$, we have:

$$\begin{aligned}\|(\mathbf{F})(t)\| \;\le\; & \sup_{t\in[0,1]}\Big[\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds\\ &+\frac{|\alpha|}{m|\Delta|}\Big\{|\beta\eta^m-m|\int_0^\xi\Big(\int_0^s\frac{(s-k)^{q-1}}{\Gamma(q)}|f(k,x(k))|dk\Big)ds\\ &+|\beta|\xi^m\int_0^\eta\Big(\int_0^s\frac{(s-k)^{q-1}}{\Gamma(q)}|f(k,x(k))|dk\Big)ds\\ &+\xi^m\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds\Big\}\\ &+\frac{|t^{m-1}|}{|\Delta|}\Big\{|\alpha(\beta\eta-1)|\int_0^\xi\Big(\int_0^s\frac{(s-k)^{q-1}}{\Gamma(q)}|f(k,x(k))|dk\Big)ds\\ &+|\beta(\alpha\xi-1)|\int_0^\eta\Big(\int_0^s\frac{(s-k)^{q-1}}{\Gamma(q)}|f(k,x(k))|dk\Big)ds\\ &+|\alpha\xi-1|\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds\Big\}\Big]\\ \le\; & \sup_{t\in[0,1]}\Big[\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}(|f(s,x(s))-f(s,0)|+|f(s,0)|)ds\end{aligned}$$

$$
\begin{aligned}
&+\frac{|\alpha|}{m|\Delta|}\Big\{|\beta\eta^m - m|\int_0^\xi \Big(\int_0^s (s-k)^{q-1}(|f(k,x(k)) - f(k,0)|\\
&+|f(k,0)|)dk\Big)ds\\
&+|\beta|\xi^m\int_0^\eta \Big(\int_0^s (s-k)^{q-1}(|f(k,x(k)) - f(k,0)| + |f(k,0)|)dk\Big)ds\\
&+\xi^m\int_0^1 (1-s)^{q-1}(|f(s,x(s)) - f(s,0)| + |f(s,0)|)ds\Big\}\\
&+\frac{|t^{m-1}|}{|\Delta|}\Big\{|\alpha(\beta\eta-1)|\int_0^\xi \Big(\int_0^s (s-k)^{q-1}(|f(k,x(k)) - f(k,0)|\\
&+|f(k,0)|)dk\Big)ds\\
&+|\beta(\alpha\xi-1)|\int_0^\eta \Big(\int_0^s (s-k)^{q-1}(|f(k,x(k)) - f(k,0)|\\
&+\|f(k,0)\|)dk\Big)ds\\
&+|(\alpha\xi-1)|\int_0^1 (1-s)^{q-1}(|f(s,x(s)) - f(s,0)| + |f(s,0)|)ds\Big\}\Big]\\
\le\ & (Lr+M)\sup_{t\in[0,1]}\Big[\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}ds + \frac{|\alpha|}{m|\Delta|}\Big\{|\beta\eta^m - m|\int_0^\xi\Big(\int_0^s \frac{(s-k)^{q-1}}{\Gamma(q)}dk\Big)ds\\
&+|\beta|\xi^m\int_0^\eta\Big(\int_0^s \frac{(s-k)^{q-1}}{\Gamma(q)}dk\Big)ds + \xi^m\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}ds\Big\}\\
&+\frac{|t^{m-1}|}{|\Delta|}\Big\{|\alpha(\beta\eta-1)|\int_0^\xi\Big(\int_0^s \frac{(s-k)^{q-1}}{\Gamma(q)}dk\Big)ds\\
&+|\beta(\alpha\xi-1)|\int_0^\eta\Big(\int_0^s \frac{(s-k)^{q-1}}{\Gamma(q)}dk\Big)ds + |\alpha\xi-1|\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}ds\Big\}\Big]\\
\le\ & \frac{(Lr+M)}{\Gamma(q+1)}\Big(1+\frac{\vartheta_1+\vartheta_2}{m|\Delta|(q+1)}\Big) = (Lr+M)\vartheta \le r.
\end{aligned}
$$

