DULAC-CHERKAS FUNCTIONS FOR GENERALIZED LIÉNARD SYSTEMS

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ABSTRACT. Dulac-Cherkas functions can be used to derive an upper bound for the number of limit cycles of planar autonomous differential systems including criteria for the non-existence of limit cycles, at the same time they provide information about their stability and hyperbolicity. In this paper, we present a method to construct a special class of Dulac-Cherkas functions for generalized Liénard systems of the type $\frac{dx}{dt} = y$, $\frac{dy}{dt} = \sum_{j=0}^{l} h_j(x)y^j$ with $l \geq 1$. In case $1 \leq l \leq 3$, linear differential equations play a key role in this process, for $l \geq 4$, we have to solve a system of linear differential and algebraic equations, where the number of equations is larger than the number of unknowns. Finally, we show that Dulac-Cherkas functions can be used to construct generalized Liénard systems with any l possessing limit cycles.

Keywords and Phrases: number of limit cycles, generalized Liénard systems, Dulac-Cherkas functions, systems of linear differential and algebraic equations

2001 Mathematical Subject Classification: 34C07, 34C05

1. INTRODUCTION

The problem of estimating the number of limit cycles for two-dimensional systems of autonomous differential equations

$$(1.1) \qquad \frac{dx}{dt} = P(x, y), \; \frac{dy}{dt} = Q(x, y)$$

in some open region $\mathcal{G} \subset R^2$ represents one of the famous problems formulated by D.Hilbert [6]. This problem is still open. There are several approaches to attack this problem, including intentions to weaken it [7]. One known method to estimate the number of limit cycles of (1.1) from above is the method of Dulac function [2]. Here, the upper bound on the number of limit cycles also depends essentially on the connectivity of the region $\mathcal{G}$. Frequently, this method is used to establish that system (1.1) has in some simply connected region no limit cycle or in a doubly connected region at most one limit cycle.

The method of Dulac function has been generalized into different directions. One promising generalization is due to the first author who introduced in 1997 a function which we call now Dulac-Cherkas function that not only permits to get an upper bound for the number of limit cycles but also provides an information about their stability (see [1]). The problem of construction of such a function has been investigated by the first and the second author in [4] with respect to the Liénard system

$$(1.2) \qquad \frac{dx}{dt} = y, \quad \frac{dy}{dt} = -g(x) - f(x)y$$

with $g(0) = 0$. In that paper, it has been shown that linear differential equations combined with the method of linear programming can be used to determine Dulac-Cherkas functions. Recently, Gasull and Giacomini used in [3] principally the same method to estimate the number of limit cycles for the Kukles system

$$(1.3) \qquad \frac{dx}{dt} = y, \quad \frac{dy}{dt} = h_0(x) + h_1(x)y + h_2(x)y^2 + y^3.$$

In the sequel we consider the problem of construction of a class of Dulac-Cherkas functions for the generalized Liénard system

$$(1.4) \qquad \frac{dx}{dt} = y, \quad \frac{dy}{dt} = \sum_{j=0}^{l} h_j(x)y^j$$

with $l \geq 1$ and

$$(1.5) \qquad h_l(x) \not\equiv 0.$$

The paper is organized as follows: In section 2 we recall some definitions and known results. In section 3 we present an algorithm to construct a special class of Dulac-Cherkas for $1 \leq l \leq 3$. We show in section 4 how this algorithm can be applied also in case $l \geq 4$ in order to derive conditions on the functions h_i implying that the corresponding system (1.4) has at most one limit cycle or no limit cycle. In the last section we demonstrate how a Dulac-Cherkas function can be used to construct a generalized Liénard system having a unique limit cycle surrounding the origin.

2. Preliminaries

First we recall the definition of a Dulac function.

Definition 2.1. *Let $P, Q \in C^1(\mathcal{G}, R)$, let X be the vector field defined by (1.1). A function $B \in C^1(\mathcal{G}, R)$ is called a Dulac-function of (1.1) in $\mathcal{G}$ if*

$$div(BX) \equiv \frac{\partial(BP)}{\partial x} + \frac{\partial(BQ)}{\partial y}$$

does not change sign in $\mathcal{G}$ and vanishes only on a set $\mathcal{N}$ of measure zero, where no oval (closed curve homeomorphic to a circle) in $\mathcal{N}$ is a limit cycle.

The existence of a Dulac function implies the following estimate on the number of limit cycles of system (1.1) in $\mathcal{G}$.

Proposition 2.1. *Let $\mathcal{G}$ be a p-connected ($p \geq 1$) region in R^2, let $P, Q \in C^1(\mathcal{G}, R)$. If there is a Dulac function B of (1.1) in $\mathcal{G}$, then (1.1) has not more than $p - 1$ limit cycles in $\mathcal{G}$.*

The method of Dulac function has been generalized in different ways. One possibility is to admit that B is not necessarily C^1 at any equilibrium provided the number of equilibria is finite in $\mathcal{G}$. This generalization has been proposed by the third author in 1968 (see [8]). Another generalization is due to the first author (see [1]). The corresponding generalized Dulac function, which we call Dulac-Cherkas function, is defined as follows.

Definition 2.2. *Let $P, Q \in C^1(\mathcal{G}, R)$. A function $\Psi \in C^1(\mathcal{G}, R)$ is called a Dulac-Cherkas function of system (1.1) in $\mathcal{G}$ if there exists a real number $k \neq 0$ such that*

$$(2.1) \quad \Phi := (grad \ \Psi, X) + k\Psi \ div \ X > 0 \quad (< 0) \quad in \quad \mathcal{G}.$$

Remark 2.1. *Condition (2.1) can be relaxed by assuming that Φ may vanish in $\mathcal{G}$ on a set of measure zero, and that no oval of this set is a limit cycle of (1.1).*

For the sequel we introduce the subset $\mathcal{W}$ of $\mathcal{G}$ by

$$\mathcal{W} := \{(x, y) \in \mathcal{G} : \Psi(x, y) = 0\}.$$

The following three theorems can be found in [1].

