

Existence of positive solutions for singular impulsive differential equations with integral boundary conditions on an infinite interval in Banach spaces*

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Abstract In this paper, the Mönch fixed point theorem is used to investigate the existence of positive solutions for the second-order boundary value problem with integral boundary conditions of nonlinear impulsive differential equations on an infinite interval in a Banach space.

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1 Introduction

The theory of boundary-value problems with integral boundary conditions for ordinary differential equations arises in different areas of applied mathematics and physics. For example, heat conduction, chemical engineering, underground water flow, thermo-elasticity, and plasma physics can be reduced to the nonlocal problems with integral boundary conditions. In recent years, the theory of ordinary differential equations in Banach space has become a new important branch of investigation (see, for example, [1-4] and references therein). In a recent paper [7], using the cone theory and monotone iterative technique, Zhang et al investigated the existence of minimal nonnegative solution of the following nonlocal boundary value problems for second-order nonlinear impulsive differential equations on an infinite interval with an infinite number of impulsive times

$$\begin{cases} -x''(t) = f(t, x(t), x'(t)), & t \in J, t \neq t_k, \\ \Delta x|_{t=t_k} = I_k(x(t_k)), & k = 1, 2, \dots, \\ \Delta x'|_{t=t_k} = \bar{I}_k(x(t_k)), & k = 1, 2, \dots, \\ x(0) = \int_0^\infty g(t)x(t)dt, & x'(\infty) = 0, \end{cases}$$

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where $J = [0, +\infty)$, $f \in C(J \times R^+ \times R^+, R^+)$, $R^+ = [0, +\infty]$, $0 < t_1 < t_2 < \dots < t_k < \dots$, $t_k \rightarrow \infty$, $I_k \in C[R^+, R^+]$, $\bar{I}_k \in C[R^+, R^+]$, $g(t) \in C[R^+, R^+]$, with $\int_0^\infty g(t)dt < 1$.

Very recently, by using Schauder fixed point theorem, Guo [6] obtained the existence of positive solutions for a class of n th-order nonlinear impulsive singular integro-differential equations in a Banach space. Motivated by Guo's work, in this paper, we shall use the cone theory and the Mönch fixed point theorem to investigate the positive solutions for a class of second-order nonlinear impulsive integro-differential equations in a Banach space.

Consider the following boundary value problem with integral boundary conditions for second-order nonlinear impulsive integro-differential equation of mixed type in a real Banach space E :

$$\begin{cases} -x''(t) = f(t, x(t), x'(t)), & t \in J, t \neq t_k, \\ \Delta x|_{t=t_k} = I_{0k}(x(t_k), x'(t_k)), \\ \Delta x'|_{t=t_k} = I_{1k}(x(t_k), x'(t_k)), & k = 1, 2, \dots, \\ x(0) = \int_0^\infty g(t)x(t)dt, & x'(\infty) = x_\infty, \end{cases} \quad (1)$$

where $J = [0, \infty)$, $J_+ = (0, \infty)$, $0 < t_1 < t_2 < \dots < t_k < \dots, t_k \rightarrow \infty$, $J_k = (t_k, t_{k+1}]$ ($k = 1, 2, \dots$), $J'_+ = J_+ \setminus \{t_1, \dots, t_k, \dots\}$, f may be singular at $t = 0$ and $x = \theta$ or $x' = \theta$. I_{0k} and I_{1k} may be singular at $x = \theta$ or $x' = \theta$, θ is the zero element of E , $g(t) \in L[0, \infty)$ with $\int_0^\infty g(t)dt < 1$, $\int_0^\infty tg(t)dt < \infty$, $x(\infty) = \lim_{t \rightarrow \infty} x'(t)$, $x_\infty \geq x_0^*$, $x_0^* \in P_+$, P_+ is the same as that defined in Section 2. $\Delta x|_{t=t_k}$ denotes the jump of $x(t)$ at $t = t_k$, i.e., $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-)$, where $x(t_k^+)$, $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k$ respectively. $\Delta x'|_{t=t_k}$ has a similar meaning for $x'(t)$.

The main features of the present paper are as follows: Firstly, compared with [7], the second-order boundary value problem we discussed here is in Banach spaces and nonlinear term permits singularity not only at $t = 0$ but also at $x, x' = \theta$. Secondly, compared with [6], the relative compact conditions we used are weaker.

2 Preliminaries and several lemmas

Let $PC[J, E] = \{x|x(t) : J \rightarrow E, x \text{ is continuous at } t \neq t_k \text{ and left continuous at } t = t_k, x(t_k^+) \text{ exists, } k = 1, 2, \dots\}$. $PC^1[J, E] = \{x|x \in PC[J, E], x'(t) \text{ exists at } t \neq t_k \text{ and } x'(t_k^+), x'(t_k^-) \text{ exist } k = 1, 2, \dots\}$.

$$FPC[J, E] = \left\{ x \in PC[J, E] : \sup_{t \in J} \frac{\|x(t)\|}{t+1} < +\infty \right\},$$

$$DPC^1[J, E] = \left\{ x \in PC^1[J, E] : \sup_{t \in J} \frac{\|x(t)\|}{t+1} < +\infty, \text{ and } \sup_{t \in J} \|x'(t)\| < +\infty \right\}.$$

Obviously, $FPC[J, E]$ is a Banach space with norm

$$\|x\|_F = \sup_{t \in J} \frac{\|x(t)\|}{t+1}.$$

and $DPC^1[J, E]$ is also a Banach space with norm

$$\|x\|_D = \max\{\|x\|_F, \|x'\|_1\},$$

where

$$\|x'\|_1 = \sup_{t \in J} \|x'(t)\|.$$

The basic space using in this paper is $DPC^1[J, E]$.

Let P be a normal cone in E with normal constant N which defines a partial ordering in E by $x \leq y$. If $x \leq y$ and $x \neq y$, we write $x < y$. Let $P_+ = P \setminus \{\theta\}$. So, $x \in P_+$ if and only if $x > \theta$. For details on cone theory, see [4].

Let $P_{0\lambda} = \{x \in P : x \geq \lambda x_0^*\}$, ($\lambda > 0$). Obviously, $P_{0\lambda} \subset P_+$ for any $\lambda > 0$. When $\lambda = 1$, we write $P_0 = P_{01}$, i.e. $P_0 = \{x \in P : x \geq x_0^*\}$. Let $P(F) = \{x \in FPC[J, E] : x(t) \geq \theta, \forall t \in J\}$, and $P(D) = \{x \in DPC^1[J, E] : x(t) \geq \theta, x'(t) \geq \theta, \forall t \in J\}$. It is clear, $P(F)$, $P(D)$ are cones in $FPC[J, E]$ and $DPC^1[J, E]$, respectively. A map $x \in DPC^1[J, E] \cap C^2[J_+, E]$ is called a positive solution of BVP (1) if $x \in P(D)$ and $x(t)$ satisfies BVP (1).

Let $\alpha, \alpha_F, \alpha_D$ denote the Kuratowski measure of non-compactness in $E, FPC[J, E], DPC^1[J, E]$. For details on the definition and properties of the measure of non-compactness, the reader is referred to references [1-4].

Denote

$$\lambda^* = \min \left\{ \frac{\int_0^\infty tg(t)dt}{1 - \int_0^\infty g(t)dt}, 1 \right\}.$$

Let us list the following assumptions, which will stand throughout this paper.

