Electronic Journal of Qualitative Theory of Differential Equations
2011, No. **42**, 1-7; **http://www.math.u-szeged.hu/ejqtde/**

Periodic solutions of a porous medium equation

Dazhi Zhang, Jiebao Sun*, Boying Wu

Department of Mathematics, Harbin Institute of Technology, Harbin 150001, P. R. China

Abstract

In this paper, we study with a periodic porous medium equation with nonlinear convection terms and weakly nonlinear sources under Dirichlet boundary conditions. Based on the theory of Leray-Shauder fixed point theorem, we establish the existence of periodic solutions.

Key words: Existence, Periodic solutions, Leray-Schauder fixed point theorem.
2000 AMS Subject Classification: 35B10, 35K55, 35K65.

1 Introduction

In this paper, we consider the following periodic porous medium equation with nonlinear convection terms and weakly nonlinear sources

$$\frac{\partial u}{\partial t} - \Delta(|u|^{m-1}u) + b(u)\cdot\nabla u = B(x,t,u) + h(x,t), \qquad (x,t)\in Q_\omega, \tag{1.1}$$

$$u(x,t) = 0, \qquad (x,t)\in\partial\Omega\times[0,\omega], \tag{1.2}$$

$$u(x,0) = u(x,\omega), \qquad x\in\Omega, \tag{1.3}$$

where $m > 1$, Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $Q_\omega = \Omega\times(0,\omega)$ and we assume that

(A1) $b(u) = (b_1(u), b_2(u), \ldots, b_N(u))$ is a $\mathbb{R}^N$-valued function on $\mathbb{R}$ with $b_i(u)\in C(\mathbb{R})\cap C(\mathbb{R}-\{0\})$, and there exist $\beta, k\geq 0$ such that $|b(u)|\leq k|u|^\beta$.

(A2) $B(x,t,u)$ is Hölder continuous in $\overline{\Omega}\times\mathbb{R}\times\mathbb{R}$, periodic in t with period ω and satisfies $B(x,t,u)u\leq b_0|u|^{\alpha+1}$ with constants $b_0\geq 0$ and $0\leq\alpha<m$.

(A3) $h(x,t)\in C_\omega(\overline{Q}_\omega)\cap L^\infty(0,\omega;W_0^{1,\infty}(\Omega))$, $h(x,t)>0$ for $\Omega\times\mathbb{R}$, where $C_\omega(\overline{Q}_\omega)$ denotes the set of functions which are continuous in $\overline{\Omega}\times\mathbb{R}$ and ω-periodic with respect to t.

In recent years, periodic problems for degenerate parabolic equations have been the subject of extensive study, see [2, 5, 7, 9, 11, 12, 13] and references therein. Among the earliest works of this aspect, we refer to Nakao [9], in which one can find the related result for the special case of the equation (1.1), that is

$$\frac{\partial u}{\partial t} - \Delta\beta(u) = B(x,t,u) + h(x,t),$$

with Dirichlet boundary value conditions, where B, h are periodic in t with period $\omega>0$, $\beta(u)$ satisfies $\beta'(u)>0$ except for $u=0$ and $\beta(u)$ is fulfilled by $|u|^{m-1}u$ if $m>1$.

*Corresponding author. *E-mail address:* sunjiebao@126.com. (J. Sun)

Under the assumption that $B(x,t,u)u \le b_0|u|$, Nakao established the existence of periodic solutions by Leray-Shauder fixed point theorem. In [12], Wang et al. considered the following porous medium equation with weakly nonlinear sources

$$\frac{\partial u}{\partial t} - \Delta(|u|^{m-1}u) = B(x,t,u) + h(x,t).$$

By Moser's technique and the Leray-Schauder fixed point theorem [6, Th2.1, pp.140], the authors established the existence of periodic solutions when the assumption (A2) holds .

The work of this paper is an extension of [9, 12], that is, we consider the porous medium equation (1.1) with weakly nonlinear sources and nonlinear convection terms. The convection term $b(u)\cdot\nabla u$ describes an effect of convection with a velocity field $b(u)$. Our aim is to establish the existence of periodic solutions of the equation (1.1) under Dirichlet boundary value conditions.