Theorem 2.1. *Any trajectory of (1.1) which meets $\mathcal{W}$ intersects $\mathcal{W}$ transversally.*

Theorem 2.2. *Let $\mathcal{G}$ be a p-connected region, let Ψ be a Dulac-Cherkas function of* (1.1) *in $\mathcal{G}$. If we additionally assume that $\mathcal{W}$ has no oval in $\mathcal{G}$, then system* (1.1) *has at most $p-1$ limit cycles in $\mathcal{G}$.*

From this theorem we immediately obtain

Corollary 2.1. *Let $\mathcal{G}$ be a simply connected region, let Ψ be a Dulac-Cherkas function of* (1.1) *in $\mathcal{G}$. If* (1.1) *has a limit cycle Γ in $\mathcal{G}$, then the region bounded by Γ contains an oval of $\mathcal{W}$ in its interior.*

Theorem 2.3. *Let Ψ be a Dulac-Cherkas function of* (1.1) *in the region $\mathcal{G}$. Then any limit cycle Γ of* (1.1) *in $\mathcal{G}$ is hyperbolic and its stability is determined by the sign of the expression $k\Phi\Psi$ on Γ.*

Theorem 2.2 has been generalized in [5] by the second and the third authors as follows.

Theorem 2.4. *Let $\mathcal{G}$ be a p-connected region, let Ψ be a Dulac-Cherkas function of* (1.1) *in $\mathcal{G}$ such that $\mathcal{W}$ has s ovals in $\mathcal{G}$. Then system* (1.1) *has at most $p-1+s$ limit cycles in $\mathcal{G}$, any existing limit cycle is hyperbolic.*

Remark 2.2. *In* [5] *it has been also shown that the differentiability conditions of Ψ in Theorem 2.4 can be weakened in the same manner as in case of a Dulac function.*

The problem to construct a Dulac-Cherkas function has been solved by the first author for the Liénard system (1.2). He uses as Ψ the function

$$(2.2) \qquad \Psi(x,y) \equiv \frac{y^2}{2} + G(x) - \alpha,$$

where α is an appropriate constant and G is defined by $G(x) := \int_0^x g(\sigma)d\sigma$. According to this choice of Ψ, the curve $\Psi(x,y) = 0$ has at most one oval. Moreover, we get from (2.1) and (1.2)

$$\Phi(x,y) \equiv -\Big(\frac{k}{2}+1\Big)f(x)y^2 - k\big(G(x)-\alpha\big)f(x).$$

Setting $k = -2$ we obtain

$$\Phi(x,y) \equiv 2\big(G(x)-\alpha\big)f(x).$$

Thus, Φ does not depend on y, and applying Theorem 2.4 we get the result:

Theorem 2.5. *Suppose $f, g : R \to R$ to be continuous. Additionally, we assume that there is a constant α such that the function Φ_1 defined by*

(2.3)
$$\Phi_1(x) := \big(G(x) - \alpha\big) f(x)$$

does not change sign in R and vanishes only at finitely many points. Then system (1.2) has at most one limit cycle Γ, and, if Γ exists, it is hyperbolic.

In the case
$$g(x) \equiv x, \quad f(x) \equiv \mu(x^2 - 1)$$
system (1.2) represents the van der Pol equation, and we get

$$\Psi(x, y) \equiv \frac{y^2}{2} + \frac{x^2}{2} - \alpha, \quad \Phi_1(x) \equiv \mu\Big(\frac{x^2}{2} - \alpha\Big)(x^2 - 1).$$

Setting $\alpha = 1/2$ we have

$$\Phi_1(x) \equiv \frac{\mu}{2}(x^2 - 1)^2,$$

that is, all conditions of Theorem 2.5 and of Theorem 2.3 are fulfilled for $\mu \neq 0$, and the curve $\Psi(x, y) = 0$ consists in R^2 of the circle $\mathcal{O} := \{(x, y) \in R^2 : y^2 + x^2 = 1\}$. Thus, we get the known result:

Proposition 2.2. *The van der Pol equation has for any $\mu \neq 0$ at most one limit cycle $\Gamma(\mu)$ which is located outside the region bounded by the circle $\mathcal{O}$. $\Gamma(\mu)$ is hyperbolic and stable (unstable) for $\mu > 0$ ($\mu < 0$).*

We note that in case of Liénard system (1.2), to the Dulac-Cherkas function Ψ in (2.2) there belongs a function Φ defined in (2.1) that does not depend on y for a special value of k. The advantage of eliminating the variable y from the function Φ consists in the fact that the inequality (2.1) must be fulfilled only on some interval. Hence, this approach makes it easier to check the validity of inequality (2.1) or to derive conditions guaranteeing that (2.1) is satisfied. This reasoning stimulated the first and second authors to develop in [4] an algorithmic way for constructing a Dulac-Cherkas function Ψ for the Liénard system (1.2) in the form

(2.4)
$$\Psi(x, y) = \sum_{j=0}^{n} \Psi_j(x) y^j$$

with

(2.5)
$$\Psi_n(x) \not\equiv 0,$$

where the coefficient functions Ψ_j can be determined by means of linear differential equations such that the corresponding function Φ in (2.1) does not depend on y. Additionally, the problem to derive conditions such that Φ is either positive or negative in the considered region was formulated as a problem of linear programming.

In [3] Gasull and Giacomini consider the class of planar autonomous systems (1.3), where the functions $h_i : R \to R, 0 \le i \le 2$, are continuous. This system represents a generalized Liénard system. They also look for a Dulac-Cherkas function in the form (2.4) and prove that to any given positive integer n there is a function Ψ as in (2.4) and a special value k such that the corresponding function Φ does not depend on y, and that the functions Ψ_j can be determined by solving linear differential equations. They did not mention that this approach in case of the Liénard system (1.2) has been introduced by the first and second author in [4], probably, they were not aware of that paper.

In the next section we consider the generalized Liénard system (1.4) and describe an algorithm to find a function Ψ and a number k such that the corresponding function Φ in (2.1) does not depend on y.

3. Construction of a class of Dulac-Cherkas functions Ψ for (1.4) in case $1 \le l \le 3$

We consider the vector field $X_l(x, y)$ defined by the differential system (1.4) in some region $\mathcal{G} \subset R^2$. For the Dulac-Cherkas function $\Psi(x, y)$ of (1.4) in $\mathcal{G}$ we make the ansatz (2.4) with $n \ge 2$. In what follows we describe an algorithm to determine the functions $\Psi_j(x)$ in (2.4) and the constant k such that the corresponding function $\Phi(x, y)$ determined by

$$(3.1) \qquad \Phi(x, y) := (grad\, \Psi(x, y), X_l(x, y)) + k\Psi(x, y)\, div\, X_l(x, y)$$

does not depend on y. We show that this algorithm works generically in the considered case.