(H₁) $f \in C[J_+ \times P_{0\lambda} \times P_{0\lambda}, P]$ for any $\lambda > 0$ and there exist $a, b, c \in L[J_+, J]$ and $z \in C[J_+ \times J_+, J]$ such that

$$\|f(t, x, y)\| \leq a(t) + b(t)z(\|x\|, \|y\|), \quad \forall t \in J_+, x \in P_{0\lambda^*}, y \in P_{0\lambda^*}$$

and

$$\frac{\|f(t, x, y)\|}{c(t)(\|x\| + \|y\|)} \rightarrow 0, \quad \text{as } x \in P_{0\lambda^*}, y \in P_{0\lambda^*}, \|x\| + \|y\| \rightarrow \infty,$$

uniformly for $t \in J_+$, and

$$\int_0^\infty a(t)dt = a^* < \infty, \quad \int_0^\infty b(t)dt = b^* < \infty, \quad \int_0^\infty c(t)(1+t)dt = c^* < \infty.$$

(H₂) $I_{ik} \in C[P_{0\lambda} \times P_{0\lambda}, P]$ for any $\lambda > 0$ and there exist $F_i \in L[J_+ \times J_+, J_+]$ and constants η_{ik}, γ_{ik} , ($i = 0, 1, k = 1, 2, \dots$) such that

$$\|I_{ik}(x, y)\| \leq \eta_{ik}F_i(\|x\|, \|y\|), \quad x \in P_{0\lambda^*}, y \in P_{0\lambda^*} \quad (i = 0, 1),$$

and

$$\frac{\|I_{ik}(t, x, y)\|}{\gamma_{ik}(\|x\| + \|y\|)} \rightarrow 0, \quad \text{as } x \in P_{0\lambda^*}, y \in P_{0\lambda^*}, \|x\| + \|y\| \rightarrow \infty,$$

uniformly for ($i = 0, 1, k = 1, 2, \dots$), here

$$0 < \eta_i^* = \sum_{k=1}^{\infty} \eta_{ik} < \infty, \quad 0 < \gamma_i^* = \sum_{k=1}^{\infty} \gamma_{ik}(1+t_k) < \infty.$$

(H₃) For any $t \in J_+$, $R > 0$ and countable bounded set $V_i \subset DPC^1[J, P_{0\lambda^*R}]$ ($i = 0, 1$), there exist $h_i(t) \in L[J, J]$ ($i = 0, 1$) and positive constants m_{ikj} ($i, j = 0, 1, k = 1, 2, \dots$) such that

$$\alpha(f(t, V_0(t), V_1(t))) \leq \sum_{i=0}^1 h_i(t)\alpha(V_i(t)), \quad \alpha(I_{ik}(V_0(t), V_1(t))) \leq \sum_{j=0}^1 m_{ikj}\alpha(V_j(t)),$$

$$h^* = \int_0^\infty h_0(t)(1+t) + h_1(t)dt < \infty, \quad m^* = \sum_{k=1}^\infty \sum_{i=0}^1 (m_{ik0}(1+t_k) + m_{ik1}) < \infty,$$

where

$$P_{0\lambda^*R} = \{x \in P : x \geq \lambda^*x_0^*, \|x\| < R\}.$$

(H₄) $t \in J_+$, $\lambda^*x_0^* \leq x_i \leq \bar{x}_i$ ($i = 0, 1$), imply $f(t, x_0, x_1) \leq f(t, \bar{x}_0, \bar{x}_1)$.

In what follows, we write $Q = \{x \in DPC^1[J, P] : x^{(i)}(t) \geq \lambda^*x_0^*, \forall t \in J, i = 0, 1\}$. Evidently, Q is a closed convex set in $DPC^1[J, E]$. We shall reduce BVP (1) to an impulsive integral equations in E . To this end, we first consider operator A defined by

$$\begin{aligned} (Ax)(t) &= \frac{1}{1 - \int_0^\infty g(t)dt} \left\{ x_\infty \int_0^\infty tg(t)dt + \int_0^\infty g(t) \left[\int_0^\infty G(t, s)f(s, x(s), x'(s))ds \right. \right. \\ &\quad \left. \left. + \sum_{k=1}^\infty G(t, t_k)I_{1k}(x(t_k), x'(t_k)) + \sum_{k=1}^\infty G'_s(t, t_k)I_{0k}(x(t_k), x'(t_k)) \right] dt \right\} + tx_\infty \\ &\quad + \int_0^\infty G(t, s)f(s, x(s), x'(s))ds + \sum_{k=1}^\infty G(t, t_k)I_{1k}(x(t_k), x'(t_k)) \\ &\quad + \sum_{k=1}^\infty G'_s(t, t_k)I_{0k}(x(t_k), x'(t_k)), \end{aligned} \quad (2)$$

where

$$G(t, s) = \begin{cases} t, & 0 \leq t \leq s < +\infty, \\ s, & 0 \leq s \leq t < +\infty, \end{cases} \quad G'_s(t, s) = \begin{cases} 0, & 0 \leq t \leq s < +\infty, \\ 1, & 0 \leq s \leq t < +\infty. \end{cases}$$

Lemma 1 *If conditions (H₁) – (H₂) are satisfied, then operator A defined by (2) is a continuous operator from Q into Q .*

Proof. Let

$$\varepsilon_0 = \frac{1}{8c^* \left(1 + \frac{\int_0^\infty g(t)dt}{1 - \int_0^\infty g(t)dt} \right)}, \quad (3)$$

and

$$r = \frac{\lambda^*\|x_0^*\|}{N} > 0. \quad (4)$$

By (H₁), there exists a $R > r$ such that

$$\|f(t, x, y)\| \leq \varepsilon_0 c(t)(\|x\| + \|y\|), \quad \forall t \in J_+, x \in P_{0\lambda^*}, y \in P_{0\lambda^*}, \|x\| + \|y\| > R,$$

and

$$\|f(t, x, y)\| \leq a(t) + Mb(t), \quad \forall t \in J_+, \quad x \in P_{0\lambda^*}, \quad y \in P_{0\lambda^*}, \quad \|x\| + \|y\| \leq R,$$

where

$$M = \max\{z(u_0, u_1) : r \leq u_i \leq R \ (i = 0, 1)\}.$$

Hence

$$\|f(t, x, y)\| \leq \varepsilon_0 c(t)(\|x\| + \|y\|) + a(t) + Mb(t), \quad \forall t \in J_+, \quad x \in P_{0\lambda^*}, \quad y \in P_{0\lambda^*}. \quad (5)$$

On the other hand, let

$$\bar{\varepsilon}_i = \frac{1}{8\gamma_i^* \left(1 + \frac{\int_0^\infty g(t)dt}{1 - \int_0^\infty g(t)dt}\right)} \quad (i = 0, 1). \quad (6)$$

We see that, by condition (H₂), there exists a $R_1 > r$ such that

$$\|I_{ik}(x, y)\| \leq \bar{\varepsilon}_i \gamma_{ik}(\|x\| + \|y\|), \quad \forall x \in P_{0\lambda^*}, \quad y \in P_{0\lambda^*}, \quad \|x\| + \|y\| > R_1 \quad (i = 0, 1, k = 1, 2, \dots),$$

and

$$\|I_{ik}(x, y)\| \leq \eta_{ik} M_1, \quad \forall x \in P_{0\lambda^*}, \quad y \in P_{0\lambda^*}, \quad \|x\| + \|y\| \leq R_1 \quad (i = 0, 1, k = 1, 2, \dots),$$

where

$$M_1 = \max\{F_i(u_0, u_1) : r \leq u_i \leq R \ (i = 0, 1)\}.$$

Hence

$$\|I_{ik}(x, y)\| \leq \bar{\varepsilon}_i \gamma_{ik}(\|x\| + \|y\|) + \eta_{ik} M_1, \quad \forall x \in P_{0\lambda^*}, \quad y \in P_{0\lambda^*}, \quad i = 0, 1, \quad k = 1, 2, \dots \quad (7)$$

Let $x \in Q$, by (5), we can get

$$\begin{aligned} \|f(t, x(t), x'(t))\| &\leq \varepsilon_0 c(t)(1+t) \left(\frac{\|x(t)\|}{t+1} + \frac{\|x'(t)\|}{t+1} \right) + a(t) + Mb(t) \\ &\leq \varepsilon_0 c(t)(1+t)(\|x\|_F + \|x'\|_1) + a(t) + Mb(t) \\ &\leq 2\varepsilon_0 c(t)(1+t)\|x\|_D + a(t) + Mb(t), \quad \forall t \in J_+, \end{aligned} \quad (8)$$

which together with condition (H₁) implies the convergence of the infinite integral

$$\int_0^\infty \|f(s, x(s), x'(s))\| ds. \quad (9)$$

On the other hand, by (7), we have

$$\begin{aligned} \|I_{ik}(x(t_k), x'(t_k))\| &\leq \bar{\varepsilon}_i \gamma_{ik}(1+t_k) \left(\frac{\|x(t_k)\|}{t_k+1} + \frac{\|x'(t_k)\|}{t_k+1} \right) + \eta_{ik} M_1 \\ &\leq \bar{\varepsilon}_i \gamma_{ik}(1+t_k)(\|x\|_F + \|x'\|_1) + \eta_{ik} M_1 \\ &\leq 2\bar{\varepsilon}_i \gamma_{ik}(1+t_k)\|x\|_D + \eta_{ik} M_1 \quad (i = 0, 1), \end{aligned} \quad (10)$$

