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Weak solvability of a hyperbolic integro-differential equation with integral condition*†

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Abstract

By using the method of semidiscretization in time also called the Rothe's method, we prove the existence, uniqueness of the weak solution and its continuous dependence upon data, for a hyperbolic integro-differential equation with initial, Neumann and integral conditions.

1 Introduction

The study of boundary value problems with non-local conditions has known a great development in the recent years. This is due to the importance of non-local conditions appearing in the mathematical modeling of various phenomena of physics, ecology, biology, etc. It is the case when the values of function on the boundary are related to values inside the domains or when the direct measurements on the boundary are not possible. Several methods are used to solve such problems as functional methods, approximation methods, a priori estimates...

The importance of approximation methods is that they don't only prove the existence and uniqueness of the solution but they also allow the construction of algorithms for numerical solutions. These methods as the Galerkin method and the method of discretization in time also called Rothe's method, are very effective tools in the study of the approximate solution and its convergence to the solution of problems [1-13]. In general it is difficult to find the exact solution in such cases, so the approximation methods provide other ways to find approximate solutions.

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The objective of this work is to apply Rothe's method to the study of an integro-differential equation with integral conditions. Several results have based on this method in the investigation of different type of equations with integral conditions like [3-6,9,10,13].

By combining the ideas from [3,9,10], we apply Rothe's method, to prove the existence, uniqueness of the weak solution and its continuous dependence upon data. Using time discretization, the posed problem is approximated by corresponding elliptic problem by means of which an approximate solution for the original evolution problem is constructed.

More precisely, we are devoted to prove, in non classical function space, the weak solvability of non-linear hyperbolic integro-differential equation

$$\tau\frac{\partial^2 v}{\partial t^2}+a\frac{\partial v}{\partial t}-b\frac{\partial^2 v}{\partial x^2}=f\left(x,t\right)+\int_0^t a\left(t-s\right)k\left(s,v\left(x,s\right)\right)ds, \tag{1.1}$$

for all $(x,t)\in(0,1)\times I$, subject to initial and Neumann conditions

$$v\left(x,0\right)=v_0\left(x\right),\frac{\partial v}{\partial t}\left(x,0\right)=v_1\left(x\right),\frac{\partial v}{\partial x}\left(0,t\right)=G\left(t\right) \tag{1.2}$$

and integral condition

$$\int_0^1 v\left(x,t\right)dx=E\left(t\right). \tag{1.3}$$

Where f,v_1,v_0,G,E are given functions and T,τ,a,b are positive constant such that $\tau>0$, $a\geq1$ and $b>0$.

Equation (1.1) represents the second order telegraph equation with constant coefficients and models mixture between diffusion and wave propagation by introducing a term that accounts for effects of finite velocity to standard heat or mass transport equation. It is also used in signal analysis for transmission and propagation of electrical signals [14]. Recently, telegraph equation becomes more suitable than ordinary diffusion equation in modeling the reaction diffusion for such branches of sciences [15]. Equation (1.1) have been extensively studied for initial and Dirichlet conditions by numerical methods, but in all these work it was assumed that the right hand side of (1.1) is function of the form $f\left(x,t\right)$ or $f\left(x,t,u\right)$ subject to some conditions, whereas in this work, the second member is a Volterra operator of the form $\int_0^t a\left(t-s\right)k\left(s,v\left(x,s\right)\right)ds$. Many mathematical formulations of physical phenomena contain integro-differential equations, these equations arise in many fields like fluid dynamics, biological models and chemical kinetics. Integro-differential equations are usually difficult to solve analytically so it is required to obtain an efficient approximate solution.

In the present work, which can be viewed as a continuation of [9], the presence of integral conditions (1.3) is the source of some great complications when applying the standard Rothe method, and to avoid this difficulties we study the problem in an appropriate nonclassical function space that is Bouziani space that we have denoted by B.

Using the transformation

$$u(x,t) = \upsilon(x,t) - r(x,t), (x,t) \in (0,1) \times I,$$

the equivalent problem of (1.1) – (1.3) can be written as:

$$\tau\frac{\partial^2 u}{\partial t^2} + a\frac{\partial u}{\partial t} - b\frac{\partial^2 u}{\partial x^2} = F(x,t) + \int_0^t a(t-s)\,k(s,u(x,s))\,ds \tag{1.4}$$

$$u(x,0) = U_0(x), \frac{\partial u}{\partial t}(x,0) = U_1(x) \tag{1.5}$$

$$\frac{\partial u}{\partial x}(0,t) = 0 \tag{1.6}$$

$$\int_0^1 u(x,t)\,dx = 0 \tag{1.7}$$

where

$$r(x,t) = G(t)\left(x - \frac{1}{2}\right) + E(t),$$

and

$$F(x,t) = f(x,t) - \tau\frac{\partial^2 r}{\partial t^2} - a\frac{\partial r}{\partial t} \tag{1.8}$$

$$\begin{aligned} U_0(x) &= \upsilon_0(x,t) - r(x,0), \\ U_1(x) &= \upsilon_1(x,t) - \frac{dr}{dt}(x,0). \end{aligned}$$

To apply Rothe's method, we proceed as follows:

