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Positive almost periodic type solutions to a class of nonlinear difference equations*

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Abstract

This paper is concerned with positive almost periodic type solutions to a class of nonlinear difference equation with delay. By using a fixed point theorem in partially ordered Banach spaces, we establish several theorems about the existence and uniqueness of positive almost periodic type solutions to the addressed difference equation. In addition, in order to prove our main results, some basic and important properties about pseudo almost periodic sequences are presented.

Keywords: almost periodic, asymptotical almost periodic, pseudo almost periodic, nonlinear difference equation, positive solution.

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1 Introduction and preliminaries

In this paper, we consider the following nonlinear difference equation with delay:

$$x(n) = h(x(n)) + \sum_{j=n-k(n)}^{n} f(j, x(j)), \quad n \in \mathbb{Z}, \tag{1.1}$$

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where $h : \mathbb{R}^+ \to \mathbb{R}^+$, $k : \mathbb{Z} \to \mathbb{Z}^+$, and $f : \mathbb{Z} \times \mathbb{R}^+ \to \mathbb{R}^+$.

Equation (1.1) can be regarded as a discrete analogue of the integral equation

$$x(t) = \int_{t-\tau(t)}^{t} f(s, x(s))ds, \quad t \in \mathbb{R}, \tag{1.2}$$

which arise in the spread of some infectious disease (cf. [1]). Since the work of Fink and Gatica [2], the existence of positive almost periodic type and positive almost automorphic type solutions to equation (1.2) and its variants has been of great interest for many authors (see, e.g., [3–10] and references therein).

On the other hand, the existence of almost periodic solutions and almost automorphic solutions has became an interesting and important topic in the study of qualitative theory of difference equations. We refer the reader to [11–18] and references therein for some recent developments on this topic. However, to the best of knowledge, there are seldom literature available about almost periodicity of the solutions to equation (1.1). That is the main motivation of this work.

Throughout the rest of this paper, we denote by $\mathbb{Z}$ ($\mathbb{Z}^+$) the set of (nonnegative) integers, by $\mathbb{N}$ the set of positive integers, by $\mathbb{R}$ ($\mathbb{R}^+$) the set of (nonnegative) real numbers, by Ω a subset of $\mathbb{R}$, by $cardE$ the number of elements for any finite set $E \subset \mathbb{R}$. In addition, for any subset $K \subset \mathbb{R}$, we denote $\mathcal{C}(\mathbb{Z} \times K)$ the set of all the functions $f : \mathbb{Z} \times K \to \mathbb{R}$ satisfying that $f(n, \cdot)$ is uniformly continuous on K uniformly for $n \in \mathbb{Z}$, i.e., $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $|f(n, x_1) - f(n, x_2)| < \varepsilon$ for all $n \in \mathbb{Z}$ and $x_1, x_2 \in K$ with $|x_1 - x_2| < \delta$.

First, let us recall some notations and basic results of almost periodic type sequences (for more details, see [19–21]).

Definition 1.1. *A function $f : \mathbb{Z} \to \mathbb{R}$ is called almost periodic if $\forall \varepsilon$, $\exists N(\varepsilon) \in \mathbb{N}$ such that among any $N(\varepsilon)$ consecutive integers there exists an integer p with the property that*

$$|f(k + p) - f(k)| < \varepsilon, \quad \forall k \in \mathbb{Z}.$$

Denote by $AP(\mathbb{Z})$ the set of all such functions.

Definition 1.2. *Let $\Omega \subset \mathbb{R}$ and f be a function from $\mathbb{Z} \times \Omega$ to $\mathbb{R}$ such that $f(n, \cdot)$ is continuous for each $n \in \mathbb{Z}$. Then f is called almost periodic in $n \in \mathbb{Z}$ uniformly for $\omega \in \Omega$ if for every $\varepsilon > 0$ and every compact $\Sigma \subset \Omega$ there corresponds an integer $N_\varepsilon(\Sigma) > 0$ such that among $N_\varepsilon(\Sigma)$ consecutive integers there exists an integer p with the property that*

$$|f(k + p, \omega) - f(k, \omega)| < \varepsilon$$

for all $k \in \mathbb{Z}$ and $\omega \in \Sigma$. Denote by $AP(\mathbb{Z} \times \Omega)$ the set of all such functions.

Lemma 1.3. *[20, Theorem 1.26] A necessary and sufficient condition for a sequence $f : \mathbb{Z} \to \mathbb{R}$ to be almost periodic is that for any integer sequence $\{n'_k\}$, one can extract a subsequence $\{n_k\}$ such that $\{f(n + n_k)\}$ converges uniformly with respect to $n \in \mathbb{Z}$.*

Next, we denote by $C_0(\mathbb{Z})$ the space of all the functions $f : \mathbb{Z} \to \mathbb{R}$ such that $\lim_{|n|\to\infty} f(n) = 0$; in addition, for each subset $\Omega \subset \mathbb{R}$, we denote by $C_0(\mathbb{Z} \times \Omega)$ the space of all the functions $f : \mathbb{Z} \times \Omega \to \mathbb{R}$ such that $f(n, \cdot)$ is continuous for each $n \in \mathbb{Z}$, and $\lim_{|n|\to\infty} f(n, x) = 0$ uniformly for x in any compact subset of Ω.