If we put (2.4) into the right hand side of (3.1) and take into account

that the vector field X_l is determined by (1.4) we get

$$\Phi(x, y) \equiv \Big(\Psi_0'(x) + \Psi_1'(x)y + ... + \Psi_n'(x)y^n \Big) y$$

$$+ \Big(\Psi_1(x) + 2\Psi_2(x)y + ... + n\Psi_n(x)y^{n-1} \Big)$$

(3.2) $$\times \Big(h_0(x) + h_1(x)y + ... + h_l(x)y^l \Big)$$

$$+ k \Big(\Psi_0(x) + \Psi_1(x)y + ... + \Psi_n(x)y^n \Big)$$

$$\times \Big(h_1(x) + 2h_2(x)y + ... + lh_l(x)y^{l-1} \Big).$$

For the sequel we represent $\Phi(x, y)$ in the form

$$(3.3) \qquad \Phi(x, y) \equiv \sum_{i=0}^{m} \Phi_i(x)y^i,$$

where $\Phi_i(x)$ are functions of the known coefficient functions $h_0(x), ..., h_l(x)$, of the unknown coefficient functions $\Psi_0(x), ..., \Psi_n(x)$, of their first derivatives $\Psi_0'(x), ..., \Psi_n'(x)$, and of k.

Concerning the highest power m of y in (3.3) we get from (3.2)

$$(3.4) \qquad m = max\{n+1, n+1+l-2\}.$$

Our goal is to determine the functions $\Psi_j(x), j = 0, ..., n$, and the real number k in such a way that we have

$$(3.5) \qquad \Phi_i(x) \equiv 0 \quad \text{for} \quad i = 1, ..., m.$$

Then it holds

$$(3.6) \qquad \Phi(x, y) \equiv \Phi_0(x) \equiv \Psi_1(x)h_0(x) + k\Psi_0(x)h_1(x).$$

If we additionally require

$$(3.7) \qquad \Phi_0(x) \geq 0 \quad (\leq 0) \quad \text{for} \quad (x, y) \in \mathcal{G}$$

and if $\Phi_0(x)$ vanishes only at finitely many points of x, then Ψ is a Dulac-Cherkas function of (1.4) in $\mathcal{G}$.

From (3.2)–(3.4) we get that for $l = 1$ and $l = 2$ the relations (3.5) represent a system of $n+1$ linear differential equations to determine the $n+1$ functions $\Psi_j, j = 0, ..., n$. In case $l = 1$ this system reads

$$0 \equiv \Psi'_n(x),$$
$$0 = \Psi'_{n-1}(x) + (k+n)h_1(x)\Psi_n(x),$$
$$0 \equiv \Psi'_{n-2}(x) + (k+n-1)h_1(x)\Psi_{n-1}(x) + nh_0(x)\Psi_n(x),$$

(3.8)

....

$$0 \equiv \Psi'_1(x) + (k+2)h_1(x)\Psi_2(x) + 3h_0(x)\Psi_3(x),$$
$$0 \equiv \Psi'_0(x) + (k+1)h_1(x)\Psi_1(x) + 2h_0(x)\Psi_2(x).$$

It is easy to see that this system can be solved successively by simple quadratures, starting with Ψ_n. The general solution depends on $n+1$ integration constants and on the constant k as well as on the functions h_i. An appropriate choice of these constants leads to efficient conditions on the functions h_i such that Ψ is a Dulac-Cherkas function for (1.4) in $\mathcal{G}$.

As an example, we consider system (1.4) with $l = 1$, i.e.

(3.9)
$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = h_0(x) + h_1(x)y.$$

We look for a Dulac-Cherkas function in the form

(3.10)
$$\Psi(x, y) = \Psi_0(x) + \Psi_1(x)y + \Psi_2(x)y^2$$

with $\Psi_2(x) \not\equiv 0$. Putting $n = 2$ in (3.8) we obtain the following system of differential equations

(3.11)
$$\Psi'_2 = 0,$$
$$\Psi'_1 = -(k+2)h_1(x)\Psi_2,$$
$$\Psi'_0 = -(k+1)h_1(x)\Psi_1 - 2h_0(x)\Psi_2.$$

Setting $k = -2$ we get from the first two equations

(3.12)
$$\Psi_2(x) \equiv c_2 \neq 0, \Psi_1(x) \equiv c_1,$$

where c_2 and c_1 are real constants. Putting $c_1 = 0$ we obtain from the last differential equation in (3.11)

(3.13)
$$\Psi_0(x) \equiv -2c_2 \int_0^x h_0(\tau)d\tau + c_0,$$

where c_0 is any real constant. Thus, we have

$$\Psi(x, y) = -2c_2 \int_0^x h_0(\tau)d\tau + c_0 + c_2 y^2,$$

$$\Phi_0(x) = -2\left(-2c_2\int_0^x h_0(\tau)d\tau + c_2\right)h_1(x)$$

$$= 4c_2\left(\int_0^x h_0(\tau)d\tau + c_0^*\right)h_1(x).$$

To guarantee the validity of one of the inequalities $\Phi_0(x) \le 0, \Phi_0(x) \ge 0$, we impose on h_0 and h_1 the following assumption.

(H). $h_0, h_1 : R \to R$ are continuous and such that there is a constant c_0^* ensuring that the function

$$\tilde{\Phi}_0(x) := \left(\int_0^x h_0(\tau)d\tau + c_0^*\right)h_1(x)$$

does not change sign in R, where $\tilde{\Phi}_0(x)$ vanishes only in finite many points x_k.

Applying Theorem 2.4 we get the result:

Proposition 3.1. *Suppose hypothesis (H) to be valid. Then system (3.9) has at most one limit cycle in the finite part of the phase plane. If system (3.9) has a limit cycle, then it is hyperbolic.*

We note that Proposition 3.1 coincides with Theorem 2.5.