which together with (2), (H₁) and (H₂) implies that

$$\begin{aligned} \|(Ax)(t)\| &\leq \frac{1}{1 - \int_0^\infty g(t)dt} \left\{ \|x_\infty\| \int_0^\infty tg(t)dt + \int_0^\infty g(t) \left[\int_0^\infty G(t,s) \|f(s, x(s), x'(s))\| ds \right. \right. \\ &+ \left. \sum_{k=1}^\infty G(t, t_k) \|I_{1k}(x(t_k), x'(t_k))\| + \sum_{k=1}^\infty G'_s(t, t_k) \|I_{0k}(x(t_k), x'(t_k))\| \right] dt \Big\} \\ &+ t \|x_\infty\| + \int_0^\infty G(t,s) \|f(s, x(s), x'(s))\| ds + \sum_{k=1}^\infty G(t, t_k) \|I_{1k}(x(t_k), x'(t_k))\| \quad (11) \\ &+ \sum_{k=1}^\infty G'_s(t, t_k) \|I_{0k}(x(t_k), x'(t_k))\|. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\|(Ax)(t)\|}{1+t} &\leq \frac{1}{1 - \int_0^\infty g(t)dt} \left\{ \|x_\infty\| \int_0^\infty tg(t)dt + \int_0^\infty g(t) \left[\int_0^\infty \|f(s, x(s), x'(s))\| ds \right. \right. \\ &+ \left. \sum_{k=1}^\infty \|I_{1k}(x(t_k), x'(t_k))\| + \sum_{k=1}^\infty \|I_{0k}(x(t_k), x'(t_k))\| \right] dt \Big\} + \|x_\infty\| \\ &+ \int_0^\infty \|f(s, x(s), x'(s))\| ds + \sum_{k=1}^\infty \|I_{1k}(x(t_k), x'(t_k))\| \\ &+ \sum_{k=1}^\infty \|I_{0k}(x(t_k), x'(t_k))\| \quad (12) \\ &\leq \left(1 + \frac{\int_0^\infty g(t)dt}{1 - \int_0^\infty g(t)dt}\right) [2\varepsilon_0 c^* \|x\|_D + a^* + Mb^*] + \left(1 + \frac{\int_0^\infty tg(t)dt}{1 - \int_0^\infty g(t)dt}\right) \|x_\infty\| \\ &+ \left(1 + \frac{\int_0^\infty g(t)dt}{1 - \int_0^\infty g(t)dt}\right) \sum_{i=0}^1 \sum_{k=1}^\infty \|I_{ik}(x(t_k), x'(t_k))\| \\ &\leq \frac{1}{2} \|x\|_D + \left(1 + \frac{\int_0^\infty g(t)dt}{1 - \int_0^\infty g(t)dt}\right) (a^* + Mb^* + \eta_0^* M_1 + \eta_1^* M_1) \\ &+ \left(1 + \frac{\int_0^\infty tg(t)dt}{1 - \int_0^\infty g(t)dt}\right) \|x_\infty\|. \end{aligned}$$

Differentiating (2), we get

$$(A'x)(t) = \int_t^\infty f(s, x(s), x'(s)) ds + \sum_{t_k \geq t} I_{1k}(x(t_k), x'(t_k)) + x_\infty. \quad (13)$$

Hence,

$$\begin{aligned} \|(A'x)(t)\| &\leq \int_0^\infty \|f(s, x(s), x'(s))\| ds + \|x_\infty\| + \sum_{k=1}^\infty \|I_{1k}(x(t_k), x'(t_k))\| \\ &\leq 2\varepsilon_0 c^* \|x\|_D + a^* + Mb^* + \|x_\infty\| + 2\bar{\varepsilon}_1 \gamma_1^* \|x\|_D + \eta_1^* M_1 \quad (14) \\ &\leq \frac{1}{2} \|x\|_D + a^* + Mb^* + \|x_\infty\| + \eta_1^* M_1, \quad \forall t \in J. \end{aligned}$$

It follows from (12) and (14) that

$$\begin{aligned} \|Ax\|_D &\leq \frac{1}{2} \|x\|_D + \left(1 + \frac{\int_0^\infty g(t)dt}{1 - \int_0^\infty g(t)dt}\right) (a^* + Mb^* + \eta_0^* M_1 + \eta_1^* M_1) \\ &+ \left(1 + \frac{\int_0^\infty tg(t)dt}{1 - \int_0^\infty g(t)dt}\right) \|x_\infty\|. \quad (15) \end{aligned}$$

So, $Ax \in DPC^1[J, E]$. On the other hand, it can be easily seen that

$$(Ax)(t) \geq \left(\frac{\int_0^\infty g(t)dt}{1 - \int_0^\infty g(t)dt} \right) x_\infty \geq \lambda^* x_\infty \geq \lambda^* x_0^*, \quad \forall t \in J,$$

$$(A'x)(t) \geq x_\infty \geq \lambda^* x_\infty \geq \lambda^* x_0^*, \quad \forall t \in J.$$

Hence, $Ax \in Q$. Thus, we have proved that A maps Q into Q and (15) holds.

Finally, we show that A is continuous. Let $(x_m, \bar{x}) \in Q$, $\|x_m - \bar{x}\|_D \rightarrow 0$ ($m \rightarrow \infty$). Then $\{x_m\}$ is a bounded subset of Q . Thus, there exists $r > 0$ such that $\|x_m\|_D < r$ for $m \geq 1$ and $\|\bar{x}\|_D \leq r$. Similar to (12) and (14), it is easy to get

$$\begin{aligned} \|Ax_m - A\bar{x}\|_D &\leq \left(1 + \frac{\int_0^\infty g(t)dt}{1 - \int_0^\infty g(t)dt} \right) \int_0^\infty \|f(s, x_m(s), x'_m(s)) - f(s, \bar{x}(s), \bar{x}'(s))\| ds \\ &+ \left(1 + \frac{\int_0^\infty g(t)dt}{1 - \int_0^\infty g(t)dt} \right) \left(\sum_{i=0}^1 \sum_{k=1}^\infty \|I_{ik}(x_m(t_k), x'_m(t_k)) - I_{ik}(\bar{x}(t_k), \bar{x}'(t_k))\| \right). \end{aligned} \quad (16)$$

It is clear that,

$$f(t, x_m(t), x'_m(t)) \rightarrow f(t, \bar{x}(t), \bar{x}'(t)) \text{ as } m \rightarrow \infty, \quad \forall t \in J_+. \quad (17)$$

By (8), we get

$$\begin{aligned} \|f(t, x_m(t), x'_m(t)) - f(t, \bar{x}(t), \bar{x}'(t))\| &\leq 4\varepsilon_0 c(t)(1+t)r + 2a(t) + 2Mb(t) \\ &= \sigma(t) \in L[J, J], \quad m = 1, 2, 3, \dots, \quad \forall t \in J_+. \end{aligned} \quad (18)$$

It follows from (17), (18) and the dominated convergence theorem that

$$\lim_{m \rightarrow \infty} \int_0^\infty \|f(s, x_m(s), x'_m(s)) - f(s, \bar{x}(s), \bar{x}'(s))\| ds = 0. \quad (19)$$

It is clear that,

$$I_{ik}(x_m(t_k), x'_m(t_k)) \rightarrow I_{ik}(\bar{x}(t_k), \bar{x}'(t_k)), \text{ as } m \rightarrow \infty, \quad i = 0, 1, \quad k = 1, 2, \dots \quad (20)$$

So,

$$\lim_{m \rightarrow \infty} \left(\sum_{i=0}^1 \sum_{k=1}^\infty \|I_{ik}(x_m(t_k), x'_m(t_k)) - I_{ik}(\bar{x}(t_k), \bar{x}'(t_k))\| \right) = 0. \quad (21)$$

It follows from (16), (19) and (21) that $\|Ax_m - A\bar{x}\|_D \rightarrow 0$ as $m \rightarrow \infty$. Therefore, the continuity of A is proved.