Definition 1.4. *A function $f : \mathbb{Z} \to \mathbb{R}$ is called asymptotically almost periodic if it admits a decomposition $f = g + h$, where $g \in AP(\mathbb{Z})$ and $h \in C_0(\mathbb{Z})$. Denote by $AAP(\mathbb{Z})$ the set of all such functions.*

Definition 1.5. *A function $f : \mathbb{Z} \times \Omega \to \mathbb{R}$ is called asymptotically almost periodic in n uniformly for $x \in \Omega$ if it admits a decomposition $f = g + h$, where $g \in AP(\mathbb{Z} \times \Omega)$ and $h \in C_0(\mathbb{Z} \times \Omega)$. Denote by $AAP(\mathbb{Z} \times \Omega)$ the set of all such functions.*

Next, we denote by $PAP_0(\mathbb{Z})$ the space of all the bounded functions $f : \mathbb{Z} \to \mathbb{R}$ such that

$$\lim_{n\to\infty} \frac{1}{2n} \sum_{k=-n}^{n} |f(k)| = 0;$$

moreover, for any subset $\Omega \subset \mathbb{R}$, we denote by $PAP_0(\mathbb{Z} \times \Omega)$ the space of all the functions $f : \mathbb{Z} \times \Omega \to \mathbb{R}$ such that $f(n, \cdot)$ is continuous for each $n \in \mathbb{Z}$, $f(\cdot, x)$ is bounded for each $x \in \Omega$, and

$$\lim_{n\to\infty} \frac{1}{2n} \sum_{k=-n}^{n} |f(k, x)| = 0$$

uniformly for x in any compact subset of Ω.

Definition 1.6. *A function $f : \mathbb{Z} \to \mathbb{R}$ is called pseudo almost periodic if it admits a decomposition $f = g + h$, where $g \in AP(\mathbb{Z})$ and $h \in PAP_0(\mathbb{Z})$. Denote by $PAP(\mathbb{Z})$ the set of all such functions.*

Definition 1.7. *A function $f : \mathbb{Z} \times \Omega \to \mathbb{R}$ is called pseudo almost periodic in n uniformly for $x \in \Omega$ if it admits a decomposition $f = g + h$, where $g \in AP(\mathbb{Z} \times \Omega)$ and $h \in PAP_0(\mathbb{Z} \times \Omega)$. Denote by $PAP(\mathbb{Z} \times \Omega)$ the set of all such functions.*

Lemma 1.8. *Let $E \in \{AP(\mathbb{Z}), AAP(\mathbb{Z}), PAP(\mathbb{Z})\}$. Then the following hold true:*
(a) $f \in E$ implies that f is bounded.
(b) $f, g \in E$ implies that $f + g, f \cdot g \in E$.

(c) If $E \in \{AP(\mathbb{Z}), AAP(\mathbb{Z})\}$, then E is a Banach space equipped with the supremum norm.

Proof. The proof is similar to that of continuous case. So we omit the details. $\square$

Now, let us recall some basic notations about cone. Let X be a real Banach space. A closed convex set P in X is called a convex cone if the following conditions are satisfied:

(i) if $x \in P$, then $\lambda x \in P$ for any $\lambda \geq 0$;

(ii) if $x \in P$ and $-x \in P$, then $x = 0$.

A cone P induces a partial ordering $\leq$ in X by

$$x \leq y \quad \text{if and only if} \quad y - x \in P.$$

For any given $u, v \in P$,

$$[u, v] := \{x \in X | u \leq x \leq v\}.$$

A cone P is called normal if there exists a constant $k > 0$ such that

$$0 \leq x \leq y \quad \text{implies that} \quad \|x\| \leq k\|y\|,$$

where $\|\cdot\|$ is the norm on X. We denote by P^o the interior of P. A cone P is called a solid cone if $P^o \neq \emptyset$.

The following theorem will be used in Section 2:

Theorem 1.9. [22, Theorem 2.1] *Let P be a normal and solid cone in a real Banach space X. Suppose that the operator $A : P^o \times P^o \times P^o \to P^o$ satisfies*

(S1) for each $x, y, z \in P^o$, $A(\cdot, y, z)$ is increasing, $A(x, \cdot, z)$ is decreasing, and $A(x, y, \cdot)$ is decreasing;

(S2) there exists a function $\phi : (0, 1) \times P^o \times P^o \to (0, +\infty)$ such that for each $x, y, z \in P^o$ and $t \in (0, 1)$, $\phi(t, x, y) > t$ and

$$A(tx, t^{-1}y, z) \geq \phi(t, x, y)A(x, y, z);$$

(S3) there exist $x_0, y_0 \in P^o$ such that $x_0 \leq y_0$, $x_0 \leq A(x_0, y_0, x_0)$, $A(y_0, x_0, y_0) \leq y_0$ and

$$\inf_{x,y \in [x_0,y_0]} \phi(t, x, y) > t, \quad \forall t \in (0, 1);$$

(S4) there exists a constant $L > 0$ such that for all $x, y, z_1, z_2 \in P^o$ with $z_1 \geq z_2$,

$$A(x, y, z_1) - A(x, y, z_2) \geq -L \cdot (z_1 - z_2).$$

Then A has a unique fixed point x^* in $[x_0, y_0]$, i.e., $A(x^*, x^*, x^*) = x^*$.

2 Main results

In this section, we assume that the function f in (1.1) admits the following decomposition

$$f(n,x)=\sum_{i=1}^{m} f_i(n,x)g_i(n,x) \tag{2.1}$$

for some $m\in\mathbb{N}$.

For convenience, we first list some assumptions:

(H1) $f_i,g_i\in AP(\mathbb{Z}\times\mathbb{R}^+)$ are nonnegative functions $(i=1,2,...,m)$, $k\in AP(\mathbb{Z})$ with $k(n)\in\mathbb{Z}^+$ for all $n\in\mathbb{Z}$, and $h:\mathbb{R}^+\to\mathbb{R}^+$ is continuous.