In case $l = 2$ we get the system

$$0 \equiv \Psi_n'(x) + (2k+n)h_2(x)\Psi_n(x),$$

$$0 \equiv \Psi_{n-1}'(x) + (2k+n-1)h_2(x)\Psi_{n-1}(x)$$
$$+ (k+n)h_1(x)\Psi_n(x),$$

$$0 \equiv \Psi_{n-2}'(x) + (2k+n-2)h_2(x)\Psi_{n-2}(x)$$
$$+ (k+n-1)h_1(x)\Psi_{n-1}(x) + nh_0(x)\Psi_n(x),$$

(3.14)
....

$$0 \equiv \Psi_1'(x) + (2k+1)h_2(x)\Psi_1(x)$$
$$+ (k+2)h_1(x)\Psi_2(x) + 3h_0(x)\Psi_3(x),$$

$$0 = \Psi_0'(x) + 2kh_2(x)\Psi_0(x)$$
$$+ (k+1)h_1(x)\Psi_1(x) + 2h_0(x)\Psi_2(x).$$

This system can also be integrated successively by solving inhomogeneous linear differential equations, starting with Ψ_n. In order to derive efficient conditions guaranteeing the validity of the inequality (3.7) we

have to choose k and the integration constants appropriately.

Next we consider the case $l = 3$. From (3.2) and (3.3) we obtain

$$0 \equiv (n + 3k)h_3(x)\Psi_n(x),$$

$$0 \equiv \Psi_n'(x) + (2k + n)h_2(x)\Psi_n(x)$$
$$+ (n - 1 + 3k)h_3(x)\Psi_{n-1}(x),$$

$$0 \equiv \Psi_{n-1}'(x) + (n - 1 + 2k)h_2(x)\Psi_{n-1}(x)$$
$$+ (n + k)h_1(x)\Psi_n(x) + (n - 2 + 3k)h_3(x)\Psi_{n-2},$$

$$0 \equiv \Psi_{n-2}'(x) + (2k + n - 2)h_2(x)\Psi_{n-2}(x)$$

(3.15)
$$+ (k + n - 1)h_1(x)\Psi_{n-1}(x) + nh_0(x)\Psi_n(x)$$
$$+ (n - 3 + 3k)h_3(x)\Psi_{n-3}(x),$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$0 \equiv \Psi_1'(x) + (1 + 2k)h_2(x)\Psi_1(x) + 3kh_3(x)\Psi_0(x)$$
$$+ (2 + k)h_1(x)\Psi_2(x) + 3h_0(x)\Psi_3(x),$$

$$0 \equiv \Psi_0'(x) + 2kh_2(x)\Psi_0(x)$$
$$+ (k + 1)h_1(x)\Psi_1(x) + 2h_0(x)\Psi_2(x).$$

The first equation is an algebraic equation which determines according to (1.5) and (2.5) the constant k uniquely as $k = -\frac{n}{3}$. The remaining equations represent a system of $n + 1$ linear differential equations. Its general solution depends on $n + 1$ integration constants which can be chosen appropriately in order to derive efficient conditions on the functions h_i satisfying the validity of the inequality (3.7).

4. Construction of Cherkas-Dulac functions in case $l \geq 4$

In case $l \geq 4$, system (3.5) consist of $n+1$ linear differential equations and $l-2$ algebraic equations to determine k and the functions $\Psi_0, ..., \Psi_n$ such that Φ does not depend on y. Thus, this system has generically no solution. In what follows we show that under additional conditions on the functions h_i system (3.5) has a nontrivial solution satisfying the inequality (3.7). We demonstrate this approach by considering the system

(4.1) $\quad \dfrac{dx}{dt} = y, \quad \dfrac{dy}{dt} = h_0(x) + h_1(x)y + h_2(x)y^2 + h_3(x)y^3 + h_4(x)y^4.$

under the assumption

($\mathbf{A_1}$). The functions $h_i : R \to R$, $0 \le i \le 4$, are continuous and such that h_4 is not identically zero.

Our aim is to construct a Dulac-Cherkas function in the form

$$(4.2) \qquad \Psi(x, y) = \sum_{j=0}^{2} \Psi_j(x) y^j$$

with

$$(4.3) \qquad \Psi_2(x) \not\equiv 0.$$

Remark 4.1. *In case that Ψ has the form (4.2), the set $\mathcal{W}$ defined by $\Psi(x, y) = 0$ contains at most one oval surrounding the origin. Taking into account Corollary 2.1, we can conclude that if Ψ has the form (4.2) and is a Dulac-Cherkas function for system (4.2) in R^2, then system (4.1) has at most one limit cycle.*

To the function Ψ with the representation (4.2) there belongs by (3.1)–(3.4) the function Φ with the representation

$$\Phi(x, y) = \sum_{j=0}^{5} \Phi_j(x) y^j.$$

Our goal is to determine the Dulac-Cherkas function Ψ in such a way that $\Phi_i(x) \equiv 0$ for $i = 1, \cdots, 5$.

Taking into account (4.1) we get from (3.5) and (3.2) the relations

$$(4.4) \qquad 2(1 + 2k)h_4(x)\Psi_2(x) \equiv 0,$$

$$(4.5) \qquad (1 + 4k)h_4(x)\Psi_1(x) + (2 + 3k)h_3(x)\Psi_2(x) \equiv 0,$$

$$(4.6) \qquad \Psi_2' + 2(1 + k)h_2(x)\Psi_2 + (1 + 3k)h_3(x)\Psi_1 + 4kh_4(x)\Psi_0 \equiv 0,$$

$$(4.7) \qquad \Psi_1' + (1 + 2k)h_2(x)\Psi_1 + (2 + k)h_1(x)\Psi_2 + 3kh_3(x)\Psi_0 \equiv 0,$$

$$(4.8) \qquad \Psi_0' + 2kh_2(x)\Psi_0 + (1 + k)h_1(x)\Psi_1 + 2h_0(x)\Psi_2 \equiv 0.$$

This system of differential and algebraic equations can be solved for $\Psi_0(x)$, $\Psi_1(x)$ and $\Psi_2(x)$ only under additional conditions on the coefficient functions $h_i(x)$. In what follows we describe an algorithm to determine the Dulac-Cherkas function in such a way that the corresponding function Φ depends only on the variable x. We describe this

approach under the additional assumption

($\mathbf{A_2}$). There is a real constant $\kappa \neq 0$ such that

(4.9) $h_3(x) \equiv \kappa\, h_4(x) \neq 0 \quad \forall x \in R.$

By (4.9) and (4.3) we get from (4.4)