We divide the time interval I into n subintervals $[t_{j-1},t_j]$, $j = 1....n$, where $t_j = j.h$ and the length $h = \frac{T}{n}$, we denote $u_j = u_j(x) = u_j(x,jh)$ the approximation of u, then we replace $\frac{\partial^2 u}{\partial t^2}$ and $\frac{\partial u}{\partial t}$ at each point $t = t_j$, $j = 1....n$, by the difference quotients respectively $\delta^2 u_j = \frac{\delta u_j - \delta u_{j-1}}{h}$ and $\delta u_j = \frac{u_j - u_{j-1}}{h}$. Consequently (1.4) becomes

$$\tau\delta^2 u_j + a\delta u_j - b\frac{\partial^2 u_j}{\partial x^2} = F_j + h\sum_{i=0}^{j-1} a_{ji}k_i \tag{1.9}$$

where $F_j = F(x,t_j)$., $a_{ji} = a(t_j - t_i)$, and $k_i = k(t_i,u_i)$ .Thereafter, we get a system of n differential equations in x with the unknown functions $u_j(x)$:

$$-b\frac{\partial^2 u_j}{\partial x^2} + \left(\frac{\tau + ah}{h^2}\right)u_j = \mathbf{F}_j \tag{1.10j}$$

$$\frac{\partial u_j}{\partial x}(0) = 0, \int_0^1 u_j(x)\,dx = 0 \tag{1.11j}$$

where

$$\mathbf{F}_j = F_j + \left(\frac{2\tau + ah}{h^2}\right) u_{j-1} - \frac{\tau}{h^2} u_{j-2} + h \sum_{i=0}^{j-1} a_{ji} k_i \tag{1.12}$$

$$u_0(x) = U_0(x), \quad u_{-1}(x) = U_0(x) - hU_1(x).$$

Using these solutions u_j, we construct piecewise linear (Rothe's) functions defined by

$$u^n(x,t) = u_{j-1} + \delta u_j (t - t_{j-1}), \quad t \in [t_{j-1}, t_j], j = 1....n.$$

$$\overline{u}^n(t) = \begin{cases} u_j, & t \in [t_{j-1}, t_j], \quad j = 1....n \\ U_0, & t \in [-h, 0] \end{cases}$$

Then we prove that $u^n(x,t)$ converges in some appropriate sense to the solution of (1.4).-(1.7)

2 Notation, function spaces and assumptions

Let $H = L^2(0,1)$ be the usual space of Lebesgue square integrable real functions on $(0,1)$ whose inner product and norm will be denoted by $(\ ,\ )$ and $\|\ .\|$ respectively. Let $H^2(0,1)$ be the real second order Sobolev space on $(0,1)$ with the norm $\|\ \|_{H^2(0,1)}$. Let B (Bouziani space) be the completion of $C_0(0,1)$, the space of real continuous functions with compact support in $(0,1)$,whose inner product and norm are defined respectively by $(u,v)_B = \int_0^1 \Im_x u.\Im_x v dx$, $\|v\|_B = \sqrt{(v,v)_B}$, where $\Im_x u = \int_0^x \phi(\xi)\, d\xi$,$\forall x \in (0,1)$.

$C(0,1)$ is the set of all continuous functions $v : I \to X$ with

$$\|v\|_{C(0,1)} = \max_{t \in I} \|v(t)\|_X.$$

$C^{0.1}(0,1)$ is the set of all Lipschitz continuous functions $v : I \to X$

$C^{1.1}(0,1)$ is the set of all $v \in C^{0.1}(0,1)$ such that $\frac{dv}{dt} \in C^{0.1}(0,1)$

We denote by V the Hilbert space $V = \left\{\phi \in L^2(0,1); \int_0^1 \phi(x)\, dx = 0\right\}$.

Now we give the assumptions:

H_1) $f(t) \in H$ and the condition $\|f(t,w) - f(',w')\|_B \le c_0 |t - t'|$ holds for some positive constant c_0.

H_2) $U_0, U_1 \in H^2(0,1)$.

H_3) $\frac{\partial U_0}{\partial x}(0) = 0$ and $\int_0^1 U_0(x)\, dx = 0$.

H_4) The mapping $K : I \times B \to H$ is continuous in both variables and satisfies

$$\|k(t,u)\|_B \le \|u(t)\|_B$$

$$\|k(t,u) - k(t,v)\| \le L(t) \|u(t) - v(t)\|_B$$

for $t \in I$ and all $u, v \in V$,where $L \in L^1(I)$ is nonnegative.

H_5) The function $a : I \to \mathbb{R}$ is Lipschitz continuous:

$$|a(t) - a(t')| \leq c_1 |t - t'| .$$

Definition 2.1. *By a weak solution of problem (1.4)-(1.7) we mean a function $u : I \to H$ such that:*
(1) $u \in C^{0,1}(I, V)$
(2) $\frac{du}{dt} \in L^{\infty}(I, V) \cap C^{0,1}(I, B)$ *and* $\frac{d^2u}{dt^2} \in L^{\infty}(I, B)$.
(3) $u(0) = U_0$ *in* V *and* $\frac{du}{dt}(0) = U_1$ *in* B.
(4) *For all all* $\phi \in V$ *and* $t \in I$, *the identity*

$$\tau \left(\frac{\partial^2 u}{\partial t^2}, \phi \right)_B + a \left(\frac{\partial v}{\partial t}, \phi \right)_B - b(u, \phi) = (f, \phi)_B + (k(t), \phi)_B \tag{2.1}$$

holds.