(H2) For each $n\in\mathbb{Z}$ and $i\in\{1,2,\dots,m\}$, $f_i(n,\cdot)$ is increasing in $\mathbb{R}^+$, $g_i(n,\cdot)$ is decreasing in $\mathbb{R}^+$, and $h(\cdot)$ is decreasing in $\mathbb{R}^+$. Moreover, there exists a constant $L>0$ such that

$$h(z_1)-h(z_2)\ge -L(z_1-z_2),\quad \forall z_1\ge z_2\ge 0. \tag{2.2}$$

(H3) There exist positive functions φ_i,ψ_i defined on $(0,1)\times(0,+\infty)$ such that

$$f_i(n,\alpha x)\ge\varphi_i(\alpha,x)f_i(n,x),\quad g_i(n,\alpha^{-1}y)\ge\psi_i(\alpha,y)g_i(n,y),$$

and

$$\varphi_i(\alpha,x)>\alpha,\quad \psi_i(\alpha,y)>\alpha$$

for all $x,y>0$, $\alpha\in(0,1)$, $n\in\mathbb{Z}$ and $i\in\{1,2,\dots,m\}$; moreover, for all $a,b\in(0,+\infty)$ with $a\le b$,

$$\inf_{x,y\in[a,b]}\varphi_i(\alpha,x)\psi_i(\alpha,y)>\alpha,\quad \alpha\in(0,1),\ i=1,2,\dots,m.$$

(H4) There exist constants $d\ge c>0$ such that

$$\inf_{n\in\mathbb{Z}}\sum_{j=n-k(n)}^{n}\sum_{i=1}^{m} f_i(j,c)g_i(j,d)\ge c.$$

and

$$\sup_{n\in\mathbb{Z}}\sum_{j=n-k(n)}^{n}\sum_{i=1}^{m} f_i(j,d)g_i(j,c)+h(d)\le d.$$

Lemma 2.1. *Let $f\in AP(\mathbb{Z}\times\mathbb{R}^+)$. Then the following hold true:*

(a) for each compact $K\subset\mathbb{R}^+$, $f\in\mathcal{C}(\mathbb{Z}\times K)$ and f is bounded on $\mathbb{Z}\times K$;

(b) *if* $x \in AP(\mathbb{Z})$ *with* $x(n) \geq 0$ *for each* $n \in \mathbb{Z}$*, then* $f(\cdot, x(\cdot)) \in AP(\mathbb{Z})$*.*

Proof. By a similar proof to continuous almost periodic functions (cf. [20, Chapter II]), one can show the conclusions. □

Lemma 2.2. *Let* $f \in AP(\mathbb{Z})$ *and* $k \in AP(\mathbb{Z})$ *with* $k(n) \in \mathbb{Z}^+$ *for all* $n \in \mathbb{Z}$*. Then* $\widetilde{f} \in AP(\mathbb{Z})$*, where*

$$\widetilde{f}(n) = \sum_{j=n-k(n)}^{n} f(j), \quad \forall n \in \mathbb{Z}.$$

Proof. Let $\{n'_l\}$ be a sequence of integers. Since $f \in AP(\mathbb{Z})$ and $k \in AP(\mathbb{Z})$, there exist a subsequence $\{n_l\} \subset \{n'_l\}$ and two functions $\overline{f} : \mathbb{Z} \to \mathbb{R}$, $\overline{k} : \mathbb{Z} \to \mathbb{R}$ satisfying that $\forall \varepsilon \in (0,1)$, $\exists N \in \mathbb{N}$ such that for all $l > N$ and $n \in \mathbb{Z}$,

$$|f(n+n_l) - \overline{f}(n)| < \varepsilon, \quad |k(n+n_l) - \overline{k}(n)| < \frac{\varepsilon}{2}.$$

Noticing that $k(n) \in \mathbb{Z}^+$ for all $n \in \mathbb{Z}$, one can obtain that $k(n+n_l) = \overline{k}(n)$ for all $l > N$ and $n \in \mathbb{Z}$. Thus, we have

$$\begin{aligned}\left|\widetilde{f}(n+n_l) - \sum_{j=n-\overline{k}(n)}^{n} \overline{f}(j)\right| &= \left|\sum_{j=n+n_l-k(n+n_l)}^{n+n_l} f(j) - \sum_{j=n-\overline{k}(n)}^{n} \overline{f}(j)\right| \\ &= \left|\sum_{j=n-\overline{k}(n)}^{n} f(j+n_l) - \sum_{j=n-\overline{k}(n)}^{n} \overline{f}(j)\right| \\ &\leq \left(\sup_{n\in\mathbb{Z}} \overline{k}(n) + 1\right)\varepsilon\end{aligned}$$

for all $l > N$ and $n \in \mathbb{Z}$, where $\sup_{n\in\mathbb{Z}} \overline{k}(n) \leq \sup_{n\in\mathbb{Z}} k(n) < +\infty$. So $\widetilde{f} \in AP(\mathbb{Z})$. □

Theorem 2.3. *Assume that* f *has the form of* (2.1) *and* (H1)-(H4) *hold. Then equation* (1.1) *has a unique almost periodic solution with a positive infimum.*

Proof. Let $P = \{x \in AP(\mathbb{Z}) : x(n) \geq 0, \forall n \in \mathbb{Z}\}$. It is not difficult to verify that P is a normal and solid cone in $AP(\mathbb{Z})$ and

$$P^o = \{x \in AP(\mathbb{Z}) : \exists \varepsilon > 0 \text{ such that } x(n) \geq \varepsilon,\ \forall n \in \mathbb{Z}\}$$

Define a nonlinear operator A on $P^o \times P^o \times P^o$ by

$$A(x,y,z)(n) = \sum_{j=n-k(n)}^{n} \sum_{i=1}^{m} f_i(j, x(j)) g_i(j, y(j)) + h(z(n)), \quad n \in \mathbb{Z}.$$