Now, for $x, y \in \mathcal{C}$ and for each $t \in [0,1]$, we obtain

$$
\begin{aligned}
&\|(\mathbf{F}x)(t) - (\mathbf{F}y)(t)\|\\
&\le \sup_{t\in[0,1]}\Big[\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}|f(s,x(s)) - f(s,y(s))|ds\\
&+\frac{|\alpha|}{m|\Delta|}\Big\{|\beta\eta^m - m|\int_0^\xi\Big(\int_0^s \frac{(s-k)^{q-1}}{\Gamma(q)}|f(k,x(k)) - f(k,y(k))|dk\Big)ds\\
&+|\beta|\xi^m\int_0^\eta\Big(\int_0^s \frac{(s-k)^{q-1}}{\Gamma(q)}|f(k,x(k)) - f(k,y(k))|dk\Big)ds
\end{aligned}
$$

$$
\begin{aligned}
& +\xi^m \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} |f(s,x(s)) - f(s,y(s))| ds \Big\} \\
& + \frac{|t^{m-1}|}{|\Delta|} \Big\{ |\alpha(\beta\eta - 1)| \int_0^\xi \Big(\int_0^s \frac{(s-k)^{q-1}}{\Gamma(q)} |f(k,x(k)) - f(k,y(k))| dk \Big) ds \\
& + |\beta(\alpha\xi - 1)| \int_0^\eta \Big(\int_0^s \frac{(s-k)^{q-1}}{\Gamma(q)} |f(k,x(k)) - f(k,y(k))| dk \Big) ds \\
& + |\alpha\xi - 1| \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} |f(s,x(s)) - f(s,y(s))| ds \Big\} \Big] \\
\le \; & L\|x-y\| \sup_{t\in[0,1]} \Big[\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} ds + \frac{|\alpha|}{m|\Delta|} \Big\{ |\beta\eta^m - m| \int_0^\xi \Big(\int_0^s \frac{(s-k)^{q-1}}{\Gamma(q)} dk \Big) ds \\
& + |\beta|\xi^m \int_0^\eta \Big(\int_0^s \frac{(s-k)^{q-1}}{\Gamma(q)} dk \Big) ds + \xi^m \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} ds \Big\} \\
& + \frac{|t^{m-1}|}{|\Delta|} \Big\{ |\alpha(\beta\eta - 1)| \int_0^\xi \Big(\int_0^s \frac{(s-k)^{q-1}}{\Gamma(q)} dk \Big) ds \\
& + |\beta(\alpha\xi - 1)| \int_0^\eta \Big(\int_0^s \frac{(s-k)^{q-1}}{\Gamma(q)} dk \Big) ds + |(\alpha\xi - 1)| \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} ds \Big\} \Big] \\
\le \; & \frac{L}{\Gamma(q+1)} \left(1 + \frac{\vartheta_1 + \vartheta_2}{m|\Delta|(q+1)} \right) \|x-y\| \\
= \; & L\vartheta \|x-y\|,
\end{aligned}
$$

where ϑ is given by (2.6). Observe that L depends only on the parameters involved in the problem. As $L < 1/\vartheta$, therefore $\mathbf{F}$ is a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle (Banach fixed point theorem).□

Our next existence result is based on Leray-Schauder degree theory.

Theorem 3.3 *Suppose that (H) holds. Furthermore, it is assumed that there exist constants $0 \le \kappa < \frac{1}{\theta}$, where θ is given by (2.6) and $M > 0$ such that $|f(t,x)| \le \kappa|x| + M$ for all $t \in [0,1], x \in X$. Then the boundary value problem (1.1) has at least one solution.*

Proof. Consider a fixed point problem

$$
x = \mathbf{F}x, \tag{3.5}
$$

where $\mathbf{F}$ is defined by (2.5). In view of the fixed point problem (3.5), we just need to prove the existence of at least one solution $x \in \mathcal{C}$ satisfying (3.5). Define a suitable ball B_R with radius $R > 0$ as

$$
B_R = \{x \in \mathcal{C} : \|x\| < R\},
$$

where R will be fixed later. Then, it is sufficient to show that $\mathbf{F} : \overline{B}_R \to \mathcal{C}$ satisfies

$$x \neq \lambda \mathbf{F} x, \quad \forall\, x \in \partial B_R \;\text{ and }\; \forall\, \lambda \in [0,1]. \tag{3.6}$$