3 Discretization schemes and a priori estimates

Theorem 3.1. *For all $n \geq 1$, and for $j = 1, ...n$, the problem (1.10j)-(1.11j) possesses a unique solution u_j in $H^2(0,1)$.*

Proof. Similarly as in [1] the Lax Milgram lemma guarantees the existence and uniqueness of a solution $u_j \in H^2(0,1)$, $\forall j = 1....n$.

Lemma 3.2. *Assume that the assumptions (H_1) and (H_5) hold. Then there exists positive constant C such that for all $n \geq n_0$, the solutions u_j of problems (1.10j)-(1.11j), $j = 1....n$; satisfy*

$$\left\| \delta^2 u_j \right\|_B + \left\| \delta u_j \right\|_B + \left\| u_j \right\| \leq C \tag{3.1}$$

Proof. For convenience, we shall denote by c a generic constant, i.e., the constant $kc, e^{kc}, ...$etc will be replace by c were k denote a positive constant independent of j, h and n. Let $n \geq n_0$, $\phi \in V$ and $x = 1$, then integrating by parts yields

$$\Im_1^2 \phi = \int_0^1 (1 - \xi) \phi(\xi) d\xi = \int_0^1 \phi(\xi) d\xi - \int_0^1 \xi \phi(\xi) d\xi = 0. \tag{3.2}$$

Now, multiplying equation (1.9) by $\Im_x^2 \phi$ for all $j = 1, ...n$, and integrating over $(0,1)$, we get

$$\tau \int_0^1 \delta^2 u_j(x) \Im_x^2 \phi dx + a \int_0^1 \delta u_j(x) \Im_x^2 \phi dx - b \int_0^1 \frac{\partial^2 u_j}{\partial x^2}(x) \Im_x^2 \phi dx$$
$$= \int_0^1 \left(F_j(x) + h \sum_{i=0}^{j-1} a_{ji} k_i \right) \Im_x^2 \phi dx, \tag{3.3}$$

using (3.2) and integrating by parts then (3.3) becomes:

$$\tau\left(\delta^2 u_j,\phi\right)_B + a\left(\delta u_j,\phi\right)_B + b\left(u_j,\phi\right) \tag{3.4j}$$

$$= \left(F_j + h\sum_{i=0}^{j-1} a_{ji}k_i,\phi\right)_B, \forall\phi\in V, \forall j = 1...,n.$$

Substituting $\phi = \delta u_j \in V$ in (3.4j) and using some elementary identities like

$$2\left(v, v-w\right) = \|v\|^2 - \|w\|^2 + \|v-w\|^2 \tag{3.5}$$

then applying Cauchy inequality for $\varepsilon = 1$, it follows

$$\tau\left\|\delta u_j\right\|_B^2 + b\left\|u_j\right\|^2 + 2h\left[a - \left(1 + T\max|a(t)|\right)\right]\left\|\delta u_j\right\|_B^2 \leq \tag{3.6j}$$

$$2h\left\|F_j\right\|_B^2 + b\left\|u_{j-1}\right\|^2 + \tau\left\|\delta u_{j-1}\right\|_B^2 + 2h^2\max|a(t)|\sum_{i=0}^{j-1}\left\|k_i\right\|_B^2$$

For $a \geq C_5 = 1 + T\max|a(t)|$, we obtain:

$$\tau\left\|\delta u_j\right\|_B^2 + b\left\|u_j\right\|^2 \leq \tag{3.7j}$$

$$2h\left\|F_j\right\|_{C(I,B)}^2 + 2h^2\max|a(t)|\sum_{i=0}^{j-1}\left\|k_i\right\|_B^2 + \tau\left\|\delta u_{j-1}\right\|_B^2 + b\left\|u_{j-1}\right\|^2$$

$$\leq hc + h^2c\sum_{k=1}^{j}\left\|u_i\right\|_B^2 + \tau\left\|\delta u_{j-1}\right\|_B^2 + b\left\|u_{j-1}\right\|^2.$$

Choose a positive integer n_0 such that $\frac{cT}{n_0} < 1$. Then for $n \geq n_0$ we get

$$(1 - Ch)\left[\tau\left\|\delta u_j\right\|_B^2 + b\left\|u_j\right\|^2\right] \leq \tag{3.8}$$

$$\leq \left(1 + Ch^2\right)\left[\tau\left\|\delta u_{j-1}\right\|_B^2 + b\left\|u_{j-1}\right\|^2\right] + h^2c\sum_{k=1}^{j}b\left\|u_i\right\|_B^2 + ch.$$

Iterating to arrive at

$$(1 - Ch)^j\left[\tau\left\|\delta u_j\right\|_B^2 + b\left\|u_j\right\|^2\right] \leq \left(1 + jCh^2\right)^j\left[\tau\left\|\delta u_0\right\|_B^2 + b\left\|U_0\right\|^2\right] + jCh \tag{3.9j}$$

Hence we get

$$\tau\left\|\delta u_j\right\|_B^2 + b\left\|u_j\right\|^2 \leq c$$

which implies

$$\left\|\delta u_j\right\|_B^2 + \left\|u_j\right\|^2 \leq \frac{c}{\min\left(\tau, b\right)}$$