Next, let us verify that A satisfy all the assumptions of Theorem 1.9. Let $x, y, z \in P^o$. It follows from (H1) and Lemma 2.1 that

$$f_i(\cdot, x(\cdot)), g_i(\cdot, y(\cdot)), h(z(\cdot)) \in AP(\mathbb{Z}), \quad i = 1, 2, ...m.$$

Then, combining Lemma 1.8 and Lemma 2.2, we get $A(x, y, z) \in AP(\mathbb{Z})$. On the other hand, there exist $\varepsilon, M > 0$ such that $x(n) \geq \varepsilon$ and $y(n) \leq M$ for all $n \in \mathbb{Z}$. If $\varepsilon < c$ and $M > d$ (the other cases are similar), we conclude by (H3) and (H4) that

$$\begin{aligned} A(x,y,z)(n) &= \sum_{j=n-k(n)}^{n} \sum_{i=1}^{m} f_i(j, x(j)) g_i(j, y(j)) + h(z(n)) \\ &\geq \sum_{j=n-k(n)}^{n} \sum_{i=1}^{m} f_i(j, \varepsilon) g_i(j, M) \\ &\geq \sum_{j=n-k(n)}^{n} \sum_{i=1}^{m} f_i(j, \frac{\varepsilon}{c} \cdot c) g_i(j, \frac{M}{d} \cdot d) \\ &\geq \sum_{j=n-k(n)}^{n} \sum_{i=1}^{m} \varphi_i(\frac{\varepsilon}{c}, c) f_i(j, c) \psi_i(\frac{d}{M}) g_i(j, d) \\ &\geq \frac{d\varepsilon}{cM} \sum_{j=n-k(n)}^{n} \sum_{i=1}^{m} f_i(j, c) g_i(j, d) \\ &\geq \frac{d\varepsilon}{cM} \cdot c = \frac{d\varepsilon}{M} > 0, \end{aligned}$$

for all $n \in \mathbb{Z}$. So A is an operator from $P^o \times P^o \times P^o$ to P^o.

By (H2), it is easy to show that for each $x, y, z \in P^o$, $A(\cdot, y, z)$ is increasing, $A(x, \cdot, z)$ is decreasing, and $A(x, y, \cdot)$ is decreasing, i.e., the assumption (S1) in Theorem 1.9 holds.

Let $x, y \in P^o$ and $\alpha \in (0, 1)$. Let

$$a(x, y) = \min\{\inf_{n \in \mathbb{Z}} x(n), \inf_{n \in z} y(n)\}, \quad b(x, y) = \max\{\sup_{n \in \mathbb{Z}} x(n), \sup_{n \in \mathbb{Z}} y(n)\}.$$

Then $0 < a(x, y) \leq b(x, y) < +\infty$ and $x(n), y(n) \in [a(x, y), b(x, y)]$ for all $n \in \mathbb{Z}$. Define

$$\phi_i(\alpha, x, y) = \inf_{u, v \in [a(x,y), b(x,y)]} \varphi_i(\alpha, u) \psi_i(\alpha, v), \quad i = 1, 2, \ldots, m$$

and

$$\phi(\alpha, x, y) = \min_{i=1,2,\ldots,m} \phi_i(\alpha, x, y).$$

By (H3), it is easy to see that $\phi_i(\alpha, x, y) > \alpha$ $(i = 1, 2, \ldots, m)$ for each $x, y \in P^o$ and $\alpha \in (0, 1)$, which gives that $\phi(\alpha, x, y) > \alpha$ for each $x, y \in P^o$ and $\alpha \in (0, 1)$. Now, We deduce by (H3) that

$$A(\alpha x, \alpha^{-1} y, z)(n) - h(z(n)) = \sum_{j=n-k(n)}^{n} \sum_{i=1}^{m} f_i[j, \alpha x(j)] g_i[j, \alpha^{-1} y(j)]$$

$$
\begin{aligned}
&\geq \sum_{j=n-k(n)}^{n}\sum_{i=1}^{m}\varphi_i[\alpha,x(j)]f_i[j,x(j)]\psi_i[\alpha,y(j)]g_i[j,y(j)]\\
&\geq \phi(\alpha,x,y)\sum_{j=n-k(n)}^{n}\sum_{i=1}^{m}f_i[j,x(j)]g_i[j,y(j)]\\
&= \phi(\alpha,x,y)[A(x,y,z)-h(z(n))]\\
&\geq \phi(\alpha,x,y)A(x,y,z)-h(z(n)),
\end{aligned}
$$

for all $x, y, z \in P^o$, $\alpha \in (0,1)$ and $n \in \mathbb{N}$, where $0 < \phi(\alpha,x,y) \leq 1$ was used. Thus ,we have

$$
A(\alpha x, \alpha^{-1}y, z) \geq \phi(\alpha,x,y)A(x,y,z), \quad \forall x,y,z \in P^o,\ \forall \alpha \in (0,1).
$$

So the assumption (S2) of Theorem 1.9 is verified.

Now, let us consider the assumptions (S3) and (S4) of Theorem 1.9. By (H4), it is easy to see that

$$
A(c,d,c) \geq c, \quad A(d,c,d) \leq d.
$$

Also, we have

$$
\begin{aligned}
\inf_{x,y\in[c,d]}\phi(\alpha,x,y) &= \min_{i=1,\dots,m}\inf_{x,y\in[c,d]}\phi_i(\alpha,x,y)\\
&= \min_{i=1,\dots,m}\phi_i(\alpha,c,d)\\
&= \phi(\alpha,c,d) > \alpha,
\end{aligned}
$$

for all $\alpha \in (0,1)$. Thus, the assumption (S3) holds. In addition, (2.2) yields that the assumption (S4) holds.