This paper is organized as follows: In Section 2, we state some necessary preliminaries including the definition of the generalized solution, some useful lemmas and the statement of the main results. In Section 3, we show the proof of the main results of this paper.

2 Preliminaries

Due to the degeneracy of the equation considered, the problem (1.1)–(1.3) admits no classical solutions in general, so we consider generalized solutions in the following sense.

Definition 2.1. *A function u is said to be a generalized solution of the problem (1.1)–(1.3), if $u \in L^2(0,\omega; H_0^1(\Omega)) \cap C_\omega(\overline{Q}_\omega)$ and for any $\varphi \in C^1(\overline{Q}_\omega)$ with $\varphi(x,0) = \varphi(x,\omega)$ and $\varphi|_{\partial\Omega\times(0,\omega)} = 0$, we have*

$$\iint_{Q_\omega}\left(-u\frac{\partial\varphi}{\partial t} + \nabla(|u|^{m-1}u)\nabla\varphi - \beta(u)\cdot\nabla\varphi - B(x,t,u)\varphi - h(x,t)\varphi\right)dxdt = 0, \quad (2.1)$$

where $\beta(u) = (\beta_1(u), \beta_2(u), \cdots, \beta_N(u))$ and $\beta_i(u) = \int_0^u b_i(s)ds$, $i = 1,\cdots,N$.

For convenience, we let $\|\cdot\|_p$ and $\|\cdot\|_{m,p}$ denote $L^p(\Omega)$ and $W^{m,p}(\Omega)$ norms, respectively. In the following, we introduce some useful lemmas which play an important role in the proof of the main results of this paper.

Lemma 2.1. *[3] (Gagliardo-Nirenberg) Let $\beta \ge 0$, $N > p \ge 1$, $\beta + 1 \le q$, and $1 \le r \le q \le (\beta+1)Np/(N-p)$, then for u such that $|u|^\beta u \in W^{1,p}(\Omega)$,*

$$\|u\|_q \le C^{\frac{1}{\beta+1}} \|u\|_r^{1-\theta} \left\||u|^\beta u\right\|_{1,p}^{\theta/(\beta+1)}, \quad (2.2)$$

with $\theta = (\beta+1)(r^{-1} - q^{-1})/\{N^{-1} - p^{-1} + (\beta+1)r^{-1}\}$, where C is a constant independent of q, r, β and θ.

Lemma 2.2. *[10] Let $y(t) \in C^1(\mathbb{R}^1)$ be a nonnegative ω periodic function satisfying the differential inequality*

$$y'(t) + Ay^{\alpha+1}(t) \le By(t) + C, \quad t \in \mathbb{R},$$

with some $\alpha, A > 0, B \ge 0$ and $C \ge 0$, then

$$y(t) \le \max\{1, (A^{-1}(B+C))^{\frac{1}{\alpha}}\}.$$

Our results will be proved by means of parabolic generalization, that is we consider the following regularized problem

$$\frac{\partial u_\varepsilon}{\partial t} - \Delta(|u_\varepsilon|^{m-1}u_\varepsilon + \varepsilon u_\varepsilon) + b(u_\varepsilon)\cdot\nabla u_\varepsilon = B(x,t,u_\varepsilon) + h(x,t), \qquad (x,t)\in Q_\omega, \tag{2.3}$$

$$u_\varepsilon(x,t) = 0, \qquad (x,t)\in\partial\Omega\times[0,\omega], \tag{2.4}$$

$$u_\varepsilon(x,0) = u_\varepsilon(x,\omega), \qquad x\in\Omega, \tag{2.5}$$

where ε is some positive constant. We will apply the Leray-Schauder fixed point theorem to establish the existence of the solution u_ε of the problem (2.3)–(2.5). The desired solution of the problem (1.1)–(1.3) will be obtained as a limit point of u_ε.

Our main results is the following theorem.

Theorem 2.1. *If (A1), (A2) and (A3) hold, then the problem (1.1)–(1.3) admits at least one periodic solution u.*

3 Proof of the Main Results

First, we establish the following a priori estimate.