Next, we will estimate$\left\|\delta^2 u_j\right\|_B$, for this ,we consider the difference of relations (3.9j)-(3.9(j-1)) for $j = 2.....,n$ and $\phi = \delta^2 u_j \in V$, we obtain

$$\tau\left(\delta^2 u_j - \delta^2 u_{j-1}, \delta^2 u_j\right)_B + a\left(\delta u_j - \delta u_{j-1}, \delta^2 u_j\right)_B + \tag{3.10}$$

$$b\left(u_j - u_{j-1}, \delta^2 u_j\right) = \left(F_j - F_{j-1}, \delta^2 u_j\right)_B + 2h\sum_{i=0}^{j-2}\left(\left(a_{ji} - a_{j-1i}\right)k_i, \delta^2 u_j\right)_B.$$

using the elementary identities and estimates, it follows

$$\tau\left\|\delta^2 u_j\right\|_B^2 + 2ah\left\|\delta^2 u_j\right\|_B^2 + b\left\|\delta u_j\right\|^2 \le 2\left\|F_j - F_{j-1}\right\|_B\left\|\delta^2 u_j\right\|_B \tag{3.11}$$

$$+\tau\left\|\delta^2 u_{j-1}\right\|_B^2 + b\left\|\delta u_{j-1}\right\|^2 + 2h^2 c_1\sum_{i=0}^{j-2}\left\|k_i\right\|_B\left\|\delta^2 u_j\right\|_B$$

$$+2h\max\left|a(t)\right|\left\|k_{j-1}\right\|_B\left\|\delta^2 u_j\right\|_B.$$

On the other hand, using Cauchy inequality for $\varepsilon = 1$, we get

$$2\left\|F_j - F_{j-1}\right\|_B\left\|\delta^2 u_j\right\|_B \le c_0 h + c_0 h\left\|\delta^2 u_j\right\|_B^2 \tag{3.12}$$

$$2h\max\left|a(t)\right|\left\|k_{j-1}\right\|_B\left\|\delta^2 u_j\right\|_B \le h\max\left|a(t)\right|\left\|u_{j-1}\right\|_B^2 + \tag{3.13}$$

$$h\max\left|a(t)\right|\left\|\delta^2 u_j\right\|_B^2 \le hc + h\max\left|a(t)\right|\left\|\delta^2 u_j\right\|_B^2$$

$$2h^2 c_1\sum_{i=0}^{j-2}\left\|k_i\right\|_B\left\|\delta^2 u_j\right\|_B \le h^2 c_1\left(j-1\right) + h^2 c_1\left(j-1\right)\left\|\delta^2 u_j\right\|_B^2$$

$$\le hc + hc_1 T\left\|\delta^2 u_j\right\|_B^2 \tag{3.14}$$

Substituting (3.12), (3.13) and (3.14) in (3.11) it yields

$$\tau\left\|\delta^2 u_j\right\|_B^2 + h(2a - (c_0 + \max\left|a(t)\right| + Tc_1))\left\|\delta^2 u_j\right\|_B^2 + b\left\|\delta u_j\right\|^2 \le \tag{3.15}$$

$$\tau\left\|\delta^2 u_{j-1}\right\|_B^2 + b\left\|\delta u_{j-1}\right\|^2 + ch + ch\left(\tau\left\|\delta^2 u_{j-1}\right\|_B^2 + b\left\|\delta u_{j-1}\right\|^2\right) +$$

$$ch^2\left(\tau\left\|\delta^2 u_{j-1}\right\|_B^2 + b\left\|\delta u_{j-1}\right\|^2\right) + ch^2\sum_{i=0}^{j-1} b\left\|\delta u_j\right\|_B^2.$$

Choose a positive integer n_0 such that $\frac{cT}{n_0} < 1$. Consequently for $n \ge n_0$and $a \ge (c_0 + \max\left|a(t)\right| + Tc_1)$ we get

$$(1 - Ch)\left[\tau\left\|\delta^2 u_j\right\|_B^2 + b\left\|\delta u_j\right\|^2\right] \le \left(1 + Ch^2\right)\left[\tau\left\|\delta^2 u_{j-1}\right\|_B^2 + b\left\|\delta u_{j-1}\right\|^2\right] \tag{3.16}$$

$$+ch+ch^2\sum_{i=0}^{j-1} b\left\|\delta u_j\right\|_B^2$$

recursively, we obtain

$$(1-Ch)^j\left[\tau\left\|\delta^2 u_j\right\|_B^2+b\left\|\delta u_j\right\|^2\right]\leq\left(1+jCh^2\right)^j\left[\tau\left\|\delta^2 u_0\right\|_B^2+b\left\|\delta u_0\right\|^2\right]+jCh \tag{3.17}$$

from where we derive

$$\left\|\delta^2 u_j\right\|_B+\left\|\delta u_j\right\|\leq C$$

This proves Lemma 3.2■

Corollary 3.3. The functions $u^n(t)$ are lipschitz continuous on I and the sequences $\{u^n(t)\}$ and $\{\overline{u}^n(t)\}$ are bounded in $C(I,B)$ uniformly in n and t :

$$\left\|u^n(t)\right\|\leq C,\left\|\overline{u}^n(t)\right\|\leq C,\left\|\frac{du^n}{dt}(t)\right\|\leq C \tag{3.18}$$

$$\left\|\overline{u}^n(t)-u^n(t)\right\|\leq C\frac{T}{n},\left\|u^n(t)-\overline{u}^n(t-h)\right\|\leq C\frac{T}{n} \tag{3.19}$$

$$\left\|\delta u^n(t)\right\|\leq C,\left\|\overline{\delta u^n}(t)\right\|\leq C,\left\|\frac{d}{dt}\delta u^n(t)\right\|_B\leq C \tag{3.20}$$

$$\left\|\overline{\delta u^n}(t)-\delta u^n(t)\right\|_B\leq C\frac{T}{n},\left\|\delta u^n(t)-\overline{\delta u^n}(t-h)\right\|_B\leq C\frac{T}{n} \tag{3.21}$$

$$\left\|\delta u^n-\frac{du^n}{dt}(t)\right\|_{L^2(I,B)}\leq C\frac{T}{n} \tag{3.22}$$

for all $t\in I$ and $n\geq n_0$.