Now Theorem 1.9 yields that A has a unique fixed point x^* in $[c,d]$, which is just an almost periodic solution with a positive infimum to Equation (1.1). It remains to show that x^* is the unique fixed point of A in P^o. Let $y^* \in P^o$ be a fixed point of A. Then, there exists $\lambda \in (0,1)$ such that $\lambda c \leq x^*, y^* \leq \lambda^{-1}d$. Denote $c' = \lambda c$ and $d' = \lambda^{-1}d$. It is not difficult to see that

$$
A(c',d',c') \geq c', \quad A(d',c',d') \leq d', \quad \inf_{x,y\in[c',d']}\phi(\alpha,x,y) > \alpha,\ \forall \alpha \in (0,1).
$$

Then, similar to the above proof, by Theorem 1.9, one know that A has a unique fixed point in $[c',d']$, which means that $x^* = y^*$. □

Next, let us consider the existence of asymptotic almost periodic solution to equation (1.1). We first establish two lemmas.

Lemma 2.4. *Let $f \in AAP(\mathbb{Z} \times \mathbb{R}^+)$ and $x \in AAP(\mathbb{Z})$ with $x(n) \geq 0$ for each $n \in \mathbb{Z}$. Then $f(\cdot, x(\cdot)) \in AAP(\mathbb{Z})$.*

Proof. Let

$$f = g + h, \quad x = y + z$$

where $g \in AP(\mathbb{Z} \times \mathbb{R}^+)$, $h \in C_0(\mathbb{Z} \times \mathbb{R}^+)$, $y \in AP(\mathbb{Z})$ and $z \in C_0(\mathbb{Z})$. We have

$$f(n, x(n)) = g(n, y(n)) + g(n, x(n)) - g(n, y(n)) + h(n, x(n)), \quad n \in \mathbb{Z}.$$

Let $K = \overline{\{x(n) : n \in \mathbb{Z}\}}$. Then, $K \subset \mathbb{R}^+$ is compact, and it is not difficult to show that $\{y(n) : n \in \mathbb{Z}\} \subset K$. Thus, by Lemma 2.1, $g(\cdot, y(\cdot)) \in AP(\mathbb{Z})$. Since $h \in C_0(\mathbb{Z} \times \mathbb{R}^+)$, we get $\lim\limits_{|n|\to\infty} h(n, x(n)) = 0$. In addition, noting that $g \in \mathcal{C}(\mathbb{Z} \times K)$, we conclude $\lim\limits_{|n|\to\infty} [g(n, x(n)) - g(n, y(n))] = 0$. So $f(\cdot, x(\cdot)) \in AAP(\mathbb{Z})$. □

Lemma 2.5. *Let $f \in AAP(\mathbb{Z})$ and $k \in AP(\mathbb{Z})$ with $k(n) \in \mathbb{Z}^+$ for all $n \in \mathbb{Z}$. Then $\widetilde{f} \in AAP(\mathbb{Z})$, where*

$$\widetilde{f}(n) = \sum_{j=n-k(n)}^{n} f(j), \quad \forall n \in \mathbb{Z}.$$

Proof. Let $f = g + h$, where $g \in AP(\mathbb{Z})$ and $h \in C_0(\mathbb{Z})$. Then

$$\widetilde{f}(n) = \sum_{j=n-k(n)}^{n} g(j) + \sum_{j=n-k(n)}^{n} h(j), \quad \forall n \in \mathbb{Z}.$$

By Lemma 2.2 and noting that k is bounded, we deduce $\widetilde{f} \in AAP(\mathbb{Z})$. □

Theorem 2.6. *Suppose that*

(H1') $f_i, g_i \in AAP(\mathbb{Z} \times \mathbb{R}^+)$ are nonnegative functions ($i = 1, 2, ..., m$), $k \in AP(\mathbb{Z})$ with $k(n) \in \mathbb{Z}^+$ for all $n \in \mathbb{Z}$, and $h : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous.

and (H2)-(H4) hold. Then equation (1.1) *has a unique asymptotic almost periodic solution with a positive infimum.*

Proof. By using Lemma 2.4 and Lemma 2.5, similar to the proof of Theorem 2.3, one can get the conclusion. □

Now, let us consider the existence of pseudo almost periodic solution to equation (1.1). Also, we first establish some lemmas.

Lemma 2.7. *Let $x \in PAP(\mathbb{Z})$ with $x = y + z$, where $y \in AP(\mathbb{Z})$ and $z \in PAP_0(\mathbb{Z})$. Then*

$$\{y(n) : n \in \mathbb{Z}\} \subset \overline{\{x(n) : n \in \mathbb{Z}\}}.$$

Proof. By contradiction, there exists $n_0 \in \mathbb{Z}$ such that

$$\inf_{n\in\mathbb{Z}} |y(n_0) - x(n)| > 0.$$

Let $\varepsilon = \inf_{n\in\mathbb{Z}} |y(n_0) - x(n)|$. Since $y \in AP(\mathbb{Z})$, $\exists N \in \mathbb{N}$ such that among any N consecutive integers there exists an integer p with the property that

$$|y(k+p) - y(k)| < \frac{\varepsilon}{2}, \quad \forall k \in \mathbb{Z}.$$

For each $n \in \mathbb{N}$, denote

$$E_n = \Big\{ p \in \mathbb{Z} \cap [-n - n_0, n - n_0] : |y(n_0 + p) - y(n_0)| < \frac{\varepsilon}{2} \Big\}.$$