(4.10) $k = -\dfrac{1}{2}.$

Taking into account (4.9) and (4.10) we obtain from (4.5)

(4.11) $\Psi_1(x) = \dfrac{\kappa}{2}\, \Psi_2(x).$

Substituting (4.11) and (4.10) into (4.6) and (4.7) we get

(4.12) $\Psi_2' + \left(h_2(x) - \dfrac{h_4(x)\kappa^2}{4} \right)\Psi_2 - 2h_4(x)\Psi_0 = 0,$

(4.13) $\Psi_2' + \dfrac{3h_1(x)}{\kappa}\Psi_2 - 3h_4(x)\Psi_0 = 0.$

A function Ψ_2 satisfying the differential equations (4.12) and (4.13) has also to obey the homogeneous equation

(4.14) $\Psi_2' + h(x)\Psi_2 = 0$

with

(4.15) $h(x) := 3\, h_2(x) - \dfrac{3\kappa^2 h_4(x)}{4} - \dfrac{6h_1(x)}{\kappa}.$

Thus, we have

(4.16) $\Psi_2(x) = c\, e^{-\int_0^x h(\sigma)d\sigma}, \quad \Psi_2(0) = c,$

where $c \neq 0$ by (4.3). From (4.9), (4.3), (4.11) and (4.16) we get that the functions Ψ_1 and Ψ_2 never take the value zero.
A solution of (4.14) satisfies the differential equation (4.12) only if the relation

(4.17) $\left(-h_2(x) + \dfrac{\kappa^2 h_4(x)}{4} + \dfrac{3h_1(x)}{\kappa} \right)\Psi_2(x) = h_4(x)\Psi_0(x)$

is valid. We get the same relation if we consider equation (4.13).
Substituting (4.16) together with (4.10) and (4.11) into (4.8) we obtain

(4.18) $\Psi_0' - h_2(x)\Psi_0 + \left(\dfrac{\kappa h_1(x)}{4} + 2h_0(x) \right)c\, e^{-\int_0^x h(\sigma)d\sigma} = 0.$

Introducing the function

$$(4.19) \qquad \hat{h}(x) := \frac{\kappa h_1(x)}{4} + 2h_0(x),$$

the differential equation (4.18) takes the form

$$(4.20) \qquad \Psi_0' - h_2(x)\Psi_0 + c\hat{h}(x)e^{-\int_0^x h(\sigma)d\sigma} = 0.$$

Its general solution reads

$$(4.21) \quad \Psi_0(x) = e^{\int_0^x h_2(\sigma)d\sigma}\left(d - c\int_0^x \hat{h}(\sigma)e^{-\int_0^\sigma \tilde{h}(\tau)d\tau}d\sigma\right), \quad \Psi_0(0) = d,$$

where

$$(4.22) \qquad \tilde{h}(x) \equiv h(x) + h_2(x)$$

and d is any real constant. If we substitute the function $\Psi_2(x)$ defined in (4.16) and the function $\Psi_0(x)$ defined in (4.21) into (4.17) we get the relation

$$(4.23) \qquad \begin{aligned} &c\left(-h_2(x) + \frac{\kappa^2 h_4(x)}{4} + \frac{3h_1(x)}{\kappa}\right)e^{-\int_0^x \tilde{h}(\sigma)d\sigma} \\ &\equiv h_4(x)\left(d - c\int_0^x \hat{h}(\sigma)e^{-\int_0^\sigma \tilde{h}(\tau)d\tau}d\sigma\right). \end{aligned}$$

That means, the functions Ψ_0, Ψ_2, and Ψ_1 defined in (4.21), (4.16) and (4.11), respectively, satisfy the equations (4.5)–(4.8) with $k = -1/2$ only if the relation (4.23) is fulfilled. This relation represents a restriction for the coefficient functions h_0, h_1, h_2 and h_4.

In order to guarantee that Ψ defined in (4.2) is a Dulac-Cherkas function we have to require that the function Φ_0 defined in (3.6) satisfies (3.7). Thus, we have the following result.

Theorem 4.1. *Consider system (4.1) under the assumptions (A_1) and (A_2). Additionally we suppose*

$(\mathbf{A_3})$. *There are constants c, d, κ and the functions h_0, h_1, h_2, h_4 are such that*
    (i). *the function*

$$(4.24) \qquad \Phi_0(x) := \Psi_1(x)h_0(x) - \frac{1}{2}\Psi_0(x)h_1(x)$$

    satisfies

$$(4.25) \qquad \Phi_0(x) \geq 0 \quad (\Phi_0(x) \leq 0) \quad \forall x \in R$$

and vanishes only in finitely many points, where, Ψ_1 is defined by (4.11),(4.16), (4.15), and Ψ_0 is defined by (4.21), (4.19), (4.22),

(ii). *the relation (4.23) is valid for $x \in R$, where $\hat{h}$ and $\tilde{h}$ are defined in (4.19) and (4.22), respectively.*

Then system (4.1) has at most one limit cycle in R^2. If system (4.1) has a limit cycle, then it is hyperbolic.

By Corollary 2.1, the existence of a limit cycle under the assumptions of Theorem 4.1 requires that the set $\mathcal{W}$ defined by

$$\Psi_0(x) + \Psi_1(x)y + \Psi_2(x)y^2 = 0$$

contains an oval surrounding the origin. That means especially that the quadratic equation

$$\Psi_0(0) + \Psi_1(0)y + \Psi_2(0)y^2 = 0$$

must have negative and positive roots. By (4.21), (4.11), (4.16) it holds

$$\Psi_0(0) = d, \Psi_1(0) = \frac{c\kappa}{2}, \Psi_2(0) = c.$$

Thus, we have

$$y_{1,2} = -\frac{\kappa}{4} \pm \sqrt{\frac{\kappa^2}{16} - \frac{d}{c}},$$

and the following result is valid.

Theorem 4.2. *Assume the hypotheses $(A_1) - - (A_3)$ are satisfied. Additionally we suppose*

$$dc > 0.$$

Then system (4.1) has no limit cycle in R^2.