Lemma 2 *If condition (H₁) and (H₂) are satisfied, then $x \in Q \cap C^2[J'_+, E]$ is a solution of BVP (1) if and only if $x \in Q$ is a solution of the following impulsive integral equation:*

$$\begin{aligned}
x(t) &= \frac{1}{1 - \int_0^\infty g(t)dt} \left\{ x_\infty \int_0^\infty tg(t)dt + \int_0^\infty g(t) \left[\int_0^\infty G(t,s)f(s,x(s),x'(s))ds \right. \right. \\
&+ \sum_{k=1}^\infty G(t,t_k)I_{1k}(x(t_k),x'(t_k)) + \sum_{k=1}^\infty G'_s(t,t_k)I_{0k}(x(t_k),x'(t_k)) \left. \left. \right] dt \right\} + tx_\infty \quad (22) \\
&+ \int_0^\infty G(t,s)f(s,x(s),x'(s))ds + \sum_{k=1}^\infty G(t,t_k)I_{1k}(x(t_k),x'(t_k)) \\
&+ \sum_{k=1}^\infty G'_s(t,t_k)I_{0k}(x(t_k),x'(t_k)).
\end{aligned}$$

Proof. Suppose that $x \in Q \cap C^2[J'_+, E]$ is a solution of BVP (1). For $t \in J$, integrating (1) from 0 to t , we have

$$-x'(t) + x'(0) = \int_0^t f(s, x(s), x'(s))ds + \sum_{t_k < t} I_{1k}(x(t_k), x'(t_k)). \quad (23)$$

Taking limit for $t \rightarrow \infty$, we get

$$-x_\infty + x'(0) = \int_0^\infty f(s, x(s), x'(s))ds + \sum_{k=1}^\infty I_{1k}(x(t_k), x'(t_k)). \quad (24)$$

Thus,

$$x'(0) = x_\infty + \int_0^\infty f(s, x(s), x'(s))ds + \sum_{k=1}^\infty I_{1k}(x(t_k), x'(t_k)). \quad (25)$$

$$\begin{aligned}
x'(t) &= x_\infty + \int_0^\infty f(s, x(s), x'(s))ds + \sum_{k=1}^\infty I_{1k}(x(t_k), x'(t_k)) - \int_0^t f(s, x(s), x'(s))ds \\
&- \sum_{t_k < t} I_{1k}(x(t_k), x'(t_k)). \quad (26)
\end{aligned}$$

$$x'(t) = x_\infty + \int_t^\infty f(s, x(s), x'(s))ds + \sum_{k=1}^\infty I_{1k}(x(t_k), x'(t_k)) - \sum_{t_k < t} I_{1k}(x(t_k), x'(t_k)). \quad (27)$$

Integrating (27) from 0 to t , we obtain

$$\begin{aligned}
x(t) &= x(0) + tx_\infty + \int_0^\infty G(t,s)f(s,x(s),x'(s))ds + \sum_{k=1}^\infty G(t,t_k)I_{1k}(x(t_k),x'(t_k)) \\
&+ \sum_{k=1}^\infty G'_s(t,t_k)I_{0k}(x(t_k),x'(t_k)), \quad (28)
\end{aligned}$$

which together with the boundary value condition implies that

$$\begin{aligned}
x(0) &= \int_0^\infty g(t)x(t)dt = x(0) \int_0^\infty g(t)dt + x_\infty \int_0^\infty tg(t)dt + \sum_{k=1}^\infty G(t,t_k)I_{1k}(x(t_k),x'(t_k)) \\
&+ \int_0^\infty g(t) \left[\int_0^\infty G(t,s)f(s,x(s),x'(s))ds + \sum_{k=1}^\infty G'_s(t,t_k)I_{0k}(x(t_k),x'(t_k)) \right] dt. \quad (29)
\end{aligned}$$

Thus,

$$\begin{aligned}
 x(0) = & \frac{1}{1 - \int_0^\infty g(t)dt} \left\{ x_\infty \int_0^\infty tg(t)dt + \int_0^\infty g(t) \left[\int_0^\infty G(t,s)f(s,x(s),x'(s))ds \right. \right. \\
 & \left. \left. + \sum_{k=1}^\infty G(t,t_k)I_{1k}(x(t_k),x'(t_k)) + \sum_{k=1}^\infty G'_s(t,t_k)I_{0k}(x(t_k),x'(t_k)) \right] dt \right\}. \quad (30)
 \end{aligned}$$

Substituting (30) into (28), we have

$$\begin{aligned}
 x(t) = & \frac{1}{1 - \int_0^\infty g(t)dt} \left\{ x_\infty \int_0^\infty tg(t)dt + \int_0^\infty g(t) \left[\int_0^\infty G(t,s)f(s,x(s),x'(s))ds \right. \right. \\
 & \left. \left. + \sum_{k=1}^\infty G(t,t_k)I_{1k}(x(t_k),x'(t_k)) + \sum_{k=1}^\infty G'_s(t,t_k)I_{0k}(x(t_k),x'(t_k)) \right] dt \right\} + tx_\infty \quad (31) \\
 & + \int_0^\infty G(t,s)f(s,x(s),x'(s))ds + \sum_{k=1}^\infty G(t,t_k)I_{1k}(x(t_k),x'(t_k)) \\
 & + \sum_{k=1}^\infty G'_s(t,t_k)I_{0k}(x(t_k),x'(t_k)).
 \end{aligned}$$

Obviously, integral $\int_0^t \int_s^\infty f(\tau, x(\tau), x'(\tau))d\tau ds$ is convergent.

Conversely, if x a solution of integral equation, then direct differentiation gives the proof.

Lemma 3 Let (H_1) be satisfied, $V \subset Q$ be a bounded set. Then $\frac{(AV)(t)}{1+t}$ and $(A'V)(t)$ are equicontinuous on any finite subinterval J_k of J and for any $\varepsilon > 0$, there exists $N > 0$ such that

$$\left\| \frac{(Ax)(t')}{1+t'} - \frac{(Ax)(t'')}{1+t''} \right\| < \varepsilon, \quad \|(A'x)(t') - (A'x)(t'')\| < \varepsilon \quad (32)$$

uniformly with respect to $x \in V$ as $t', t'' \geq N$.

Proof. For $x \in V$, $t'' > t'$, $t'', t' \in J_k$, we have

$$\begin{aligned}
 & \left\| \frac{(Ax)(t')}{1+t'} - \frac{(Ax)(t'')}{1+t''} \right\| \\
 & \leq |t' - t''| \cdot \left(1 + \frac{\int_0^\infty tg(t)dt}{1 - \int_0^\infty g(t)dt} \right) \|x_\infty\| + \left(1 + \frac{\int_0^\infty g(t)dt}{1 - \int_0^\infty g(t)dt} \right) \\
 & \quad \left\{ \left\| \frac{t'}{1+t'} \int_{t'}^\infty f(s,x(s),x'(s))ds - \frac{t''}{1+t''} \int_{t''}^\infty f(s,x(s),x'(s))ds \right\| \right. \\
 & \quad \left. + \left\| \int_0^{t'} \frac{s}{1+t'} f(s,x(s),x'(s))ds - \int_0^{t''} \frac{s}{1+t''} f(s,x(s),x'(s))ds \right\| \right\} \quad (33) \\
 & \quad + \left| \frac{1}{1+t'} - \frac{1}{1+t''} \right| \int_0^\infty g(t) \left[\sum_{k=1}^\infty G(t,t_k) \|I_{1k}(x(t_k),x'(t_k))\| \right. \\
 & \quad \left. + \sum_{k=1}^\infty G'_s(t,t_k) \|I_{0k}(x(t_k),x'(t_k))\| \right] dt + \left| \frac{1}{1+t'} - \frac{1}{1+t''} \right| \sum_{k=1}^\infty G(t',t_k) \|I_{1k}(x(t_k),x'(t_k))\| \\
 & \quad + \left| \frac{1}{1+t'} - \frac{1}{1+t''} \right| \sum_{k=1}^\infty G'_s(t',t_k) \|I_{0k}(x(t_k),x'(t_k))\|
 \end{aligned}$$

$$\begin{aligned}
&\leq |t' - t''| \cdot \left(1 + \frac{\int_0^\infty tg(t)dt}{1 - \int_0^\infty g(t)dt}\right) \|x_\infty\| + \left(1 + \frac{\int_0^\infty g(t)dt}{1 - \int_0^\infty g(t)dt}\right) \cdot \\
&\quad \left\{ \left| \frac{t'}{1+t'} - \frac{t''}{1+t''} \right| \cdot \left\| \int_0^\infty f(s, x(s), x'(s))ds \right\| + \left\| \int_{t'}^{t''} sf(s, x(s), x'(s))ds \right\| \right. \\
&\quad + \frac{t''}{1+t''} \left\| \int_{t'}^{t''} f(s, x(s), x'(s))ds \right\| + \left| \frac{1}{1+t'} - \frac{1}{1+t''} \right| \cdot \left\| \int_0^{t'} sf(s, x(s), x'(s))ds \right\| \\
&\quad \left. + \left| \frac{t'}{1+t'} - \frac{t''}{1+t''} \right| \cdot \left\| \int_0^{t'} f(s, x(s), x'(s))ds \right\| \right\} \\
&\quad + \left| \frac{1}{1+t'} - \frac{1}{1+t''} \right| \int_0^\infty g(t) \left[\sum_{k=1}^\infty G(t, t_k) \|I_{1k}(x(t_k), x'(t_k))\| + \sum_{k=1}^\infty G'_s(t, t_k) \|I_{0k}(x(t_k), x'(t_k))\| \right] dt \\
&\quad + |t' - t''| \left[\sum_{k=1}^\infty G(t', t_k) \|I_{1k}(x(t_k), x'(t_k))\| + \sum_{k=1}^\infty G'_s(t', t_k) \|I_{0k}(x(t_k), x'(t_k))\| \right],
\end{aligned}$$

which implies that $\{\frac{AV(t)}{1+t} : x \in V\}$ is equicontinuous on any finite subinterval J_k of J .