Lemma 3.1. *Let u_ε be a solution of*

$$\frac{\partial u_\varepsilon}{\partial t} - \Delta(|u_\varepsilon|^{m-1}u_\varepsilon + \varepsilon u_\varepsilon) + b(u_\varepsilon)\cdot\nabla u_\varepsilon = \sigma B(x,t,u_\varepsilon) + \sigma h(x,t), \qquad (x,t)\in Q_\omega, \tag{3.1}$$

$$u_\varepsilon(x,t) = 0, \qquad (x,t)\in\partial\Omega\times[0,\omega], \tag{3.2}$$

$$u_\varepsilon(x,0) = u_\varepsilon(x,\omega), \qquad x\in\Omega, \tag{3.3}$$

with $\sigma\in[0,1]$, then

$$\|u_\varepsilon(t)\|_\infty < R, \tag{3.4}$$

where $u_\varepsilon(t) = u_\varepsilon(\cdot,t)$ and R is a positive constant independent of ε and σ.

Proof. Suppose u_ε is a solution of the problem (3.1)–(3.3). Multiplying the equation (3.1) by $|u_\varepsilon|^p u_\varepsilon$ ($p\ge 0$) and integrating the resulting relation over Ω, noticing that

$$\begin{aligned}&\int_\Omega b(u_\varepsilon)\cdot\nabla u_\varepsilon |u_\varepsilon|^p u_\varepsilon dx\\ =&\int_\Omega \sum_{i=1}^N b_i(u_\varepsilon)|u_\varepsilon|^p u_\varepsilon \frac{\partial u_\varepsilon}{\partial x_i}dx = \sum_{i=1}^N\int_\Omega\left(\int_0^{u_\varepsilon} b_i(s)|s|^p s ds\right)_{x_i}dx\\ =&\sum_{i=1}^N\int_{\partial\Omega}\left(\int_0^{u_\varepsilon} b_i(s)|s|^p s ds\right)\cos(\mathrm{n},x_i)dx = 0,\end{aligned}$$

where n is the outer normal to $\partial\Omega$, we have

$$\begin{aligned}&\frac{1}{p+2}\frac{d}{dt}\int_\Omega |u_\varepsilon(t)|^{p+2}dx + \int_\Omega \Delta(|u_\varepsilon(t)|^{m-1}u_\varepsilon(t) + \varepsilon u_\varepsilon(t))|u_\varepsilon(t)|^p u_\varepsilon(t)dx\\ \le& b_0\int_\Omega |u_\varepsilon(t)|^{p+\alpha+1}dx + \int_\Omega |u_\varepsilon(t)|^p u_\varepsilon(t) h dx.\end{aligned} \tag{3.5}$$

Notice that the second term on the left hand can be written by

$$\int_\Omega \Delta\left[|u_\varepsilon(t)|^{m-1}u_\varepsilon(t) + \varepsilon u_\varepsilon(t)\right]|u_\varepsilon(t)|^p u_\varepsilon(t)dx$$

$$
\begin{aligned}
&=\int_\Omega \nabla\left[|u_\varepsilon(t)|^{m-1}u_\varepsilon(t)\right]\cdot\nabla\left[|u_\varepsilon(t)|^p u_\varepsilon(t)\right]+\varepsilon\nabla u_\varepsilon(t)\cdot\nabla\left[|u_\varepsilon(t)|^p u_\varepsilon(t)\right]dx\\
&=\int_\Omega \left[m|u_\varepsilon(t)|^{m-1}\nabla u_\varepsilon(t)\right]\cdot\left[(p+1)|u_\varepsilon(t)|^p\nabla u_\varepsilon(t)\right]dx+\varepsilon(p+1)|u_\varepsilon(t)|^p|\nabla u_\varepsilon(t)|^2dx\\
&\geq m(p+1)\int_\Omega |u_\varepsilon(t)|^{p+m-1}|\nabla u_\varepsilon(t)|^2dx\\
&=\frac{4m(p+1)}{(p+m+1)^2}\int_\Omega\left|\nabla\left[|u_\varepsilon(t)|^{\frac{m+p-1}{2}}u_\varepsilon(t)\right]\right|^2dx,
\end{aligned}
$$

and

$$
\int_\Omega |u_\varepsilon(t)|^p u_\varepsilon(t)h\,dx\leq\left(\int_\Omega |u_\varepsilon(t)|^{p+2}dx\right)^{\frac{p+1}{p+2}}\left(\int_\Omega h^{p+2}dx\right)^{\frac{1}{p+2}},
$$