Proof. The proof is a consequence of Lemma 3.2.

4 Convergence and existence results

For all $n\geq n_0$, we define the sequences $\left\{\overline{F}^n\right\}$ and $\{K^n\}$ of step functions respectively by

$$\overline{F}^n(t)=\begin{cases} F_J, & t\in[t_{j-1},t_j], \quad j=1....n \\ 0, & t\in[-h,0]\end{cases}$$

and

$$\begin{cases} K^n(t)=h\sum_{i=0}^{j-1} a_{ji}k_i \\ K^n(0)=ha_{10}k_0,\end{cases}$$

The variational equation (3.4j) may be written as:

$$\tau\left(\frac{d}{dt}\delta u^n(t),\phi\right)_B + a\left(\delta u^n(t),\phi\right)_B - b\left(\overline{u}^n(t),\phi\right) = \tag{4.1}$$

$$\left(\overline{F}^n(t) + K^n(t),\phi\right)_B, \qquad \forall\phi\in V, t\in I.$$

Theorem 4.1. *Under the assumptions* (H_1) *and* (H_5)*, there exists a function* $u\in C^{0,1}(I,V)$ *such* $\frac{du}{dt}\in L^\infty(I,V)\cap C^{0,1}(I,B)$ *and* $\frac{d^2u}{dt^2}\in L^\infty(I,B)$ *satisfying*

(i) $u^n\to u$ *in* $C(I,V)$.

(ii) $\overline{u}^n(t)\to u(t)$ *in* V *for all* $t\in I$.

(iii) $\delta u^n\to\frac{du}{dt}$ *in* $C(I,B)$.

(iv) $\overline{\delta u}^n(t)\rightharpoonup\frac{du}{dt}(t)$ *in* V *for all* $t\in I$.

(v) $\frac{du^n}{dt}\rightharpoonup\frac{du}{dt}$ *in* $L^2(I,V)$.

(vi) $\frac{d}{dt}\delta u^n\rightharpoonup\frac{d^2u}{dt^2}$ *in* $L^2(I,B)$.

Moreover, the error estimate is

$$\|u^n-u\|_{C(I,V)} + \left\|\delta u^n - \frac{du}{dt}\right\|_{C(I,B)} \le Ch^{\frac{1}{2}} \tag{4.2}$$

for all $n\ge n_0$.

Proof. Let u^n and u^m be the Rothe functions corresponding to the step $h_n=\frac{T}{n}$ and $h_m=\frac{T}{m}$ respectively, with $m>n\ge n_0$. Considering the difference of (4.1) for n and m, with $\phi=\overline{\delta u}^{n,m}=\overline{\delta u}^n-\overline{\delta u}^m\in V$, we get for all $t\in I$.

$$\tau\left(\frac{d}{dt}(\delta u^n(t)-\delta u^m(t)),\overline{\delta u}^{n,m}\right)_B + a\left\|\overline{\delta u}^n(t)-\overline{\delta u}^m(t)\right\|_B^2 \tag{4.3}$$

$$+b\left(\overline{u}^n(t)-\overline{u}^m(t),\overline{\delta u}^{n,m}\right) = \left((\overline{F}^n(t)-\overline{F}^n(t)),\overline{\delta u}^{n,m}\right)_B$$

$$+\left(K^n(t)-K^m(t),\overline{\delta u}^{n,m}\right)_B$$

Similarly as in [1], we obtain

$$\frac{\tau}{2}\frac{d}{dt}\|\delta u^n(t)-\delta u^m(t)\|_B^2 + \frac{b}{2}\frac{d}{dt}\|u^n(t)-u^m(t)\|^2 = \tag{4.4}$$

$$\begin{aligned}
&\tau\left(\tfrac{d}{dt}(\delta u^n(t)-\delta u^m(t)),\left(\delta u^n(t)-\overline{\delta u}^n(t)\right)\right)_B\\
&+\tau\left(\tfrac{d}{dt}(\delta u^n(t)-\delta u^m(t)),\left(\overline{\delta u}^m(t)-\delta u^m(t)\right)\right)_B\\
&+b\left((u^n(t)-\overline{u}^n(t))+(\overline{u}^m(t)-u^m(t)),\overline{\delta u}^{n,m}\right)\\
&+(\overline{F}^n(t)-\overline{F}^m\left(t),\overline{\delta u}^{n,m}\right)_B+\left(K^n(t)-K^m(t),\overline{\delta u}^{n,m}\right)_B\\
&-a\left\|\overline{\delta u}^n(t)-\overline{\delta u}^m(t)\right\|_B^2.
\end{aligned}$$