Since $V \subset Q$ is bounded, there exists $r > 0$ such that for any $x \in V$, $\|x\|_D \leq r$. By (13), $t'', t' \in J_k$, we get

$$\begin{aligned}
\|(A'x)(t') - (A'x)(t'')\| &= \left\| \int_{t'}^{t''} f(s, x(s), x'(s))ds + \sum_{\substack{t_k \geq t' \\ t_k \geq t''}} I_{1k}(x(t_k), x'(t_k)) + x_\infty \right. \\
&\quad \left. - \sum_{t_k \geq t''} I_{1k}(x(t_k), x'(t_k)) - x_\infty \right\| \\
&\leq \int_{t'}^{t''} [2\varepsilon_0 rc(s)(1+s) + a(s) + Mb(s)]ds.
\end{aligned} \tag{34}$$

In the following, we are in position to show that for any $\varepsilon > 0$, there exists $N > 0$ such that

$$\left\| \frac{(Ax)(t')}{1+t'} - \frac{(Ax)(t'')}{1+t''} \right\| < \varepsilon, \quad \|(A'x)(t') - (A'x)(t'')\| < \varepsilon$$

uniformly with respect to $x \in V$ as $t', t'' \geq N$.

Combining with (33), we need only to show that for any $\varepsilon > 0$, there exists sufficiently large $N > 0$ such that

$$\left\| \int_0^{t'} \frac{s}{1+t'} f(s, x(s), x'(s))ds - \int_0^{t''} \frac{s}{1+t''} f(s, x(s), x'(s))ds \right\| < \varepsilon$$

for all $x \in V$ as $t', t'' \geq N$. The rest part of the proof is very similar to Lemma 2.3 in [5], we omit the details.

Lemma 4 *Let (H₁) and (H₂) are satisfied, V be a bounded set in $DPC^1[J, E]$. Then*

$$\alpha_D(AV) = \max \left\{ \sup_{t \in J} \alpha \left(\frac{(AV)(t)}{1+t} \right), \sup_{t \in J} \alpha((AV)'(t)) \right\}.$$

Proof. The proof is similar to that of Lemma 2.4 in [5], we omit it.

Lemma 5 ([1,2]) *Mönch Fixed-Point Theorem. Let Q be a closed convex set of E and u ∈ Q. Assume that the continuous operator F : Q → Q has the following property: V ⊂ Q countable, V ⊂ $\overline{co}(\{u\} \cup F(V)) \implies V$ is relatively compact. Then F has a fixed point in Q.*

Lemma 6 *If (H₄) is satisfied, then for $x, y \in Q$, $x^{(i)} \leq y^{(i)}$, $t \in J$ ($i = 0, 1$) imply that $(Ax)^{(i)} \leq (Ay)^{(i)}$, $t \in J$ ($i = 0, 1$).*

Proof. It is easy to see that this lemma follows from (2), (13) and condition (H₄). The proof is obvious.

3 Main results

Theorem 1 *Assume conditions (H₁), (H₂) and (H₃) are satisfied. If $\left(1 + \frac{\int_0^\infty g(t)dt}{1 - \int_0^\infty g(t)dt}\right) \cdot (2h^* + m^*) < 1$, then BVP (1) has a positive solution $\bar{x} \in DPC^1[J, E] \cap C^2[J'_+, E]$ satisfying $(\bar{x})^{(i)}(t) \geq \lambda^* x_0^*$ for $t \in J$ ($i = 0, 1$).*

Proof. By Lemma 1, operator A defined by (2) is a continuous operator from Q into Q , and by Lemma 2, we need only to show that A has a fixed point \bar{x} in Q . Choose

$$R > 2 \left\{ \left(1 + \frac{\int_0^\infty g(t)dt}{1 - \int_0^\infty g(t)dt}\right) (a^* + Mb^* + \eta_0^* M_1 + \eta_1^* M_1) + \left(1 + \frac{\int_0^\infty tg(t)dt}{1 - \int_0^\infty g(t)dt}\right) \|x_\infty\| \right\}, \quad (35)$$

and let $Q_1 = \{x \in Q : \|x\|_D \leq R\}$. Obviously, Q_1 is a bounded closed convex set in space $DPC^1[J, E]$. It is easy to see that Q_1 is not empty since $\lambda^*(1+t)x_\infty \in Q_1$. It follows from (15) and (35) that $x \in Q_1$ implies that $Ax \in Q_1$, i.e., A maps Q_1 into Q_1 . Now, we are in position to show that $A(Q_1)$ is relatively compact. Let $V = \{x_m : m = 1, 2, \dots\} \subset Q_1$ satisfying $V \subset \overline{\text{co}}\{u\} \cup AV$ for some $u \in Q_1$. Then $\|x_m\|_D \leq R$. We have, by (2) and (13)

$$\begin{aligned} (Ax_m)(t) &= \frac{1}{1 - \int_0^\infty g(t)dt} \left\{ x_\infty \int_0^\infty tg(t)dt + \int_0^\infty g(t) \left[\int_0^\infty G(t,s)f(s, x_m(s), x'_m(s))ds \right. \right. \\ &\quad \left. \left. + \sum_{k=1}^\infty G(t, t_k)I_{1k}(x_m(t_k), x'_m(t_k)) + \sum_{k=1}^\infty G'_s(t, t_k)I_{0k}(x_m(t_k), x'_m(t_k)) \right] dt \right\} + tx_\infty \\ &\quad + \int_0^\infty G(t,s)f(s, x_m(s), x'_m(s))ds + \sum_{k=1}^\infty G(t, t_k)I_{1k}(x_m(t_k), x'_m(t_k)) \\ &\quad + \sum_{k=1}^\infty G'_s(t, t_k)I_{0k}(x_m(t_k), x'_m(t_k)), \end{aligned} \quad (36)$$

and

$$(A'x_m)(t) = \int_t^\infty f(s, x_m(s), x'_m(s))ds + \sum_{t_k \geq t} I_{1k}(x_m(t_k)) + x_\infty. \quad (37)$$

By Lemma 4, we have

$$\alpha_D(AV) = \max \left\{ \sup_{t \in J} \alpha((AV)'(t)), \sup_{t \in J} \alpha\left(\frac{(AV)(t)}{1+t}\right) \right\}, \quad (38)$$

where $(AV)(t) = \{(Ax_m)(t) : m = 1, 2, 3, \dots\}$, and $(AV)'(t) = \{(A'x_m)(t) : m = 1, 2, 3, \dots\}$.