Then $cardE_n \geq [\frac{2n}{N}] - 1$. Also, for all $p \in E_n$, we have

$$\begin{aligned} |z(n_0+p)| &= |x(n_0+p) - y(n_0+p)| \\ &\geq |x(n_0+p) - y(n_0)| - |y(n_0) - y(n_0+p)| \\ &\geq \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2}. \end{aligned}$$

Then, we obtain

$$\frac{1}{2n} \sum_{k=-n}^{n} |z(k)| \geq \frac{\varepsilon}{2} \cdot \frac{cardE_n}{2n} \geq \frac{\varepsilon}{2} \cdot \frac{[\frac{2n}{N}] - 1}{2n} \to \frac{\varepsilon}{2N} > 0, \quad (n \to \infty).$$

This contradicts with $z \in PAP_0(\mathbb{Z})$. ☐

Remark 2.8. By using Lemma 2.7, it is not difficult to show that $PAP(\mathbb{Z})$ is a Banach space equipped with the supremum norm.

Lemma 2.9. *Let $f : \mathbb{Z} \to \mathbb{R}$ be bounded. Then $f \in PAP_0(\mathbb{Z})$ if and only if $\forall \varepsilon > 0$,*

$$\lim_{n\to\infty} \frac{cardE_f(n,\varepsilon)}{2n} = 0,$$

where $E_f(n,\varepsilon) = \{k \in [-n,n] \bigcap \mathbb{Z} : |f(k)| \geq \varepsilon\}$.

Proof. "Necessity". Let $f \in PAP_0(\mathbb{Z})$. The conclusion follows from

$$\frac{1}{2n} \sum_{k=-n}^{n} |f(k)| \geq \frac{cardE_f(n,\varepsilon)}{2n} \cdot \varepsilon \geq 0.$$

"Sufficiency". Let $M = \sup_{n\in\mathbb{Z}} |f(n)|$. Then, $\forall \varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\frac{cardE_f(n,\varepsilon)}{2n} < \varepsilon, \quad n > N,$$

which yields that

$$\frac{1}{2n}\sum_{k=-n}^{n}|f(k)| \le M\varepsilon + \varepsilon = (M+1)\varepsilon, \quad n > N.$$

Thus,

$$\lim_{n\to\infty}\frac{1}{2n}\sum_{k=-n}^{n}|f(k)| = 0,$$

i.e., $f \in PAP_0(\mathbb{Z})$. □

Lemma 2.10. *Let $f \in PAP_0(\mathbb{Z}\times\Omega)$ and $K \subset \Omega$ be compact. If $f \in \mathcal{C}(\mathbb{Z}\times K)$, then $\widetilde{f} \in PAP_0(\mathbb{Z})$, where*

$$\widetilde{f}(k) = \sup_{x\in K}|f(k,x)|, \quad k \in \mathbb{Z}.$$

Proof. Since $f \in \mathcal{C}(\mathbb{Z}\times K)$, $\forall\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(k,u) - f(k,v)| < \varepsilon$$

for all $k \in \mathbb{Z}$ and $u, v \in K$ with $|u - v| < \delta$. Also, noting that K is compact, for the above $\delta > 0$, there exist $x_1, \ldots, x_m \in K$ such that

$$K \subset \bigcup_{i=1}^{m} B(x_i, \delta).$$

In addition, thanks to $f \in PAP_0(\mathbb{Z}\times\Omega)$, for the above $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$,

$$\frac{1}{2n}\sum_{k=-n}^{n}|f(k,x_i)| < \frac{\varepsilon}{m}, \quad i = 1, \ldots, m.$$

For each $x \in K$, there exists $i \in \{1, 2, \ldots, m\}$ such that $|x - x_i| < \delta$. Thus, we have

$$|f(k,x)| \le |f(k,x_i)| + |f(k,x_i) - f(k,x)| \le |f(k,x_i)| + \varepsilon, \quad k \in \mathbb{Z},$$

which gives that

$$\widetilde{f}(k) = \sup_{x\in K}|f(k,x)| \le \sum_{i=1}^{m}|f(k,x_i)| + \varepsilon, \quad k \in \mathbb{Z}.$$

Thus, $\widetilde{f}$ is bounded, and we have

$$\frac{1}{2n}\sum_{k=-n}^{n}|\widetilde{f}(k)| \le \sum_{i=1}^{m}\left(\frac{1}{2n}\sum_{k=-n}^{n}|f(k,x_i)|\right) + \varepsilon < 2\varepsilon, \quad n > N,$$

which means that $\widetilde{f} \in PAP_0(\mathbb{Z})$. □

Now, we are ready to establish a composition theorem of pseudo almost periodic sequences.

Lemma 2.11. *Let $f \in PAP(\mathbb{Z} \times \mathbb{R}^+)$, $x \in PAP(\mathbb{Z})$ with $x(n) \geq 0$ for each $n \in \mathbb{Z}$, and $K = \overline{\{x(n) : n \in \mathbb{Z}\}}$. If $f \in \mathcal{C}(\mathbb{Z} \times K)$, then $f(\cdot, x(\cdot)) \in PAP(\mathbb{Z})$.*

Proof. Let

$$f = g + h, \quad x = y + z$$

where $g \in AP(\mathbb{Z} \times \mathbb{R}^+)$, $h \in PAP_0(\mathbb{Z} \times \mathbb{R}^+)$, $y \in AP(\mathbb{Z})$ and $z \in PAP_0(\mathbb{Z})$. We denote $f(n, x(n)) = I_1(n) + I_2(n) + I_3(n),\ n \in \mathbb{Z}$, where

$$I_1(n) = g(n, y(n)), \quad I_2(n) = g(n, x(n)) - g(n, y(n)), \quad I_3(n) = h(n, x(n)).$$