In what follows we consider (4.1) under the additional assumption

(4.26) $$\tilde{h}(x) := h(x) + h_2(x) \equiv 0.$$

In that case we have by (4.11), (4.16), and (4.21)

(4.27) $$\Psi_1(x) := \frac{c\kappa}{2} e^{\int_0^x h_2(\sigma)d\sigma},$$

(4.28) $$\Psi_0(x) := e^{\int_0^x h_2(\sigma)d\sigma} \left(d - c \int_0^x \hat{h}(\sigma)d\sigma \right).$$

Substituting these relations into (4.24) we get

(4.29) $$\Phi_0(x) = \frac{c}{2} e^{\int_0^x h_2(\sigma)d\sigma} \left(\kappa h_0(x) - h_1(x) \left[\frac{d}{c} - \int_0^x \hat{h}(\sigma)d\sigma \right] \right).$$

Thus, introducing the function

(4.30) $$\tilde{\Phi}_0(x) := \kappa h_0(x) - h_1(x)\Big[\frac{d}{c} - \int_0^x \hat{h}(\sigma)d\sigma\Big]$$

we have

(4.31) $$\Phi_0(x) = \frac{c}{2}\, e^{\int_0^x h_2(\sigma)d\sigma}\, \tilde{\Phi}_0(x),$$

and the inequality $\Phi_0(x) \geq 0 \quad (\Phi_0(x) \leq 0)$ fulfilled, if it holds

(4.32) $$\tilde{\Phi}_0(x) \geq 0 \quad (\tilde{\Phi}_0(x) \leq 0).$$

We note that the assumption (4.26) implies

(4.33) $$h_2(x) \equiv \frac{3}{16}\kappa^2 h_4(x) + \frac{3h_1(x)}{2\kappa}.$$

Taking into account (4.26),(4.33) and (4.9), relation (4.23) takes the form

(4.34) $$\frac{\kappa^2}{16} - \frac{d}{c} + \frac{3h_1(x)}{2h_4(x)\kappa} \equiv -\int_0^x \Big(\frac{\kappa h_1(\sigma)}{4} + 2h_0(\sigma)\Big)d\sigma.$$

Using this relation we obtain from (4.30)

(4.35) $$\tilde{\Phi}_0(x) \equiv \kappa h_0(x) - h_1(x)\Big(\frac{\kappa^2}{16} + \frac{3h_1(x)}{2\kappa h_4(x)}\Big).$$

Hence, analogously to Theorem 4.1 and Theorem 4.2 we have

Theorem 4.3. *Consider system (4.1) under the assumptions (A_1) and (A_2). Additionally we suppose:*
There are real constants c, d, κ and functions h_0, h_1, h_2, h_4 such that

  (i). *the relations (4.33) and (4.34) are valid for $x \in R$.*
  (ii). *the function $\tilde{\Phi}_0$ defined in (4.35) is positive or negative semi-definite on R and vanishes only in finitely many points.*

Then system (4.1) has at most one limit cycle in R^2. If system (4.1) has a limit cycle, then it hyperbolic.
If we additionally suppose $dc > 0$, then system (4.1) has no limit cycle in R^2.

  For the following we additionally assume

(4.36) $$h_4(x) \equiv \lambda \neq 0.$$

If we substitute (4.36) and (4.19) into (4.34) we get the integral equation

$$(4.37) \qquad \frac{\kappa^2}{16} - \frac{d}{c} + \frac{3h_1(x)}{2\kappa\lambda} = -\int_0^x \left[\frac{\kappa h_1(\sigma)}{4} + 2h_0(\sigma)\right]d\sigma$$

which is equivalent to the initial value problem

$$(4.38) \qquad \begin{aligned} & h_1' + \frac{\kappa^2\lambda}{6}h_1 + \frac{4\kappa\lambda}{3}h_0(x) = 0, \\ & h_1(0) = \frac{2\kappa\lambda}{3}\left(\frac{d}{c} - \frac{\kappa^2}{16}\right). \end{aligned}$$

The initial value problem (4.38) can be used to determine $h_1(x)$ as a function of $h_0(x)$. Its explicit solution reads

$$(4.39) h_1(x) = e^{-\frac{\kappa^2\lambda x}{6}}\left[\frac{2\kappa\lambda}{3}\left(\frac{d}{c} - \frac{\kappa^2}{16}\right) - \frac{4\kappa}{3}\int_0^x e^{\frac{\kappa^2\lambda\sigma}{6}}h_0(\sigma)d\sigma\right].$$

Thus, we get from Theorem 4.3

Theorem 4.4. *Consider system (4.1) under the hypotheses $(A_1), (A_2)$. Additionally we assume:*
The functions h_1, h_2, h_4 are defined by (4.33),(4.39)(4.36). There are real constants c, d, κ, λ and the function h_0 is such that the function $\tilde{\Phi}_0$ defined in (4.35) is positive or negative semidefinite on R and vanishes only in finitely many points.
Then system (4.1) has at most one limit cycle in R^2. If system (4.1) has a limit cycle, then it is hyperbolic.
If we additionally require $dc > 0$, then system (4.1) has no limit cycle in R^2.

Taking into account assumption (A_1) and using (4.30) we get from (4.31)

$$\Phi_0(0) = -\frac{d}{2}\, h_1(0) = -\frac{d\kappa\lambda}{3}\left(\frac{d}{c} - \frac{\kappa^2}{16}\right).$$

Thus, we have the following corollary.

Corollary 4.1. *Under the assumptions of Theorem 4.4 and under the additional condition*

$$d\kappa\lambda\left(\frac{d}{c} - \frac{\kappa^2}{16}\right) \neq 0$$

there exist an interval I containing the origin such that system (4.1) has in the region $I \times R$ at most one limit cycle.

The following example shows that the interval I can coincide with the real axis.