By (9), we know that the infinite integral $\int_0^\infty \|f(t, x(t), x'(t))\|dt$ is convergent uniformly for $m = 1, 2, 3, \dots$. So, for any $\varepsilon > 0$, we can choose a sufficiently large $T > 0$ such that

$$\int_T^\infty \|f(t, x(t), x'(t))\|dt < \varepsilon. \quad (39)$$

Then, by Guo et al. [1, Theorem 1.2.3], (2), (36), (37), (39) and (H₃), we obtain

$$\begin{aligned}
 \alpha\left(\frac{(AV)(t)}{1+t}\right) &\leq \frac{1}{1+t}\left(1 + \frac{\int_0^\infty g(t)dt}{1 - \int_0^\infty g(t)dt}\right)\left\{2 \int_0^T \alpha(f(t, V(t), V'(t)))dt + 2\varepsilon\right\} \\
 &\quad + \frac{1}{1+t}\left(1 + \frac{\int_0^\infty g(t)dt}{1 - \int_0^\infty g(t)dt}\right)\sum_{k=1}^\infty \sum_{i=0}^1 \alpha(I_{ik}(V(t_k), V'(t_k))) \\
 &\leq 2\left(1 + \frac{\int_0^\infty g(t)dt}{1 - \int_0^\infty g(t)dt}\right)\int_0^\infty \alpha(f(t, V(t), V'(t)))dt + 2\varepsilon \\
 &\quad + \left(1 + \frac{\int_0^\infty g(t)dt}{1 - \int_0^\infty g(t)dt}\right)\sum_{i=0}^1 \sum_{k=1}^\infty \alpha(I_{ik}(V(t_k), V'(t_k))) \tag{40} \\
 &\leq 2\left(1 + \frac{\int_0^\infty g(t)dt}{1 - \int_0^\infty g(t)dt}\right)\alpha_D(V)\int_0^\infty h_0(t)(1+t) + h_1(t)dt \\
 &\quad + \left(1 + \frac{\int_0^\infty g(t)dt}{1 - \int_0^\infty g(t)dt}\right)\alpha_D(V)\sum_{k=1}^\infty \sum_{i=0}^1 (m_{ik0}(1+t_k) + m_{ik1}) + 2\varepsilon. \\
 &\leq 2\left(1 + \frac{\int_0^\infty g(t)dt}{1 - \int_0^\infty g(t)dt}\right)h^*\alpha_D(V) + m^*\left(1 + \frac{\int_0^\infty g(t)dt}{1 - \int_0^\infty g(t)dt}\right)\alpha_D(V) + 2\varepsilon,
 \end{aligned}$$

and

$$\alpha((AV)'(t)) \leq 2 \int_0^\infty \alpha(f(s, V(s), V'(s)))ds + 2\varepsilon \leq 2\left(1 + \frac{\int_0^\infty g(t)dt}{1 - \int_0^\infty g(t)dt}\right)h^*\alpha_D(V) + 2\varepsilon. \tag{41}$$

By (38), (40) and (41) that

$$\alpha_D(AV) \leq \left(1 + \frac{\int_0^\infty g(t)dt}{1 - \int_0^\infty g(t)dt}\right)(2h^* + m^*)\alpha_D(V). \tag{42}$$

On the other hand, $\alpha_D(V) \leq \alpha_D\{\overline{\text{co}}(\{u\} \cup (AV))\} = \alpha_D(AV)$. Then, (42) implies $\alpha_D(V) = 0$, i.e., V is relatively compact in $DPC^1[J, E]$. Hence, the Mönch fixed point theorem guarantees that A has a fixed point \bar{x} in Q_1 . Thus, Theorem 1 is proved.

Theorem 2 *Let cone P be normal and conditions (H₁) – (H₄) be satisfied. Then BVP (1) has a positive solution $y \in Q \cap [J'_+, E]$ which is minimal in the sense that $x^{(i)}(t) \geq y^{(i)}(t)$, $t \in J$ ($i = 0, 1$) for any positive solution $x \in Q \cap [J'_+, E]$ of BVP (1). Moreover, $\|y\|_D \leq 2\gamma + \|x_0\|_D$, where*

$$\gamma = \left\{ \left(1 + \frac{\int_0^\infty g(t)dt}{1 - \int_0^\infty g(t)dt}\right)(a^* + Mb^* + \eta_0^*M_1 + \eta_1^*M_1) + \left(1 + \frac{\int_0^\infty tg(t)dt}{1 - \int_0^\infty g(t)dt}\right)\|x_\infty\| \right\},$$

and there exists a monotone iterative sequence $\{x_m(t)\}$ such that $x_m^{(i)}(t) \rightarrow y^{(i)}(t)$ as $m \rightarrow \infty$ ($i = 0, 1$) uniformly on J and $x_m''(t) \rightarrow y''(t)$ as $m \rightarrow \infty$ for any $t \in J_+$, where

$$\begin{aligned}
 x_0(t) &= \frac{1}{1 - \int_0^\infty g(t)dt}\left\{x_\infty \int_0^\infty tg(t)dt + \int_0^\infty g(t)\left[\int_0^\infty G(t, s)f(s, \lambda^*x_0^*, \lambda^*x_0^*)ds\right.\right. \\
 &\quad \left. + \sum_{k=1}^\infty G(t, t_k)I_{1k}(\lambda^*x_0^*, \lambda^*x_0^*) + \sum_{k=1}^\infty G'_s(t, t_k)I_{0k}(\lambda^*x_0^*, \lambda^*x_0^*)\right]dt\left\} + tx_\infty \\
 &\quad + \int_0^\infty G(t, s)f(s, \lambda^*x_0^*, \lambda^*x_0^*)ds + \sum_{k=1}^\infty G(t, t_k)I_{1k}(\lambda^*x_0^*, \lambda^*x_0^*) \tag{43} \\
 &\quad + \sum_{k=1}^\infty G'_s(t, t_k)I_{0k}(\lambda^*x_0^*, \lambda^*x_0^*),
 \end{aligned}$$

and

$$\begin{aligned}
 x_m(t) &= \frac{1}{1 - \int_0^\infty g(t)dt} \left\{ x_\infty \int_0^\infty tg(t)dt + \int_0^\infty g(t) \left[\int_0^\infty G(t,s)f(s, x_{m-1}(s), x'_{m-1}(s))ds \right. \right. \\
 &+ \left. \sum_{k=1}^\infty G(t, t_k)I_{1k}(x_{m-1}(t_k), x'_{m-1}(t_k)) + \sum_{k=1}^\infty G'_s(t, t_k)I_{0k}(x_{m-1}(t_k), x'_{m-1}(t_k)) \right] dt \Big\} + tx_\infty \\
 &+ \int_0^\infty G(t,s)f(s, x_{m-1}(s), x'_{m-1}(s))ds + \sum_{k=1}^\infty G(t, t_k)I_{1k}(x_{m-1}(t_k), x'_{m-1}(t_k)) \\
 &+ \sum_{k=1}^\infty G'_s(t, t_k)I_{0k}(x_{m-1}(t_k), x'_{m-1}(t_k)), \quad \forall t \in J \quad (m = 1, 2, 3, \dots).
 \end{aligned} \tag{44}$$

Proof. From (43), we can see that $x_0 \in C[J, E]$ and

$$x'_0(t) = \int_t^\infty f(s, \lambda^* x_0^*, \lambda^* x_0^*)ds + \sum_{t_k \geq t} I_{1k}(\lambda^* x_0^*, \lambda^* x_0^*) + x_\infty. \tag{45}$$

By (43) and (45), we have that $x_0^{(i)} \geq \lambda^* x_\infty \geq \lambda^* x_0^*$ ($i = 0, 1$) and

$$\begin{aligned}
 \|x_0(t)\| &\leq \frac{1}{1 - \int_0^\infty g(t)dt} \left\{ \|x_\infty\| \int_0^\infty tg(t)dt + \int_0^\infty g(t) \left[\int_0^\infty \|f(s, \lambda^* x_0^*, \lambda^* x_0^*)\|ds \right. \right. \\
 &+ \left. \sum_{k=1}^\infty \|I_{1k}(\lambda^* x_0^*, \lambda^* x_0^*)\| + \sum_{k=1}^\infty \|I_{0k}(\lambda^* x_0^*, \lambda^* x_0^*)\| \right] dt \Big\} + \|x_\infty\| \\
 &+ \int_0^\infty \|f(s, \lambda^* x_0^*, \lambda^* x_0^*)\|ds + \sum_{k=1}^\infty \|I_{1k}(\lambda^* x_0^*, \lambda^* x_0^*)\| \\
 &+ \sum_{k=1}^\infty \|I_{0k}(\lambda^* x_0^*, \lambda^* x_0^*)\| \\
 &\leq \left(1 + \frac{\int_0^\infty g(t)dt}{1 - \int_0^\infty g(t)dt} \right) \int_0^\infty a(s) + b(s)z(\|\lambda^* x_0^*\|, \|\lambda^* x_0^*\|)ds \\
 &+ \left(1 + \frac{\int_0^\infty g(t)dt}{1 - \int_0^\infty g(t)dt} \right) \sum_{i=0}^1 \sum_{k=1}^\infty \eta_{ik} F_i(\|\lambda^* x_0^*\|, \|\lambda^* x_0^*\|) \\
 &+ \left(1 + \frac{\int_0^\infty tg(t)dt}{1 - \int_0^\infty g(t)dt} \right) \|x_\infty\|,
 \end{aligned}$$

and

$$\begin{aligned}
 \|x'_0(t)\| &\leq \int_t^\infty \|f(s, \lambda^* x_0^*, \lambda^* x_0^*)\|ds + \sum_{t_k \geq t} \|I_{1k}(\lambda^* x_0^*, \lambda^* x_0^*)\| + \|x_\infty\| \\
 &\leq \int_0^\infty a(s) + b(s)z(\|\lambda^* x_0^*\|, \|\lambda^* x_0^*\|)ds + \sum_{k=1}^\infty \eta_{ik} F_i(\|\lambda^* x_0^*\|, \|\lambda^* x_0^*\|) + \|x_\infty\|,
 \end{aligned}$$

which imply that $\|x_0\|_D < \infty$. Thus, $x_0 \in DPC^1[J, E]$. It follows from (2) and (44) that

$$x_m(t) = (Ax_{m-1})(t), \quad \forall t \in J, \quad m = 1, 2, 3, \dots. \tag{46}$$