Let us set

$$H(\lambda, x) = \lambda \mathbf{F} x, \quad x \in X, \quad \lambda \in [0,1].$$

Then, by the Arzelá-Ascoli Theorem, $h_\lambda(x) = x - H(\lambda, x) = x - \lambda \mathbf{F} x$ is completely continuous. If (3.6) is true, then the following Leray-Schauder degrees are well defined and by the homotopy invariance of topological degree, it follows that

$$\begin{aligned} \deg(h_\lambda, B_R, 0) &= \deg(I - \lambda \mathbf{F}, B_R, 0) = \deg(h_1, B_R, 0) \\ &= \deg(h_0, B_R, 0) = \deg(I, B_R, 0) = 1 \neq 0, \quad 0 \in B_r, \end{aligned}$$

where I denotes the identity operator. By the nonzero property of Leray-Schauder degree, $h_1(t) = x - \lambda \mathbf{F} x = 0$ for at least one $x \in B_R$. In order to prove (3.6), we assume that $x = \lambda \mathbf{F} x$ for some $\lambda \in [0,1]$ and for all $t \in [0,1]$ so that

$$\begin{aligned} |x(t)| &= |\lambda \mathbf{F} x(t)| \leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s, x(s))| ds \\ &\quad + \frac{|\alpha|}{m|\Delta|} \Big[|\beta \eta^m - m| \int_0^\xi \Big(\int_0^s \frac{(s-k)^{q-1}}{\Gamma(q)} |f(k, x(k))| dk \Big) ds \\ &\quad + |\beta| \xi^m \int_0^\eta \Big(\int_0^s \frac{(s-k)^{q-1}}{\Gamma(q)} |f(k, x(k))| dk \Big) ds \\ &\quad + \xi^m \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} |f(s, x(s))| ds \Big] \\ &\quad + \frac{|t^{m-1}|}{|\Delta|} \Big[|\alpha(\beta\eta - 1)| \int_0^\xi \Big(\int_0^s \frac{(s-k)^{q-1}}{\Gamma(q)} |f(k, x(k))| dk \Big) ds \\ &\quad + |\beta(\alpha\xi - 1)| \int_0^\eta \Big(\int_0^s \frac{(s-k)^{q-1}}{\Gamma(q)} |f(k, x(k))| dm \Big) ds \\ &\quad + |\alpha\xi - 1| \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} |f(s, x(s))| ds \Big] \\ &\leq (\kappa |x| + M) \Big\{ \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} ds + \frac{|\alpha|}{m|\Delta|} \Big[|\beta \eta^m - m| \int_0^\xi \Big(\int_0^s \frac{(s-k)^{q-1}}{\Gamma(q)} dk \Big) ds \\ &\quad + |\beta| \xi^m \int_0^\eta \Big(\int_0^s \frac{(s-k)^{q-1}}{\Gamma(q)} dk \Big) ds + \xi^m \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} ds \Big] \\ &\quad + \frac{|t^{m-1}|}{|\Delta|} \Big[|\alpha(\beta\eta - 1)| \int_0^\xi \Big(\int_0^s \frac{(s-k)^{q-1}}{\Gamma(q)} dk \Big) ds \\ &\quad + |\beta(\alpha\xi - 1)| \int_0^\eta \Big(\int_0^s \frac{(s-k)^{q-1}}{\Gamma(q)} dk \Big) ds + |(\alpha\xi - 1)| \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} ds \Big] \Big\} \end{aligned}$$

$$\leq \frac{(\kappa|x| + M)}{\Gamma(q+1)}\Big(1 + \frac{\vartheta_1 + \vartheta_2}{m|\Delta|(q+1)}\Big) = (\kappa|x| + M)\,\vartheta,$$

which, on taking norm and solving for $\|x\|$, yields

$$\|x\| \leq \frac{M\vartheta}{1 - \kappa\vartheta}.$$

Letting $R = \dfrac{M\vartheta}{1-\kappa\vartheta} + 1$, (3.6) holds. This completes the proof. □

4 Examples

Let us define

$$X = \{x = (x_1, x_2, ..., x_p, ...) : x_p \to 0\}$$

with the norm $\|x\| = \sup_p |x_p|$.