from (3.5) we have

$$
\begin{aligned}
&\frac{1}{p+2}\frac{d}{dt}\|u_\varepsilon(t)\|_{p+2}^{p+2}+\frac{4m(p+1)}{(p+m+1)^2}\|\nabla(|u_\varepsilon(t)|^{\frac{m+p-1}{2}}u_\varepsilon(t))\|_2^2\\
&\leq b_0\|u_\varepsilon(t)\|_{p+\alpha+1}^{p+\alpha+1}+\|u_\varepsilon\|_{p+2}^{p+1}\|h\|_{p+2}.
\end{aligned}
\tag{3.6}
$$

Then we have

$$
\frac{d}{dt}\|u_\varepsilon(t)\|_{p+2}^{p+2}+C_1\|\nabla(|u_\varepsilon(t)|^{\frac{m+p-1}{2}}u_\varepsilon(t))\|_2^2\leq C_2(p+2)\left(\|u_\varepsilon(t)\|_{p+\alpha+1}^{p+\alpha+1}+\|u_\varepsilon\|_{p+2}^{p+1}\right),\tag{3.7}
$$

where C_1, C_2 are positive constants independent of $u_\varepsilon(t), p$.

First, we consider the case of $1\leq\alpha<m$. If $N>2$, by Hölder's inequality, we have

$$
\begin{aligned}
\int_\Omega |u_\varepsilon(t)|^{p+\alpha+1}dx&=\int_\Omega |u_\varepsilon(t)|^{\frac{(p+2)(m-\alpha)}{m-1}}|u_\varepsilon(t)|^{\frac{(\alpha-1)(p+m+1)}{m-1}}dx\\
&\leq\left(\int_\Omega |u_\varepsilon(t)|^{\frac{(p+2)(m-\alpha)}{m-1}\cdot\frac{m-1}{m-\alpha}}dx\right)^{\frac{m-\alpha}{m-1}}\left(\int_\Omega |u_\varepsilon(t)|^{\frac{(\alpha-1)(p+m+1)}{m-1}\cdot\frac{m-1}{\alpha-1}}dx\right)^{\frac{\alpha-1}{m-1}}\\
&\leq\left(\int_\Omega |u_\varepsilon(t)|^{p+2}dx\right)^{\frac{m-\alpha}{m-1}}\left(\int_\Omega |u_\varepsilon(t)|^{p+m+1}dx\right)^{\frac{\alpha-1}{m-1}}\\
&\leq\left(\int_\Omega |u_\varepsilon(t)|^{p+2}dx\right)^{\frac{m-\alpha}{m-1}}\left(\int_\Omega |u_\varepsilon(t)|^{\frac{(p+m+1)N}{N-2}}dx\right)^{\frac{(\alpha-1)(N-2)}{N(m-1)}}|\Omega|^{\frac{2(\alpha-1)}{N(m-1)}}\\
&=\|u_\varepsilon(t)\|_{p+2}^{\frac{(p+2)(m-\alpha)}{m-1}}\|u_\varepsilon(t)\|_q^{\frac{(\alpha-1)(m+p+1)}{m-1}}|\Omega|^{\frac{2(\alpha-1)}{N(m-1)}},
\end{aligned}
\tag{3.8}
$$

with $q=\frac{N(m+p+1)}{N-2}$. By Sobolev's imbeding theorem in [1, Th5.4, pp.114]

$$
\|u_\varepsilon(t)\|_q^{m+p+1}\leq C\|\nabla(|u_\varepsilon(t)|^{\frac{m+p-1}{2}}u_\varepsilon(t))\|_2^2
$$

and Young's inequality, we have

$$
\begin{aligned}
&\|u_\varepsilon(t)\|_{p+\alpha+1}^{p+\alpha+1}\\
&\leq C\|u_\varepsilon(t)\|_{p+2}^{\frac{(p+2)(m-\alpha)}{m-1}}\|\nabla(|u_\varepsilon(t)|^{\frac{m+p-1}{2}}u_\varepsilon(t))\|_2^{\frac{2(\alpha-1)}{m-1}}\\
&\leq\frac{2m(p+1)}{(m+p+1)^2}\|\nabla(|u_\varepsilon(t)|^{\frac{m+p-1}{2}}|u_\varepsilon(t))\|_2^2+C\left[\frac{2m(p+1)}{(m+p+1)^2}\right]^{\frac{1-\alpha}{m-\alpha}}\|u_\varepsilon(t)\|_{p+2}^{p+2}.
\end{aligned}
\tag{3.9}
$$