Using some estimates, we see that each term in the right hand-side of (4.4) is estimate respectively by

$$\tau\left(\frac{d}{dt}(\delta u^n(t)-\delta u^m(t)),\left(\left(\delta u^n(t)-\overline{\delta u}^n(t)\right)+\left(\overline{\delta u}^m(t)-\delta u^m(t)\right)\right)_B\right)$$

$$\leq C\left(\frac{1}{n}+\frac{1}{m}\right) \tag{4.5}$$

$$b\left((u^n(t)-\overline{u}^n(t))+(\overline{u}^m(t)-u^m(t)),\overline{\delta u}^{n,m}\right)\leq c\left(\frac{1}{n}+\frac{1}{m}\right) \tag{4.6}$$

$$\begin{aligned}(\overline{F}^n(t)-\overline{F}^m\left(t),\overline{\delta u}^{n,m}\right)_B &\leq \left\|\overline{F}^n(t)-\overline{F}^m(t)\right\|_B\left\|\overline{\delta u}^{n,m}\right\|_B\\ &\leq \|F(t_k)-F(t_i)\|_B\left\|\overline{\delta u}^{n,m}\right\|_B\\ &\leq c_0|t_k-t_j|\left\|\overline{\delta u}^{n,m}\right\|_B\\ &\leq c\left(\frac{1}{n}+\frac{1}{m}\right)\left\|\overline{\delta u}^{n,m}\right\|_B.\end{aligned}$$

On the other hand we have

$$\begin{aligned}\left\|\overline{\delta u}^{n,m}\right\|_B &\leq cT\left(\frac{1}{n}+\frac{1}{m}\right)+\|\delta u^n-\delta u^m\|_B\\ &\leq cT\left(\frac{1}{n}+\frac{1}{m}\right)+\|\delta u^n\|_B+\|\delta u^m\|_B\\ &\leq c\left(\frac{1}{n}+\frac{1}{m}\right)+c,\end{aligned}$$

consequently

$$(\overline{F}^n(t)-\overline{F}^m\left(t),\overline{\delta u}^{n,m}\right)_B\leq c\left(\frac{1}{n}+\frac{1}{m}\right)^2+c\left(\frac{1}{n}+\frac{1}{m}\right) \tag{4.7}$$

Therefore, we obtain

$$\begin{aligned}&\left(K^n(t)-K^m(t),\overline{\delta u}^{n,m}\right)_B\\ \leq\ &\|K^n(t)-K^m(t)\|_B\left\|\overline{\delta u}^{n,m}\right\|_B\\ \leq\ &\left\|\frac{T}{n}\sum_{i=0}^{j-1}a_{ji}k_i-\frac{T}{m}\sum_{i=0}^{j-1}a_{ji}k_i\right\|_B\left\|\overline{\delta u}^{n,m}\right\|_B\end{aligned}$$

$$\begin{aligned}
&\leq T\left(\frac{1}{n}+\frac{1}{m}\right)\left\|\sum_{i=0}^{j-1} a_{ji}k_i\right\|_B \left\|\overline{\delta u}^{n,m}\right\|_B \\
&\leq T\left(\frac{1}{n}+\frac{1}{m}\right)\max|a(t)|\left\|\overline{\delta u}^{n,m}\right\|_B \\
&\leq c\left(\frac{1}{n}+\frac{1}{m}\right)\left[c\left(\frac{1}{n}+\frac{1}{m}\right)+c\right].
\end{aligned}$$

Then

$$\left(K^n(t)-K^m(t),\overline{\delta u}^{n,m}\right)_B \leq c\left(\frac{1}{n}+\frac{1}{m}\right)^2+c\left(\frac{1}{n}+\frac{1}{m}\right) \tag{4.8}$$

Summing up the inequalities (4.5)-(4.8) it follows that (4.4) becomes

$$\frac{\tau}{2}\frac{d}{dt}\left\|\delta u^n(t)-\delta u^m(t)\right\|_B^2+\frac{b}{2}\frac{d}{dt}\left\|u^n(t)-u^m(t)\right\|^2 \leq \tag{4.9}$$

$$c\left(\frac{1}{n}+\frac{1}{m}\right)^2+c\left(\frac{1}{n}+\frac{1}{m}\right)$$

which implies

$$\begin{aligned}
&\left[\frac{d}{dt}\left\|\delta u^n(t)-\delta u^m(t)\right\|_B^2+\frac{b}{2}\frac{d}{dt}\left\|u^n(t)-u^m(t)\right\|^2\right] \\
\leq\ & \frac{2}{\min(\tau,b)}\left[c\left(\frac{1}{n}+\frac{1}{m}\right)^2+c\left(\frac{1}{n}+\frac{1}{m}\right)\right].
\end{aligned}$$

Since $u^n(0)=u^m(0)=u_0$ and $\delta u^n(0)=\delta u^m(0)=u_1$, integrating over $(0,t)$, it yields

$$\begin{aligned}
&\left\|\delta u^n(t)-\delta u^m(t)\right\|_B^2+\left\|u^n(t)-u^m(t)\right\|^2 \\
\leq\ & \left[C_9T\left(\frac{1}{n}+\frac{1}{m}\right)+C_{10}T\left(\frac{1}{n}+\frac{1}{m}\right)^2\right]\exp\left(C_{11}T\right)
\end{aligned}$$