We consider system (4.1) under the assumptions $(A_1)-(A_4)$ of Theorem 4.4. As function h_0 we choose

$$h_0(x) \equiv -x.$$

Then, the function h_1 reads

(4.40) $\qquad h_1(x) = e^{-\frac{\kappa^2 \lambda x}{6}}\left[h_1(0) + \frac{48}{\kappa^3}\right] + \frac{8}{\kappa}\left(x - \frac{6}{\kappa^2}\right).$

Setting

(4.41) $\qquad\qquad\qquad\qquad \kappa = \lambda = c = d = 1$

we have

(4.42) $\qquad\qquad\qquad h_1(x) = \frac{389}{8}e^{-\frac{x}{6}} + 8(x - 6).$

The following relations can be easily verified

(4.43) $\qquad\qquad\qquad \lim_{x\to\pm\infty} h_1(x) = +\infty,$

(4.44) $\qquad h_1'(x) = -\frac{389}{48}e^{-\frac{x}{6}} + 8, \quad h_1''(x) = \frac{389}{288}e^{-\frac{x}{6}}.$

Hence, we have

(4.45) $\qquad\qquad\qquad h_1''(x) > 0 \quad \forall x \in R,$

and we can conclude that $h_1(x)$ has a unique minimum at $x = x_m$. From

$$h_1'(x_m) = -\frac{389}{48}e^{-\frac{x_m}{6}} + 8 = 0$$

and from (4.42) we get

$$x_m = -6\ln\frac{384}{389} > 0, \quad h_1(x_m) = 8x_m > 0.$$

Thus, we have

(4.46) $\qquad\qquad\qquad h_1(x) > 0 \quad \forall x \in R.$

Especially, we obtain from (4.42) and (4.43)

(4.47) $\qquad\qquad\qquad h_1(0) = 5/8, \quad h_1'(0) = -5/48,$

(4.48) $\qquad h_1\left(\frac{-1}{2}\right) = \frac{1}{8}\left(389e^{\frac{1}{12}} - 416\right) > 0.85,$

(4.49) $\qquad h_1'\left(\dfrac{-1}{2}\right) = \dfrac{1}{48}\left(-389e^{\frac{1}{12}} + 384\right) < -0.80.$

From (4.35) and (4.41) we obtain

(4.50) $\qquad \tilde{\Phi}_0(x) = -x - h_1(x)\left(\dfrac{1}{16} + \dfrac{3}{2}h_1(x)\right),$

(4.51) $\qquad \tilde{\Phi}_0'(x) = -1 - \dfrac{1}{16}h_1'(x) - 3h_1(x)h_1'(x),$

(4.52) $\qquad \tilde{\Phi}_0''(x) = -\dfrac{1}{16}h_1''(x) - 3h_1(x)h_1''(x) - 3(h_1'(x))^2.$

From (4.42), (4.43), and (4.50) we get

$$\lim_{x \to \pm\infty} \tilde{\Phi}_0(x) = -\infty.$$

From (4.45), (4.46), and (4.52) it follows $\tilde{\Phi}_0''(x) < 0$, that is $\tilde{\Phi}_0(x)$ has a unique maximum at $x = x_M$. By (4.51) and (4.47) we have $\tilde{\Phi}_0'(0) = -613/768 < 0$, that is $x_M < 0$. By (4.48), (4.49) we have

$$-3h_1\left(-\dfrac{1}{2}\right)h_1'\left(-\dfrac{1}{2}\right) > 2.04$$

such that by (4.51) it holds $\tilde{\Phi}_0'(-\frac{1}{2}) > 0$, that is $x_M > -1/2$. By (4.50) and our results about $h_1(x)$ we have

$$\tilde{\Phi}_0(x) < 0.5 - \dfrac{3}{2}\left(\dfrac{5}{8}\right)^2 < 0 \quad \text{for} \quad -1/2 \le x \le 0$$

such that all conditions of Theorem 4.4 are fulfilled and we have the result

Corollary 4.2. *The system*

(4.53) $\quad \dfrac{dx}{dt} = y, \quad \dfrac{dy}{dt} = -x + h_1(x)y + h_2(x)y^2 + y^3 + y^4$

with

$$h_1(x) \equiv \dfrac{389}{8}e^{-\frac{x}{6}} + 8(x-6), \ h_2(x) \equiv \dfrac{3}{16} + \dfrac{3}{2}h_1(x)$$

has no limit cycle in R^2.

5. Construction of a generalized Liénard system having a unique limit cycle by means of a Dulac-Cherkas function

We consider the generalized Liénard system (1.4) with $l = 5$

(5.1)
$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = h_0(x) + h_1(x)y + h_2(x)y^2 + h_3(x)y^3 + h_4(x)y^4 + h_5(x)y^5.$$

Our aim is to determine the functions h_i, $0 \leq i \leq 5$, by means of a Dulac-Cherkas function Ψ such that (5.1) has a unique limit cycle surrounding the origin. We note that the following algorithm is not restricted to the case $l = 5$.

For Ψ we use the ansatz

(5.2)
$$\Psi(x, y) = \Psi_0(x) + \Psi_1(x)y + \Psi_2(x)y^2,$$

where we assume

(5.3)
$$\Psi_2(x) \neq 0 \quad \forall\, x \in R.$$

First we determine the constant k and the functions h_i such that the equations (3.5) hold, that is, the function Φ defined in (3.1) has the form

(5.4)
$$\Phi(x, y) \equiv \Phi_0(x) = \Psi_1(x)h_0(x) + k\Psi_0(x)h_1(x).$$

From (3.2), (5.2), (5.3) and (3.5) we get

(5.5)
$$k = -\frac{2}{5},$$

(5.6)
$$h_4(x) = \frac{5\Psi_1(x)h_5(x)}{2\Psi_2(x)},$$

(5.7)
$$h_3(x) = \frac{5}{4\Psi_2(x)}\left(2\Psi_0(x) + \frac{3\Psi_1^2(x)}{\Psi_2(x)}\right)h_5(x),$$

(5.8)
$$h_2(x) = \frac{5}{6\Psi_2(x)}\left[\left(\frac{9}{2}\Psi_0(x) + \frac{3\Psi_1^2(x)}{8\Psi_2(x)}\right)\frac{\Psi_1(x)h_5(x)}{\Psi_2(x)} - \Psi_2'(x)\right],$$

$$(5.9) \quad h_1(x) = \frac{5}{8\Psi_2(x)}\Big[-\Psi_1'(x) + \frac{\Psi_1(x)\Psi_2'(x)}{6\Psi_2(x)}$$

$$+ \Big(3\Psi_0^2(x) + \frac{3\Psi_0(x)\Psi_1^2(x)}{2\Psi_2(x)} - \frac{\Psi_1^4(x)}{16\Psi_2^2(x)}\Big)\frac{h_5(x)}{\Psi_2(x)}\Big],$$

$$(5.10) \quad h_0(x) = \frac{1}{2\Psi_2(x)}\Big[-\Psi_0'(x) + \frac{3\Psi_1(x)\Psi_1'(x)}{8\Psi_2(x)}$$

$$- \frac{\Psi_2'(x)}{\Psi_2(x)}\Big(\frac{\Psi_1^2(x)}{16\Psi_2(x)} + \frac{2\Psi_0(x)}{3}\Big)$$

$$+ \Big(\frac{15}{8}\Psi_0^2(x) - \frac{5\Psi_0(x)\Psi_1^2(x)}{16\Psi_2(x)} + \frac{3\Psi_1^4(x)}{128\Psi_2^2(x)}\Big)\frac{\Psi_1(x)h_5(x)}{\Psi_2^2(x)}\Big].$$