By Lemma 1, we have $x_m \in Q$ and

$$\|x_m\|_D = \|Ax_{m-1}\|_D \leq \frac{1}{2} \|x_{m-1}\|_D + \gamma. \tag{47}$$

By Lemma 6 and (46), we get

$$\lambda^* x_0^* \leq x_0^{(i)}(t) \leq x_1^{(i)}(t) \leq \dots \leq x_m^{(i)}(t) \leq \dots, \quad \forall t \in J \quad (i = 0, 1). \quad (48)$$

It follows from (47), by induction, that

$$\begin{aligned} \|x_m\|_D &\leq \gamma + \frac{1}{2}\gamma + \dots + \left(\frac{1}{2}\right)^{m-1} \gamma + \left(\frac{1}{2}\right)^m \|x_0\|_D \leq \frac{\gamma[1 - (\frac{1}{2})^m]}{1 - \frac{1}{2}} + \|x_0\|_D \\ &\leq 2\gamma + \|x_0\|_D \quad (m = 1, 2, 3, \dots). \end{aligned} \quad (49)$$

Let $K = \{x \in Q : \|x\|_D \leq 2\gamma + \|x_0\|_D\}$. Then, K is a bounded closed convex set in space $DPC^1[J, E]$ and operator A maps K into K . Clearly, K is not empty since $x_0 \in K$. Let $W = \{x_m : m = 0, 1, 2, \dots\}$, $AW = \{Ax_m : m = 0, 1, 2, \dots\}$. Obviously, $W \subset K$ and $W = \{x_0\} \cup A(W)$. Similar to above proof of Theorem 1, we can obtain $\alpha_D(AW) = 0$, i.e., W is relatively compact in $DPC^1[J, E]$. So, there exists a $y \in DPC^1[J, E]$ and a subsequence $\{x_{m_j} : j = 1, 2, 3, \dots\} \subset W$ such that $\{x_{m_j}^{(i)}(t) : j = 1, 2, 3, \dots\}$ converges to $y^{(i)}(t)$ uniformly on J ($i = 0, 1$). Since that P is normal and $\{x_m^{(i)}(t) : m = 1, 2, 3, \dots\}$ is nondecreasing, it is easy to see that the entire sequence $\{x_m^{(i)}(t) : m = 1, 2, 3, \dots\}$ converges to $y^{(i)}(t)$ uniformly on J ($i = 0, 1$). Since $x_m \in K$ and K is a closed convex set in space $DPC^1[J, E]$, we have $y \in K$. It is clear,

$$f(s, x_m(s), x'_m(s)) \rightarrow f(s, y(s), y'(s)), \quad \text{as } m \rightarrow \infty, \quad \forall s \in J_+. \quad (50)$$

By (H_1) and (49), we have

$$\begin{aligned} \|f(s, x_m(s), x'_m(s)) - f(s, y(s), y'(s))\| &\leq 4\varepsilon_0 c(s)(1+s)\|x_m\|_D + 2a(s) + 2Mb(s) \\ &\leq 4\varepsilon_0 c(s)(1+s)(2\gamma + \|x_0\|_D) + 2a(s) + 2Mb(s). \end{aligned} \quad (51)$$

Noticing (50) and (51) and taking limit as $m \rightarrow \infty$ in (44), we obtain

$$\begin{aligned} y(t) &= \frac{1}{1 - \int_0^\infty g(t)dt} \left\{ x_\infty \int_0^\infty tg(t)dt + \int_0^\infty g(t) \left[\int_0^\infty G(t, s)f(s, y(s), y'(s))ds \right. \right. \\ &\quad \left. \left. + \sum_{k=1}^\infty G(t, t_k)I_{1k}(y(t_k), y'(t_k)) + \sum_{k=1}^\infty G'_s(t, t_k)I_{0k}(y(t_k), y'(t_k)) \right] dt \right\} + tx_\infty \\ &\quad + \int_0^\infty G(t, s)f(s, y(s), y'(s))ds + \sum_{k=1}^\infty G(t, t_k)I_{1k}(y(t_k), y'(t_k)) \\ &\quad + \sum_{k=1}^\infty G'_s(t, t_k)I_{0k}(y(t_k), y'(t_k)), \end{aligned} \quad (52)$$

which implies by Lemma 2 that $y \in K \cap C^2[J_+, E]$ and $y(t)$ is a positive solution of BVP (1). Differentiating (44) twice, we get

$$x''_m(t) = -f(t, x_{m-1}(t), x'_{m-1}(t)), \quad \forall t \in J_+, \quad m = 1, 2, 3, \dots.$$

Hence, by (50), we obtain

$$\lim_{m \rightarrow \infty} x''_m(t) = -f(t, y(t), y'(t)) = y''(t), \quad \forall t \in J_+.$$

Let $u(t)$ be any positive solution of BVP (1). By Lemma 2, we have $u \in Q$ and $u(t) = (Au)(t)$, for $t \in J$. It is clear that $u^{(i)}(t) \geq \lambda^* x_0^* > \theta$ for any $t \in J$ ($i = 0, 1$). So, by Lemma 6, we have

$u^{(i)}(t) \geq x_0^{(i)}(t)$ for any $t \in J$ ($i = 0, 1$). Assume that $u^{(i)}(t) \geq x_{m-1}^{(i)}(t)$ for $t \in J, m \geq 1$ ($i = 0, 1$). Then, it follows from Lemma 6 that $(Au)^{(i)}(t) \geq Ax_{m-1}^{(i)}(t)$ for $t \in J$ ($i = 0, 1$), i.e. $u^{(i)}(t) \geq x_m^{(i)}(t)$ for $t \in J$ ($i = 0, 1$). Hence, by induction, we get

$$u^{(i)}(t) \geq x_m^{(i)}(t) \quad \forall t \in J \quad (i = 0, 1; m = 0, 1, 2, \dots). \quad (53)$$

Now, taking limits in (53), we get $u^{(i)}(t) \geq y^{(i)}(t)$ for $t \in J$ ($i = 0, 1$). The proof is proved.

Theorem 3 *Let cone P be fully regular and conditions (H_1) , (H_2) and (H_4) be satisfied. Then the conclusion of Theorem 2 holds.*

Proof. The proof is almost the same as that of Theorem 2. The only difference is that, instead of using condition (H_3) , the conclusion $\alpha_D(W) = 0$ is implied directly by (48) and (49), the full regularity of P and Lemma 4.