Example 4.1 *Consider the problem*

$$\begin{cases} {}^C D^q x_p(t) = (121 + x_p^3(t))^{\frac{1}{2}} + 2(t+1)(x_p - \sin x_p(t)) - 11, \quad 0 < t < 1, \\ x_p(0) = \alpha \displaystyle\int_0^\xi x_p(s)ds, \ x_p'(0) = 0, \ x_p''(0) = 0, \ldots, x_p^{(m-2)}(0) = 0, \\ \quad x_p(1) = \beta \displaystyle\int_0^\eta x_p(s)ds, \quad 0 < \xi, \eta < 1, \end{cases} \tag{4.1}$$

where $m - 1 < q \leq m, \ m \geq 2$.

It can easily be verified that all the assumptions of Theorem 3.1 hold. Consequently, the conclusion of Theorem 3.1 implies that the problem (4.1) has at least one solution.

Example 4.2 *Consider the following four-point integral fractional boundary value problem*

$$\begin{cases} {}^c D^{13/2} x_p(t) = \dfrac{512}{p(t+2)^2} \dfrac{|x_p|}{(1+|x_p|)}, \quad t \in [0,1], \\ x_p(0) = \dfrac{1}{2}\displaystyle\int_0^{1/4} x_p(s)ds, \ x_p'(0) = 0, \ldots, x_p^{(v)}(0) = 0, \ x_p(1) = \displaystyle\int_0^{3/4} x_p(s)ds. \end{cases} \tag{4.2}$$

Here, $q = 13/2$, $m = 7$, $\alpha = 1/2$, $\beta = 1$, $\xi = 1/4$, $\eta = 3/4$, and $f_p(t,x) = \dfrac{512}{p(t+2)^2}\dfrac{|x_p|}{(1+|x_p|)}$, $f = (f_1, f_2, ..., f_p, ...)$. As $|f_p(t,x) - f_p(t,y)| \leq 128|x_p - y_p|$, therefore, $\|f(t,x) - f(t,y)\| \leq 128\|x - y\|$ with $L = 128$. Further,

$$L\vartheta \ = \tfrac{L}{\Gamma(q+1)}\Big(1 + \tfrac{\vartheta_1+\vartheta_2}{m|\Delta|(q+1)}\Big) = 128 \times (1.087591 \times 10^{-03}) = 0.139212 < 1.$$

Thus, by the conclusion of Theorem 3.2, the boundary value problem (4.2) has a unique solution on $[0,1]$.

Example 4.3 *Consider the following boundary value problem*

$$\begin{cases} {}^cD^{13/2}x_p(t) = \dfrac{1}{(4\pi)}\sin(2\pi x_p) + \dfrac{|x_p|}{1+|x_p|}, \quad t \in [0,1], \\ x_p(0) = 2\displaystyle\int_0^{1/4} x_p(s)ds, \ x'_p(0) = 0, \ldots, x_p^{(v)}(0) = 0, \ x_p(1) = 3\displaystyle\int_0^{3/4} x_p(s)ds. \end{cases} \tag{4.3}$$

Here, $q = 13/2, \ \alpha = 2, \ \beta = 3, \xi = 1/3, \ \eta = 2/3$, and

$$\Big|f_p(t,x)\Big| = \left|\frac{1}{(4\pi)}\sin(2\pi x_p) + \frac{|x_p|}{1+|x_p|}\right| \le \frac{1}{2}|x_p| + 1.$$

So $\|f(t,x)\| \le \frac{1}{2}\|x\| + 1$. Clearly $M = 1, \ \kappa = 1/2, \ \vartheta = 1.081553 \times 10^{-03}$, and $\kappa < 1/\vartheta$. Thus, all the conditions of Theorem 3.3 are satisfied and consequently the problem (4.3) has at least one solution.