Combining (3.6) with (3.9), we have

$$\begin{aligned}&\frac{d}{dt}\|u_\varepsilon(t)\|_{p+2}^{p+2}+C_1\|\nabla(|u_\varepsilon(t)|^{\frac{m+p-1}{2}}u_\varepsilon(t))\|_2^2\\ \le& C_2\left((p+2)\|u_\varepsilon(t)\|_{p+2}^{p+1}+(p+1)^{1+\frac{\alpha-1}{m-\alpha}}\|u_\varepsilon(t)\|_{p+2}^{p+2}\right),\end{aligned} \tag{3.10}$$

where C_1, C_2 are positive constants independent of u, p.

If $N \le 2$, by Hölder's inequality, we have

$$\begin{aligned}&\int_\Omega |u_\varepsilon(t)|^{p+\alpha+1}dx=\int_\Omega |u_\varepsilon(t)|^{\frac{(p+2)(m-\alpha)}{m-1}}|u_\varepsilon(t)|^{\frac{(\alpha-1)(p+m+1)}{m-1}}dx\\ \le&\left(\int_\Omega |u_\varepsilon(t)|^{\frac{(p+2)(m-\alpha)}{m-1}\cdot\frac{m-1}{m-\alpha}}dx\right)^{\frac{m-\alpha}{m-1}}\left(\int_\Omega |u_\varepsilon(t)|^{\frac{(\alpha-1)(p+m+1)}{m-1}\cdot\frac{m-1}{\alpha-1}}dx\right)^{\frac{\alpha-1}{m-1}}\\ =&\|u_\varepsilon(t)\|_{p+2}^{\frac{(p+2)(m-\alpha)}{m-1}}\|u_\varepsilon(t)\|_{p+m+1}^{\frac{(p+m+1)(\alpha-1)}{m-1}}.\end{aligned}$$

Then also by Sobolev's imbeding theorem for $\|u_\varepsilon(t)\|_{m+p+1}$, we can also get (3.9) and (3.10).

Now we consider the case of $0 \le \alpha < 1$. By Hölder's inequality and Young's inequality, we have

$$\begin{aligned}\int_\Omega |u_\varepsilon(t)|^{p+\alpha+1}dx\le&\left(\int_\Omega |u_\varepsilon(t)|^{p+2}dx\right)^{\frac{p+\alpha+1}{p+2}}|\Omega|^{\frac{1-\alpha}{p+2}}\\ \le&\max\{1,|\Omega|^{\frac12}\}\|u_\varepsilon(t)\|_{p+2}^{p+\alpha+1}\\ =&\max\{1,|\Omega|^{\frac12}\}\|u_\varepsilon(t)\|_{p+2}^{(p+2)\alpha}\|u_\varepsilon(t)\|_{p+2}^{(p+1)(1-\alpha)}\\ \le&\|u_\varepsilon(t)\|_{p+2}^{p+2}+C\|u_\varepsilon(t)\|_{p+2}^{p+1}.\end{aligned} \tag{3.11}$$

Combining (3.11) with (3.6), we can also obtain (3.10).

By Young's inequality, from (3.10) that we have

$$\frac{d}{dt}\|u_\varepsilon(t)\|_{p+2}^{p+2}+C_1\|\nabla(|u_\varepsilon(t)|^{\frac{m+p-1}{2}}u_\varepsilon(t))\|_2^2\le C_2((p+1)^\sigma\|u_\varepsilon(t)\|_{p+2}^{p+2}+1), \tag{3.12}$$

where C_1, C_2 are various positive constants independent of p.