4 An example

Consider the infinite system of scalar second order impulsive singular integro-differential equations

$$\left\{ \begin{array}{l} -x_n''(t) = \frac{1}{4n^3 \sqrt[3]{e^{2t}(2+5t)^9}} \left(5 + x_n(t) + x'_{2n}(t) + \frac{2}{3n^2 x_n(t)} + \frac{4}{7n^5 x'_{2n}(t)} \right)^{\frac{1}{2}} \\ \quad + \frac{1}{4\sqrt[6]{t}(1+3t)^2} \ln[(1+3t)x_n(t)], \\ \Delta x|_{t=t_k} = \frac{1}{n^3} \cdot \frac{k}{2^{k+1}} \left(\frac{1}{x_n(t_k)} + x'_{2n}(t_k) \right)^{\frac{1}{5}}, \quad k = 1, 2, \dots, \\ \Delta x'|_{t=t_k} = \frac{1}{n^4} \cdot \frac{1}{(k+1)^3} \left(\frac{1}{x_{n+1}(t_k)} + x'_{n+2}(t_k) \right)^{\frac{1}{7}}, \quad k = 1, 2, \dots, \\ x_n(0) = \int_0^\infty e^{-t^2} x_n(t) dt, \quad x'_n(\infty) = \frac{1}{n}, \quad n = 1, 2, \dots \end{array} \right. \quad (54)$$

Proposition 1 *Infinite system (54) has a minimal positive solution $x_n(t)$ satisfying $x_n(t), x'_n(t) \geq \frac{1}{n}$ for $0 \leq t < +\infty$ ($n = 1, 2, 3, \dots$), and this minimal solution can be obtained by taking limits from some iterative sequences.*

Proof. Let $E = c_0 = \{x = (x_1, \dots, x_n, \dots) : x_n \rightarrow 0\}$ with the norm $\|x\| = \sup_n |x_n|$. Obviously, $(E, \|\cdot\|)$ is a real Banach space. Choose $P = \{x = (x_n) \in c_0 : x_n \geq 0, n = 1, 2, 3, \dots\}$. It is easy to verify that P is a normal cone in E with normal constant 1. Now we consider infinite system (54), which can be regarded as a BVP of form (1) in E with $x_\infty = (1, \frac{1}{2}, \frac{1}{3}, \dots)$. In this situation, $x = (x_1, \dots, x_n, \dots), y = (y_1, \dots, y_n, \dots), f = (f_1, \dots, f_n, \dots)$, and $I_{ik} = (I_{ik1}, \dots, I_{ikn}, \dots)$ ($i = 0, 1$), in which

$$\begin{aligned} f_n(t, x, y) = & \frac{1}{4n^3 \sqrt[3]{e^{2t}(2+5t)^9}} \left(5 + x_n + y_{2n} + \frac{2}{3n^2 x_n} + \frac{4}{7n^5 y_{2n}} \right)^{\frac{1}{2}} \\ & + \frac{1}{4\sqrt[6]{t}(1+3t)^2} \ln[(1+3t)x_n], \end{aligned} \quad (55)$$

and

$$I_{0kn} = \frac{1}{n^3} \cdot \frac{k}{2^{k+1}} \left(\frac{1}{x_n} + y_{2n} \right)^{\frac{1}{5}}, \quad I_{1kn} = \frac{1}{n^4} \cdot \frac{1}{(k+1)^3} \left(x_{n+1} + \frac{1}{y_{n+2}} \right)^{\frac{1}{7}}. \quad (56)$$

Let $x_0^* = x_\infty = (1, \frac{1}{2}, \frac{1}{3}, \dots)$. Then $P_{0\lambda} = \{x = (x_1, x_2, \dots, x_n, \dots) : x_n \geq \frac{\lambda}{n}, n = 1, 2, 3, \dots\}$, for $\lambda > 0$. It is clear, $f \in C[J_+ \times P_{0\lambda} \times P_{0\lambda}, P]$ for any $\lambda > 0$. Noticing that $\sqrt[3]{e^{2t}} > \sqrt[6]{t}$ for $t > 0$, by (55), we get

$$\|f(t, x, y)\| \leq \frac{1}{4\sqrt[6]{t}(1+3t)^2} \left\{ \left(\frac{143}{21} + \|x\| + \|y\| \right)^{\frac{1}{2}} + \ln[(1+3t)\|x\|] \right\}, \quad (57)$$

which imply (H₁) is satisfied for $a(t) = 0$, $b(t) = c(t) = \frac{1}{4\sqrt[6]{t}(1+3t)^2}$, and

$$z(u_0, u_1) = \left(\frac{143}{21} + u_0 + u_1 \right)^{\frac{1}{2}} + \ln[(1+3t)u_0].$$

On the other hand, for $x \in P_{0\lambda^*}, y \in P_{0\lambda^*}$, we have, by (56)

$$\|I_{0k}(x, y)\| \leq \frac{k}{2^{k+1}} (1 + \|y\|)^{\frac{1}{5}}, \quad \|I_{1k}(x, y)\| \leq \frac{1}{(k+1)^3} (\|x\| + 1)^{\frac{1}{7}},$$

which imply (H₂) is satisfied for

$$F_0(u_0, u_1) = (1 + u_1)^{\frac{1}{5}}, \quad F_1(u_0, u_1) = (1 + u_0)^{\frac{1}{7}},$$

and

$$\eta_{0k} = \frac{k}{2^{k+1}}, \quad \eta_{1k} = \frac{1}{(k+1)^3}, \quad \gamma_{0k} = \frac{k}{2^{k+1}(1+t_k)}, \quad \gamma_{1k} = \frac{1}{(k+1)^3(1+t_k)}.$$

Let $f^1 = \{f_1^1, f_2^1, \dots, f_n^1, \dots\}$, $f^2 = \{f_1^2, f_2^2, \dots, f_n^2, \dots\}$, where

$$f_n^1(t, x, y) = \frac{1}{4n^3 \sqrt[3]{e^{2t}}(2+5t)^9} \left(5 + x_n + y_{2n} + \frac{2}{3n^2 x_n} + \frac{4}{7n^5 y_{2n}} \right)^{\frac{1}{2}}, \quad (58)$$

$$f_n^2(t, x, y) = \frac{1}{4\sqrt[6]{t}(1+3t)^2} \ln[(1+3t)x_n]. \quad (59)$$

Let $t \in J_+$, and $R > 0$ be given and $\{z^{(m)}\}$ be any sequence in $f^1(t \times P_{0\lambda^*R}, P_{0\lambda^*R})$, where $z^{(m)} = (z_1^{(m)}, \dots, z_n^{(m)}, \dots)$. By (58), we have

$$0 \leq z_n^{(m)} \leq \frac{1}{4n^3 \sqrt[3]{e^{2t}}(2+5t)^9} \left(\frac{143}{21} + 2R \right)^{\frac{1}{2}} \quad (n, m = 1, 2, 3, \dots). \quad (60)$$

So, $\{z_n^{(m)}\}$ is bounded and by the diagonal method together with the method of constructing subsequence, we can choose a subsequence $\{m_i\} \subset \{m\}$ such that

$$\{z_n^{(m_i)}\} \rightarrow \bar{z}_n \quad \text{as } i \rightarrow \infty \quad (n = 1, 2, 3, \dots), \quad (61)$$

which implies by (60)

$$0 \leq \bar{z}_n \leq \frac{1}{4n^3 \sqrt[3]{e^{2t}}(2+5t)^9} \left(\frac{143}{21} + 2R \right)^{\frac{1}{2}} \quad (n = 1, 2, 3, \dots). \quad (62)$$

Hence $\bar{z} = (\bar{z}_1, \dots, \bar{z}_n, \dots) \in c_0$. It is easy to see from (60)-(62) that

$$\|z^{(m_i)} - \bar{z}\| = \sup_n |z_n^{(m_i)} - \bar{z}_n| \rightarrow 0 \text{ as } i \rightarrow \infty.$$

Thus, we have proved that $f^1(t \times P_{0\lambda^*R}, P_{0\lambda^*R})$ is relatively compact in c_0 .

For any $t \in J_+$, $R > 0$, $x, y, \bar{x}, \bar{y} \in D \subset P_{0\lambda^*R}$, we have by (59)

$$\begin{aligned} |f_n^2(t, x, y) - f_n^2(t, \bar{x}, \bar{y})| &= \frac{1}{4\sqrt[6]{t}(1+3t)^2} |\ln[(1+3t)x_n] - \ln[(1+3t)\bar{x}_n]| \\ &\leq \frac{1}{4\sqrt[6]{t}(1+3t)} \frac{|x_n - \bar{x}_n|}{(1+3t)\xi_n}, \end{aligned} \quad (63)$$

where ξ_n is between x_n and \bar{x}_n . By (63), we get

$$\|f^2(t, x, y) - f^2(t, \bar{x}, \bar{y})\| \leq \frac{1}{4\sqrt[6]{t}(1+3t)} \|x - \bar{x}\|, \quad x, y, \bar{x}, \bar{y} \in D. \quad (64)$$

Thus, by (64), it is easy to see that (H_3) holds for $h_0(t) = \frac{1}{4\sqrt[6]{t}(1+3t)}$. Our conclusion follows from Theorem 2. This completes the proof.

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