Taking the upper bound with respect to $t\in I$ in the left hand side of the last inequality, we obtain

$$\begin{aligned}
&\sup_{t\in I}\left\|\delta u^n(t)-\delta u^m(t)\right\|_B^2+\sup_{t\in I}\left\|u^n(t)-u^m(t)\right\|^2 \\
\leq\ & cT\left(\frac{1}{n}+\frac{1}{m}\right)+cT\left(\frac{1}{n}+\frac{1}{m}\right)^2
\end{aligned}$$

that implies

$$\left\|\delta u^n-\delta u^m\right\|_{C(I,B)}^2+\left\|u^n-u^m\right\|_{C(I,V)}^2 \leq C \tag{4.10}$$

From (4.10) we deduce that both $\{u^n\}_n, \{\delta u^n\}_n$ are Cauchy sequences in the Banach spaces $C(I,V)$, $C(I,B)$ respectively. Consequently there exist two functions $u\in C(I,V)$ and $w\in C(I,B)$ such that

$$u^n \to u \text{ in } C(I,V), \delta u^n \to w \text{ in } C(I,B). \tag{4.11}$$

Now, on the basis of estimations (3.18), (3.19) and (4.11), Lemma 3.2 we state the following assertions:

(i) $u \in C^{0,1}(I,V)$

(ii) u is differentiable in I and $\frac{du}{dt} \in L^\infty(I,V)$.

(iii) *$\overline{u}^n(t) \to u(t)$ in V for all $t \in I$.*

(iv) *$\frac{du^n}{dt} \rightharpoonup \frac{du}{dt}$ in $L^2(I,V)$.*

From the estimates (3.20), (3.21), (4.11) and Lemma 3.2, the following statements are true for the functions δu^n and the corresponding step function $\overline{\delta u}^n$:

(v) $w \in C^{0,1}(I,B)$

(vi) w is differentiable in I and $\frac{dw}{dt} \in L^\infty(I,B)$

(vii) *$\overline{\delta u}^n(t) \rightharpoonup w(t)$ in V for all $t \in I$;*

$(viii)$ *$\frac{d}{dt}\delta u^n \rightharpoonup \frac{dw}{dt}$ in $L^2(I,B)$.*

Using the same steps as in [1] we can easily prove that w coincides with $\frac{du}{dt}$ *for all $v \in L^2(I,B)$,*

Now, we give an existence result.

Theorem 4.2. *Suppose that the conditions $H_1 - H_3$ hold, then problem (1.5)-(1.8) has a unique weak solution.*

Proof. First, for existence we have to show that the properties $(1)-(4)$ in the Definition 2.1are fulfilled.

From Theorem 4.1 we conclude directly that the two first statements of Definition 2.1 are satisfied, the third one is true since $u^n \to u$ in $C(I,V)$, $\delta u^n \to \frac{du}{dt}$ in $C(I,B)$ as $n \to \infty$, $u^n(0) = u_0$ and $\frac{du}{dt}(0) = u_1$ are in V and B respectively. It remains to prove the last statement. Integrating over $(0,t)$ the relation (4.1) for all $\phi \in V$, we get

$$\tau(\delta u^n(t) - u_1, \phi)_B + a\int_0^t \left(\overline{\delta u}^n(s), \phi\right)_B ds + b\int_0^t (\overline{u}^n(s), \phi)\, ds = \tag{4.12}$$

$$\int_0^t \left(\overline{F}^n(s), \phi\right)_B ds + \int_0^t (K^n(s), \phi)_B\, ds$$

the third statement in Theorem 4.1 implies that

$$(\delta u^n(t) - u_1, \phi)_B \underset{n\to\infty}{\to} \left(\frac{du}{dt} - u_1, \phi\right)_B, \forall \phi \in V, \forall t \in I. \tag{4.13}$$

It is easy to see that the expressions $|(\overline{u}^n(s), \phi)|$, $\left|\left(\overline{\delta u}^n(s), \phi\right)\right|$ and $\left|\left(\overline{F}^n(s), \phi\right)_B\right|$ are uniformly bounded with respect to both n and s, then by bounded convergence theorem it yields

$$\int_0^t \left(\overline{\delta u}^n(s), \phi\right)_B ds \underset{n\to\infty}{\to} \int_0^t \left(\frac{du}{dt}(s), \phi\right)_B ds, \forall \phi \in V,\ \forall t \in I. \tag{4.14}$$

$$\int_0^t (\overline{u}^n(s), \phi)\, ds \underset{n\to\infty}{\to} \int_0^t (u(s), \phi)\, ds, \forall\phi \in V,\ \forall t \in I \tag{4.15}$$

$$\int_0^t \left(\overline{F}^n(s), \phi\right)_B ds \underset{n\to\infty}{\to} \int_0^t (F(s), \phi)_B ds, \forall\phi \in V,\ \forall t \in I. \tag{4.16}$$

On the other hand we have the following Lemma

Lemma 4.3. *Under the assumptions of Theorem 4.2, the sequence* $\{K^n(t)\}$ *is uniformly bounded and* $K^n(t) \underset{n\to\infty}{\to} K(u)(t)$ *in* $L^2(I, B)$.

Proof. Is the same as the proof of Lemma 2.4 in [3].