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References

[1] R.P. Agarwal, M. Benchohra, S. Hamani, Boundary value problems for fractional differential equations, *Georgian Math. J.* **16** (2009), 401-411.

[2] R.P. Agarwal, B. Ahmad, Existence of solutions for impulsive anti-periodic boundary value problems of fractional semilinear evolution equations, *Dyn. Contin. Discrete Impuls. Syst., Ser. A, Math. Anal.,* to appear.

[3] B. Ahmad, J.J. Nieto, Existence of solutions for nonlocal boundary value problems of higher order nonlinear fractional differential equations, *Abstr. Appl. Anal. 2009, Art. ID 494720, 9 pp.*

[4] B. Ahmad, J.J. Nieto, Existence results for nonlinear boundary value problems of fractional integrodifferential equations with integral boundary conditions, *Bound. Value Probl. 2009, Art. ID 708576, 11 pp.*

[5] B. Ahmad, A. Alsaedi, B. Alghamdi, Analytic approximation of solutions of the forced Duffing equation with integral boundary conditions, *Nonlinear Anal. Real World Appl.* **9** (2008), 1727-1740.

[6] B. Ahmad, J.J. Nieto, Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions, *Comput. Math. Appl.* **58** (2009) 1838-1843.

[7] B. Ahmad, S. Sivasundaram, On four-point nonlocal boundary value problems of nonlinear integro-differential equations of fractional order, *Appl. Math. Comput.* **217** (2010), 480-487.

[8] B. Ahmad, Existence of solutions for irregular boundary value problems of nonlinear fractional differential equations, *Appl. Math. Lett.* **23** (2010), 390-394.

[9] B. Ahmad, Existence of solutions for fractional differential equations of order $q \in (2, 3]$ with anti-periodic boundary conditions, *J. Appl. Math. Comput.* **34** (2010), 385-391.

[10] B. Ahmad, S.K. Ntouyas, A. Alsaedi, New existence results for nonlinear fractional differential equations with three-point integral boundary conditions, *Adv. Difference Equ.* 2011, Art. ID 107384, 11 pp.

[11] Z. Bai, On positive solutions of a nonlocal fractional boundary value problem, *Nonlinear Anal.* **72** (2010), 916-924.

[12] M. Benchohra, S. Hamani, Nonlinear boundary value problems for differential inclusions with Caputo fractional derivative, *Topol. Methods Nonlinear Anal.* **32**(2008), 115-130.

[13] M. Benchohra, S. Hamani, S.K. Ntouyas, Boundary value problems for differential equations with fractional order and nonlocal conditions, *Nonlinear Anal.* **71** (2009) 2391-2396.

[14] R.P Agarwal, M. Benchohra and S. Hamani, A survey on existence result for boundary value problems of nonlinear fractional differential equations and inclusions, *Acta. Appl. Math.* **109** (2010), 973-1033.

[15] Y.-K. Chang, J.J. Nieto, Some new existence results for fractional differential inclusions with boundary conditions, *Math. Comput. Model.* **49** (2009), 605-609.

[16] S. Hamani, M. Benchohra, John R. Graef, Existence results for boundary value problems with nonlinear fractional inclusions and integral conditions, *Electronic J. Differential Equations, Vol. 2010(2010), No. 20, pp. 1-16.*

[17] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204, Elsevier Science B.V., Amsterdam, 2006.

[18] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[19] J. Sabatier, O.P. Agrawal, J.A.T. Machado (Eds.), *Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering*, Springer, Dordrecht, 2007.

[20] S.G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional Integrals and Derivatives, Theory and Applications*, Gordon and Breach, Yverdon, 1993.

[21] D.R. Smart, Fixed Point Theorems. Cambridge University Press, 1980.

[22] S. Zhang, Positive solutions to singular boundary value problem for nonlinear fractional differential equation, *Comput. Math. Appl.* **59** (2010), 1300-1309.

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