By Lemma 2.7, $\{y(n) : n \in \mathbb{Z}\} \subset K$. Then, it follows from Lemma 2.1 that $I_1 \in AP(\mathbb{Z})$. In addition, by Lemma 2.1, we know that $g \in \mathcal{C}(\mathbb{Z} \times K)$, and thus $h \in \mathcal{C}(\mathbb{Z} \times K)$. Then, Lemma 2.10 yields that

$$\lim_{n\to\infty} \frac{1}{2n} \sum_{k=-n}^{n} \sup_{x\in K} |h(k, x)| = 0.$$

Combining this with

$$|I_3(k)| = |h(k, x(k))| \leq \sup_{x\in K} |h(k, x)|, \quad k \in \mathbb{Z},$$

we conclude $I_3 \in PAP_0(\mathbb{Z})$. Next, let us to show that $I_2 \in PAP_0(\mathbb{Z})$. It is easy to see that I_2 is bounded. Since $\forall\varepsilon > 0$, $\exists\delta > 0$ such that

$$|g(k, u) - g(k, v)| < \varepsilon$$

for all $k \in \mathbb{Z}$ and $u, v \in K$ with $|u - v| < \delta$, we conclude that

$$E_{I_2}(n, \varepsilon) \subset E_z(n, \delta).$$

By Lemma 2.9, we have

$$\lim_{n\to\infty} \frac{card E_z(n, \delta)}{2n} = 0,$$

which gives that

$$\lim_{n\to\infty} \frac{card E_{I_2}(n, \varepsilon)}{2n} = 0.$$

Again by Lemma 2.9, $I_2 \in PAP_0(\mathbb{Z})$. □

Lemma 2.12. *Let $f \in PAP(\mathbb{Z})$ and $k \in PAP(\mathbb{Z})$ with $k(n) \in \mathbb{Z}^+$ for all $n \in \mathbb{Z}$. Then $\widetilde{f} \in PAP(\mathbb{Z})$, where*

$$\widetilde{f}(n) = \sum_{j=n-k(n)}^{n} f(j), \quad \forall n \in \mathbb{Z}.$$

Proof. Let $f = f_1 + f_2$ and $k = k_1 + k_2$, where $f_1, k_1 \in AP(\mathbb{Z})$, $f_2, k_2 \in PAP_0(\mathbb{Z})$. We denote

$$\widetilde{f}(n) = J_1(n) + J_2(n) + J_3(n), \quad n \in \mathbb{Z},$$

where

$$J_1(n) = \sum_{j=n-k_1(n)}^{n} f_1(j), \quad J_3(n) = \sum_{j=n-k(n)}^{n} f_2(j),$$

and

$$J_2(n) = \begin{cases} \sum\limits_{j=n-k(n)}^{j=n-k_1(n)-1} f_1(j)\ , & k_2(n) > 0, \\ 0 \qquad\qquad , & k_2(n) = 0, \\ \sum\limits_{j=n-k_1(n)}^{j=n-k(n)-1} -f_1(j), & k_2(n) < 0. \end{cases}$$

By Lemma 2.2, $J_1 \in AP(\mathbb{Z})$. Since $|J_2(n)| \leq \|f_1\|_{AP(\mathbb{Z})} \cdot |k_2(n)|$ for all $n \in \mathbb{Z}$, we get $J_2 \in PAP_0(\mathbb{Z})$. Next, let us show that $J_3 \in PAP_0(\mathbb{Z})$. Let $q = \sup\limits_{n\in\mathbb{Z}} |k(n)|$. We have

$$\begin{aligned} \frac{1}{2m}\sum_{n=-m}^{m} |J_3(n)| &= \frac{1}{2m}\sum_{n=-m}^{m} \left| \sum_{j=n-k(n)}^{n} f_2(j) \right| \\ &\leq \frac{1}{2m}\sum_{n=-m}^{m} \sum_{j=n-q}^{n} |f_2(j)| \\ &= \frac{1}{2m}\sum_{n=-m}^{m} \sum_{j=-q}^{0} |f_2(n+j)| \\ &= \sum_{j=-q}^{0} \frac{1}{2m}\sum_{n=-m}^{m} |f_2(n+j)| \\ &= \sum_{j=-q}^{0} \frac{1}{2m}\sum_{n=j-m}^{j+m} |f_2(n)| \\ &\leq \sum_{j=-q}^{0} \frac{1}{2m}\sum_{n=-(m-j)}^{m-j} |f_2(n)|, \quad m \in \mathbb{N}. \end{aligned}$$

This together with $f_2 \in PAP_0(\mathbb{Z})$, we get

$$\lim_{m\to\infty} \frac{1}{2m}\sum_{n=-m}^{m} |J_3(n)| = 0.$$

□

Theorem 2.13. *Suppose that*

(H1") $f_i, g_i \in PAP(\mathbb{Z}\times\mathbb{R}^+)$ are nonnegative functions $(i = 1, 2, ..., m)$, $k \in PAP(\mathbb{Z})$ with $k(n) \in \mathbb{Z}^+$ for all $n \in \mathbb{Z}$, and $h : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous; moreover, for each $i \in \{1, 2, ..., m\}$ and compact subset $K \subset \mathbb{R}^+$, $f_i, g_i \in \mathcal{C}(\mathbb{Z}\times K)$.

and (H2)-(H4) hold. Then equation (1.1) *has a unique pseudo almost periodic solution with a positive infimum.*

Proof. By using Lemma 2.11 and Lemma 2.12, similar to the proof of Theorem 2.3, one can get the conclusion. □

Next, we give a simple example, which does not aim at generality but illustrate how our results can be used.