From Lemma 4.3 and bounded convergence theorem we conclude that

$$\int_0^t (K^n(s), \phi)_B\, ds \underset{n\to\infty}{\to} \int_0^t (K(s)\, u, \phi)_B\, ds \tag{4.17}$$

Finally taking into account (4.13)-(4.16), then passing to the limit as $n \to \infty$ in (4.12) we get

$$\begin{array}{l} \tau\left(\frac{du}{dt} - u_1, \phi\right)_B + a\int_0^t \left(\frac{du}{dt}(s), \phi\right)_B ds + b\int_0^t (u(s), \phi)\, ds \\ = \int_0^t (F(s), \phi)_B ds + \int_0^t (K(s)\, u, \phi)_B\, ds., \forall\phi \in V \end{array} \tag{4.18}$$

Differentiating this identity we obtain the required relation.

To prove the uniqueness of the weak solution we suppose that it exists two weak solutions $\hat{u}$ and $\check{u}$ of problem (1.5)-(1.8). Let $u = \hat{u} - \check{u}$ and $\phi = \frac{du}{dt}(t)$. From (2.1) we obtain

$$\tau\int_I \left(\frac{d^2u}{dt^2}(t), \frac{du}{dt}(t)\right)_B dt + a\int_I \left(\frac{du}{dt}(t), \frac{du}{dt}(t)\right)_B dt + b\int_I \left(u(t), \frac{du}{dt}(t)\right) dt = \tag{4.19}$$

$$\int_I \left(\int_0^t a(t-s)\left[k(s,u_1) - k(s,u_2)\right] ds, \frac{du}{dt}(t)\right)_B dt.$$

Let p be a length of a finite number corresponding to the subdivision of time such that

$$w.p < \frac{\min(\tau, b)}{2}, w = \max|a(t)| \int_0^T L(t)\, dt, \tag{4.20}$$

testing (4.18) with

$$\phi = \begin{cases} \frac{du}{dt}, t \in [0, p] \\ 0, t \in \,]p, T] \end{cases}$$

we get

$$\tau \int_0^p \left\| \frac{du}{dt}(t) \right\|_B^2 dt + a \int_0^p \left\| \frac{du}{dt}(t) \right\|_B^2 dt + b \int_0^p \|u(t)\|^2 dt$$
$$= \int_I \left(\int_0^t a(t-s)\left[k(s,u_1) - k(s,u_2)\right] ds, \frac{du}{dt}(t) \right)_B dt.$$

Using the Cauchy Schwarz inequality and the condition (H_4) we obtain

$$\tau \int_0^p \left\| \frac{du}{dt}(t) \right\|_B^2 dt + b \int_0^p \|u(t)\|^2 dt \leq \tag{4.21}$$

$$\int_I \left\| \int_0^t a(t-s)\left[k(s,u_1) - k(s,u_2)\right] ds \right\|_B^2 \left\| \frac{du}{dt}(t) \right\|_B^2$$

then

$$\int_0^p \left\| \frac{du}{dt}(t) \right\|_B^2 dt + \int_0^p \|u(t)\|^2 dt \leq \tag{4.22}$$

$$\frac{2}{\min(\tau,b)} \max_I |a(t)| .p. \int_0^T L(t)\, dt. \|u(t)\| \times \left\| \frac{du}{dt} \right\|_B$$
$$\leq wp \frac{2}{\min(\tau,b)} \left[\left(\max_{t\in[0,p]} \|u(t)\| \right)^2 + \left(\max_{t\in[0,p]} \left\| \frac{du}{dt} \right\|_B \right)^2 \right].$$

Let $t_1, t_2 \in [0,p]$ be such that

$$\left\| \frac{du}{dt}(t_1) \right\|_B = \max_{[0,p]} \left\| \frac{du}{dt}(t) \right\|_B \tag{4.23}$$

$$\|u(t_2)\| = \max_{[0,p]} \|u(t)\| \tag{4.24}$$

since $\frac{du}{dt}(0) = u(0) = 0$ it result that

$$\int_0^{t_1} \frac{d}{dt} \left\| \frac{du}{dt} \right\|_B^2 dt + \int_0^{t_2} \frac{d}{dt} \|u(t)\|^2 dt = \left\| \frac{du}{dt}(t_1) \right\|_B^2 + \|u(t_2)\|^2 \leq \tag{4.25}$$

$$\int_0^p \frac{d}{dt} \left\| \frac{du}{dt} \right\|_B^2 dt + \int_0^p \frac{d}{dt} \|u(t)\|^2 dt$$

from inequalities (4.22), (4.20) and (4.25) we obtain

$$\frac{du}{dt}(t) = u(t) = 0, \forall t \in [0, p]$$

Repeating the same procedure on the $[ip, (i+1)p]$, $i = 1,...,$ we get

$$u(t) = 0, \forall t \in I$$

this achieves the proof of Theorem 4.2.■

Conclusion. A new nonlocal problem generated by an integro-differential equation subject to integral condition has been studied by applying the time discretization method. It may be concluded that this technique is very powerful and efficient in finding the approximate solutions for a large class of integro-differential equations. As a continuation of this work we propose the investigation of the following time fractional integro-differential equation for $(x, t) \in (0, 1) \times I$:

$$\tau\frac{\partial^{2\alpha}\upsilon}{\partial t^{2\alpha}} + a\frac{\partial^{\alpha}\upsilon}{\partial t^{\alpha}} - b\frac{\partial^{2}\upsilon}{\partial x^{2}} = f(x,t) + \int_0^t a(t-s)\,k(s, \upsilon(x,s))\,ds,$$

that has many applications in various fields of science and engineering. The fractional derivatives appearing in the above equation must be understood in the sense of Caputo fractional derivative.

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