Set

$$k_1=m+1, k_i=2k_{i-1}, \alpha_i=\frac{m+(1-\theta_i)k_i-1}{\theta_i}>0, \quad \theta_i=\frac{N(m+k_i-1)}{2k_i+Nk_i+2N(m-1)}.$$

By Gagliardo-Nirenberg inequality, we have

$$\|u_\varepsilon(t)\|_{k_i}\le C\|u_\varepsilon(t)\|_{k_{i-1}}^{1-\theta_i}\|\nabla(|u_\varepsilon(t)|^{\frac{m+k_i-3}{2}}u_\varepsilon(t))\|_2^{\frac{2\theta_i}{m+k_i-1}}. \tag{3.13}$$

Set $p+2=k_i$ in (3.12), by (3.13) we have

$$\frac{d}{dt}\|u_\varepsilon(t)\|_{k_i}^{k_i}+C_1\|u_\varepsilon(t)\|_{k_{i-1}}^{\frac{(\theta_i-1)(m+k_i-1)}{\theta_i}}\|u_\varepsilon(t)\|_{k_i}^{\frac{m+k_i-1}{\theta_i}}\le C_2((p+1)^\sigma\|u_\varepsilon(t)\|_{k_i}^{k_i}+1),$$

that is

$$\frac{d}{dt}\|u_\varepsilon(t)\|_{k_i}+C_1C^{-\frac{m+k_i-1}{\theta_i}}k_i^{-1}\|u_\varepsilon(t)\|_{k_{i-1}}^{\frac{(\theta_i-1)(m+k_i-1)}{\theta_i}}\|u_\varepsilon(t)\|_{k_i}^{1+\alpha_i}\le C((p+1)^\sigma\|u_\varepsilon(t)\|_{k_i}+1).$$

Let $\chi_i = \sup_t \|u_\varepsilon(t)\|_{k_i}$, by Lemma 2.2, we obtain

$$\chi_i \le \max\left\{1, \left(C\chi_{i-1}^{\frac{(1-\theta_i)(m+k_i-1)}{\theta_i}}\right)^{\frac{1}{\alpha i}} = B_i^{\frac{1}{\alpha i}}\right\}.$$

We set without loss of generality that $B_i > 1$, which implies $\chi_i \le B_i^{1/\alpha_i}$. It is easy to verify that $\{\alpha_i\}$ is bounded (see [9]), and

$$\sup_t \|u_\varepsilon(t)\|_\infty \le \overline{\lim_{i\to\infty}} \chi_i \le C < \infty.$$

The proof is completed. □

Now, we show the proof of the main results.

Proof of Theorem 2.1 First, we introduce a map by considering the following problem

$$\frac{\partial u_\varepsilon}{\partial t} - \Delta(|u|^{m-1}u + \varepsilon u_\varepsilon) + b(u_\varepsilon)\cdot\nabla u_\varepsilon = g(x,t), \qquad (x,t)\in Q_\omega, \tag{3.14}$$

$$u_\varepsilon(x,t) = 0, \qquad (x,t)\in\partial\Omega\times[0,\omega], \tag{3.15}$$

$$u_\varepsilon(x,0) = u_\varepsilon(x,\omega), \qquad x\in\Omega, \tag{3.16}$$

where $g\in C_\omega(\overline{Q}_\omega)$. With a similar method of [9], we conclude that for any $g\in C_\omega(\overline{Q}_\omega)$, the problem (3.14)–(3.16) admits a unique solution $u_\varepsilon \in L^\infty(0,\omega;H_0^1(\Omega))$ and $\frac{\partial u_\varepsilon}{\partial t}\in L^1(Q_\omega)$. Define a map $T: C_\omega(\overline{Q}_\omega)\to C_\omega(\overline{Q}_\omega)$ by $u_\varepsilon = Tg$. Then we can infer that the map $u_\varepsilon = Tg$ is compact and continuous.

In fact, by [4] and the periodicity of u_ε, we have

$$|u_\varepsilon(x_1,t_1) - u_\varepsilon(x_2,t_2)| \le \gamma(|x_1-x_2| + |t_1-t_2|^{1/2})^\beta,$$

for every pair of points $(x_1,t_1),(x_2,t_2)\in\overline{Q}_\omega$, where the constants $\gamma,\beta\in(0,1)$ are independent of ε. By Ascoli-Arezela theorem, we can see that T maps any bounded set of $C_\omega(\overline{Q}_\omega)$ into a compact set of $C_\omega(\overline{Q}_\omega)$. Suppose that $g_k\to g$ as $k\to\infty$ and denote $u_k = Tg_k$, then there exist a subsequence of u_k and a function $u_\varepsilon\in C_\omega(\overline{Q}_\omega)$ such that

$$u_k(x,t)\to u_\varepsilon(x,t), \quad \text{uniformly in } Q_\omega.$$

Noticing that

$$\int_\Omega b(u_\varepsilon)\cdot\nabla u_\varepsilon u_\varepsilon dx = 0.$$

So we can prove that $u_\varepsilon = Tg$ is compact and continuous by using the argument similar to [13].