We note that the continuous function h_5 can be chosen arbitrarily. In order to guarantee that the origin is an equilibrium point we have to assume

$$(5.11) \qquad\qquad h_0(0) = 0.$$

From (5.4), (5.5), (5.9), and (5.10) we obtain

(5.12)
$$\Phi_0(x) = \frac{3h_5(x)}{4\Psi_2^2(x)}\Big(-\Psi_0^3(x) + \frac{3\Psi_0^2(x)\Psi_1^2(x)}{4\Psi_2(x)} - \frac{3\Psi_0(x)\Psi_1^4(x)}{16\Psi_2^2(x)} + \frac{\Psi_1^6(x)}{64\Psi_2^3(x)}\Big)$$

$$+ \frac{1}{2\Psi_2(x)}\Big(-\Psi_1(x)\Psi_0'(x) + \frac{\Psi_1'(x)\Psi_0(x)}{2} + \frac{3\Psi_1'(x)\Psi_1^2(x)}{8\Psi_2(x)}$$

$$- \frac{\Psi_2'(x)\Psi_1^3(x)}{16\Psi_2^2(x)} - \frac{3\Psi_2'(x)\Psi_1(x)\Psi_0(x)}{4\Psi_2(x)}\Big).$$

Thus, under the condition

$$(5.13) \qquad\qquad \Phi_0(x) > 0(< 0) \qquad \text{for} \quad x \in R,$$

the function $\Psi(x, y)$ with the form (5.2) is a Dulac-Cherkas function of system (5.1) in R^2.

By Corollary 2.1, the existence of a limit cycle Γ of system (5.1) requires that the set

$$\mathcal{W} := \{(x, y) \in R^2 : \Psi_0(x) + \Psi_1(x)y + \Psi_2(x)y^2 = 0\}$$

contains an oval $\mathcal{O}$ surrounding the origin.

If we choose

$$(5.14) \qquad \Psi_2(x) \equiv 1, \ \Psi_1(x) \equiv ax, \Psi_0(x) \equiv bx^2 - c$$

and assume that the constants a, b, c satisfy

$$(5.15) \qquad b > \frac{a^2}{4} > 0, \quad c > 0,$$

then the equation $\Psi(x, y) = 0$ describes a unique oval $\mathcal{O}$ surrounding the origin. For the following we put

$$(5.16) \qquad b = c = 1, \ a = 10^{-2}.$$

In that case the corresponding oval $\mathcal{O}_1 := \{(x, y) \in R^2 : x^2 + 0.01xy + y^2 = 1\}$ represents a slightly perturbed unit circle. In order to ensure that Φ_0 has constant sign and that system (5.1) has a unique limit cycle we have to choose the function h_5 in an appropriate way. For the following, we set

$$(5.17) \qquad h_5(x) \equiv 10^{-3}.$$

Thus, using (5.16) and (5.17), we get from (5.6)–(5.10)

(5.18)

$$h_0(x) \equiv -\frac{31.9991}{32} x - \frac{11.9999 \times 10^{-4}}{64} x^3 + \frac{23.999600003 \times 10^{-4}}{256} x^5,$$

that is, h_0 satisfies (5.11). Furthermore we have

(5.19)

$$h_1(x) \equiv -\frac{7 \times 10^{-2}}{16} - \frac{12.0003 \times 10^{-2}}{32} x^2 + \frac{48.00239999 \times 10^{-2}}{256} x^4,$$

$$(5.20) \qquad h_2(x) \equiv -\frac{3 \times 10^{-4}}{8} x + \frac{12.0001 \times 10^{-4}}{32} x^3,$$

$$(5.21) \qquad h_3(x) \equiv -\frac{10^{-2}}{4} + \frac{4.0003 \times 10^{-2}}{16} x^2,$$

$$(5.22) \qquad h_4(x) \equiv \frac{10^{-4}}{4} x.$$

Thus, in case that the functions h_0–h_5 are determined by the relations (5.17)–(5.22), system (5.1) has three equilibria: a stable focus at the origin and saddles at the points $(\pm 18.0997; 0)$, moreover, any limit cycle

must surround the origin.

The corresponding function Φ_0 reads

$$\Phi_0(x) \equiv -\frac{56}{125} - \frac{1559.961}{625}\,x^2 + \frac{14.399280009}{25}\,x^4 - 0.191985600359997\,x^6.$$

Now we show that Φ_0 is negative for any $x \in R$. Its derivative

$$\Phi_0'(x) = -\frac{15599.61}{3125}\,x + \frac{1439.9280009}{625}\,x^3 - \frac{5.75956801079991}{5}\,x^5$$

has unique real root at $x = 0$, where $\Phi_0'(x)$ is positive for $x < 0$ and negative for $x > 0$. Taking into account $\Phi_0(0) = -7/4000$ we can conclude that $\Phi_0(x)$ is negative for any x. Thus, the function

$$\Psi(x,y) \equiv x^2 - 1 + 0.01xy + y^2$$

is a Dulac-Cherkas function for system (5.1) in R^2.

In what follows we construct a Bendixson annular region containing at least one limit cycle. As inner boundary we can use the oval $\mathcal{O}_1$ which intersect all trajectories of (5.1) transversally by Theorem 2.1, they enter the region bounded by $\mathcal{O}_1$ for increasing t. As outer boundary we can choose the circle $x^2 + y^2 = 9$. It can be verified that a trajectory of (5.1) which meets this circle intersects it transversally, and leaves the annulus for increasing t. Thus we have the following result

Theorem 5.1. *System (5.1) with h_i defined by (5.17) - (5.22) has a unique limit cycle which is hyperbolic and unstable.*

6. Acknowledgement

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(Received October 8, 2010)

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