Example 2.14. Let $m = 1$,

$$f_1(n, x) = (1 + \sin^2 n\pi + sin^2 n)\sqrt{\ln(x + 1)}, \quad g_1(n, x) \equiv \frac{1}{\sqrt{x + 1}}$$

and

$$h(x) = e^{-x}, \quad k(n) = \begin{cases} 1, & n \text{ is odd,} \\ 2, & n \text{ is even.} \end{cases}$$

Let

$$\varphi_1(\alpha, x) = \sqrt{\frac{\ln(\alpha x + 1)}{\ln(x + 1)}}, \quad \psi_1(\alpha, x) \equiv \sqrt{\alpha}.$$

Then, it is not difficult to verify that (H1)-(H3) are satisfied. In addition, we have

$$\inf_{n\in\mathbb{Z}} \sum_{j=n-k(n)}^{n} f_1(j, \frac{1}{100})g_1(j, 99) \geq \frac{\sqrt{\ln\frac{101}{100}}}{5} \geq \frac{1}{100}$$

and

$$\sup_{n\in\mathbb{Z}} \sum_{j=n-k(n)}^{n} f_1(j, 99)g_1(j, \frac{1}{100}) + h(99) \leq 9\sqrt{\ln 100} + e^{-99} \leq 99,$$

which means that (H4) holds with $c = \frac{1}{100}$ and $d = 99$. Then, by Theorem 2.3, the following equation

$$x(n) = e^{-x(n)} + \sum_{j=n-k(n)}^{n} (1 + \sin^2 \pi j + sin^2 j)\frac{\sqrt{\ln[x(j) + 1]}}{\sqrt{x(j) + 1}}$$

has a unique almost periodic solution with a positive infimum.

References

[1] H. L. Smith, On periodic solutions of a delay integral equation modelling epidemics, Journal of Mathematical Biology 4 (1977), 69–80.

[2] A. M. Fink, J. A. Gatica, Positive almost periodic solutions of some delay integral equations, J. Differential Equations 83 (1990), 166–178.

[3] K. Ezzinbi, M. A. Hachimi, Existence of positive almost periodic solutions of functional equations via Hilbert's projective metric, Nonlinear Anal. TMA 26 (1996), 1169–1176.

[4] E. Ait Dads, K. Ezzinbi, Almost periodic solution for some neutral nonlinear integral equation, Nonlinear Anal. TMA 28 (1997), 1479–1489.

[5] E. Ait Dads, K. Ezzinbi, Existence of positive pseudo-almost-periodic solution for some nonlinear infinite delay integral equations arising in epidemic problems, Nonlinear Anal. TMA 41 (2000), 1–13.

[6] E. Ait Dads, P. Cieutat, L. Lhachimi, Positive almost automorphic solutions for some nonlinear infinite delay integral equations, Dynamic Systems and Applications 17 (2008), 515–538.

[7] E. Ait Dads, P. Cieutat, L. Lhachimi, Positive pseudo almost periodic solutions for some nonlinear infinite delay integral equations, Mathematical and Computer Modelling 49 (2009), 721–739.

[8] H. S. Ding, J. Liang, G. M. N'Guérékata, T. J. Xiao, Existence of positive almost automorphic solutions to neutral nonlinear integral equations, Nonlinear Anal. TMA 69 (2008), 1188–1199.

[9] H. S. Ding, T. J. Xiao, J. Liang, Existence of positive almost automorphic solutions to nonlinear delay integral equations, Nonlinear Anal. TMA 70 (2009), 2216–2231.

[10] H. S. Ding, J. Liang, T. J. Xiao, Positive almost automorphic solutions for a class of nonlinear delay integral equations, Applicable Analysis 88 (2009), 231–242.

[11] J. Blot, D. Pennequin, Existence and structure results on almost periodic solutions of difference equations, J. Differ. Equations Appl. 7 (2001), 383–402.

[12] D. Pennequin, Existence of almost periodic solutions of discrete time equations, Discrete Contin. Dynam. Systems 7 (2001), 51–60.

[13] D. Araya, R. Castro, C. Lizama, Almost automorphic solutions of difference equations. Adv. Difference Equ. 2009 (2009), Art. ID 591380, 15 pp.

[14] C. Cuevas, H. Henríquez, C. Lizama, On the existence of almost automorphic solutions of volterra difference equations, submitted. (http://netlizama.usach.cl/Cuevas-Henriquez-Lizama(Final)(DCDS)(2010).pdf)

[15] Nguyen Van Minh, On the asymptotic behavior of volterra difference equations, preprint. (http://arxiv.org/abs/1010.5454)

[16] Y. Hamaya, Existence of an almost periodic solution in a difference equation with infinite delay, J. Differ. Equations Appl., 9 (2003), 227–237.

[17] R. P. Agarwal, D. O'Regan, P. J. Y. Wong, Constant-sign periodic and almost periodic solutions of a system of difference equations, Compu. Math. Appl. 50 (2005), 1725–1754.

[18] Y. Song, Positive almost periodic solutions of nonlinear discrete systems with finite delay, Compu. Math. Appl. 58 (2009), 128–134.

[19] A. M. Fink, Almost Periodic Differential Equations, Lecture Notes in Mathematics, Springer-Verlag, New York-Berlin, 1974.

[20] C. Corduneanu, Almost periodic functions, 2nd edition, Chelsea, New york, 1989.

[21] C. Zhang, Almost Periodic Type Functions and Ergodicity, Kluwer Academic Publishers, Dordrecht/Boston/London, 2003.

[22] H. S. Ding, J. Liang, T. J. Xiao, Fixed point theorems for nonlinear operators with and without monotonicity in partially ordered Banach spaces, Fixed Point Theory and Applications, Volume 2010 (2010), Article ID 108343, 11 pages.

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