Let $\Phi(u_\varepsilon) = B(x,t,u_\varepsilon) + h(x,t)$, by (A2)–(A3) and the estimates above, we can see that $T(\sigma\Phi)$ is also the complete continuous map for $\sigma\in[0,1]$. By Lemma 3.1, we can see that any fixed point u_ε of the map $T(\sigma\Phi)$ satisfies

$$\|u_\varepsilon\|_\infty \le C,$$

where C is a constant independent of ε,σ. So we conclude that the generalized problem (2.3)–(2.5) admits a periodic solution u_ε by Leray-Schauder fixed point theorem [6, Th2.1, pp.140]. Then by using a similar argument as that in [8, 14], we can obtain a periodic solution u of the problem (1.1)–(1.3) as a limit point of u_ε. The proof is completed.

Acknowledgments

The authors express their thanks to the editor and the referees for their very helpful suggestions to improve some results in this paper. This work is supported by "the Fundamental Research Funds for the Central Universities" (Grant No. HIT. NSRIF. 2011006) and also by the 985 project of Harbin Institute of Technology.

References

[1] R.A. Adams, Sobolev Spaces (Qixiao Ye et al., Trans.), People's Publishing House, Beijing, 1983 (in Chinese).

[2] M. Badii, Periodic solutions for a class of degenerate evolution problems, Nonlinear Anal. 44 (4)(2001) 499–508.

[3] C.S. Chen, R.Y. Wang, L^∞ estimates of solution for the evolution m-laplacian equation with initial value in $L^\infty(\Omega)$, Nonlinear Anal. 48 (4)(2002) 607–616.

[4] E. DiBenedetto, Degenerate Parabolic Equations, Universitext, Springer, New York, NY, USA, 1993.

[5] A. Favini, G. Marinoschi, Periodic behavior for a degenerate fast diffusion equation, J. Math. Anal. Appl. 351(2)(2009) 509–521.

[6] D.J. Guo, Nonlinear Functional Analysis, Shandong Science and Technology Press, 2001.

[7] J.B. Sun, B.Y. Wu, J. Li, D.Z. Zhang, A class of doubly degenerate parabolic equations with periodic sources, Discrete. Contin. Dyn. Syst. Ser. B, 14 (3)(2010) 1199–1210.

[8] M. Nakao, On solutions to the initial-boundary problem for $\frac{\partial u}{\partial t} - \Delta\beta(u) = f$, J. Math. Soc. Japan. 35 (1983) 71–83.

[9] M. Nakao, Periodic solutions of some nonlinear degenerate parabolic equations, J. Math. Anal. Appl. 104 (2)(1984) 554–567.

[10] M. Nakao, Y. Ohara, Gradient estimates of periodic solutions for some quasilinear parabolic equations, J. Math. Anal. Appl. 204 (3)(1996) 868–883.

[11] J.B. Sun, J.X. Yin, Y.F. Wang, Asymptotic bounds of solutions for a periodic doubly degenerate parabolic equation, Nonlinear Anal. TMA, 74 (2011) 2415-2424.

[12] Y.F. Wang, J.X. Yin, Z.Q. Wu, Periodic solutions of porous medium equations with weakly nonlinear sources, Communications in Mathematical Research 16 (4)(2000) 475–483.

[13] Y.F. Wang, J.X. Yin, Z.Q. Wu, Periodic solutions of evolution p-Laplacian equations with nonlinear sources, J. Math. Anal. Appl. 219 (1)(1998) 76–96.

[14] Z.Q. Wu, J.N. Zhao, J.X. Yin, H.L. Li, Nonlinear Diffusion Equations, World Scientific, Singapore, 2001.

(Received June 10, 2010)