

LINEAR DIFFERENTIAL EQUATIONS WITH COEFFICIENTS IN FOCK TYPE SPACE

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ABSTRACT. In this paper we deal with complex differential equations of the form

$$f^{(k)} + a_{k-1}(z)f^{(k-1)} + \cdots + a_1(z)f' + a_0(z)f = 0$$

with the coefficients in Fock type space. The relation between the solutions and coefficients in Fock type space is obtained.

1. INTRODUCTION

Motivated by the work in [6], [7] and [8], we will study complex differential equations of the form

$$(1) \quad f^{(k)} + a_{k-1}(z)f^{(k-1)} + \cdots + a_1(z)f' + a_0(z)f = 0$$

where the coefficients are entire functions.

In [8], equations of the form (1) with coefficients in weighted Bergman or Hardy spaces are studied. The *direct problem* is proved, that is, if the coefficients $a_j(z)$, $j = 0, \dots, k - 1$ of (1) belong to the weighted Bergman space, then all solutions are of finite order of growth and belong to weighted Bergman space. The *inverse problem* is also investigated, that is, if all solutions are of finite order of growth, then the coefficient is proved to belong to weighted Bergman space.

The Bargmann-Fock space (see [1], [2]) is the Hilbert space of entire functions equipped with the inner product

$$\langle f, g \rangle = \frac{1}{\pi} \int_{\mathbb{C}} f(z) \overline{g(z)} e^{-z\bar{z}} dx dy,$$

normed by $\|f\| = \sqrt{\langle f, f \rangle}$. This space has been studied by many authors and it is rooted from mathematical problems of relativistic physics (see [12]) or from quantum optics (see [10]). In physics the Bargmann-Fock space contains the canonical coherent states, so it is the main tool for studying the bosonic coherent state theory of radiation field (see [11]). The Bargmann-Fock space has also been proved invaluable in the

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theory of the wavelets. In fact, the Bargmann transform is a unitary map from $L^2(\mathbb{R})$ onto the Bargmann-Fock space which transforms the family of evaluation functionals at a point into canonical coherent states which are nothing but the Gabor wavelets.

The Fock-type space F_α (see [3]) is the Hilbert space of entire functions equipped with the inner product

$$\langle f, g \rangle_\alpha = \int_{\mathbb{C}} f(z) \overline{g(z)} e^{-\alpha(|z|^2)} dx dy,$$

normed by $\|f\|_\alpha = \sqrt{\langle f, f \rangle_\alpha}$, where $\alpha(r)$ is a nonnegative and non-decreasing function of r . It's obvious that $|z|^2$ is such a function. Thus the Fock-type space F_α is a generalization of the Bargmann-Fock space.

In this paper, we will consider the growth relation between the coefficients and the solutions of (1). We are particularly interested in the Fock-type spaces F_α and F_{e^α} case:

(i) Find the conditions imposed on the coefficients $a_j(z)$, $j = 0, \dots, k-1$ of (1) which make all of the solutions belong to the Fock-type space F_{e^α} .

(ii) Suppose that all solutions of (1) belong to the Fock-type space F_{e^α} , find out whether all of the coefficients $a_j(z)$, $j = 0, \dots, k-1$ belong to the Fock-type space F_α .

Hereafter, problems (i) and (ii) will be referred to as the *direct problem* and the *inverse problem*, respectively.

Throughout this paper, A will denote positive constants, it may be different at each occurrence.

2. DIRECT PROBLEM

In this section, sufficient conditions for all of the solutions of (1) belong to F_{e^α} will be obtained. We need the following result on growth estimate for solutions of (1) in [6].

Lemma 2.1. *Let f be a solution of (1) in the disk $\{z \in \mathbb{C} : |z| < r\}$, where $0 < r \leq \infty$, let $n_c \in \{1, \dots, k\}$ be the number of nonzero coefficients $a_j(z)$, $j = 0, \dots, k-1$, and let $\theta \in [0, 2\pi)$. If $z_\theta = r_0 e^{i\theta} \in \{z \in \mathbb{C} : |z| < r\}$ is such that $a_j(z_\theta) \neq 0$ for some $j = 0, \dots, k-1$, then for all $r_0 < r_1 < r$,*

$$|f(r_1 e^{i\theta})| \leq A \exp \left\{ n_c \int_{r_0}^{r_1} \max_{j=0, \dots, k-1} |a_j(t e^{i\theta})|^{1/(k-j)} dt \right\}$$

where A is some positive constant depends on the values of the derivatives of f and the values of $a_j(z_\theta)$ at z_θ .

The main result of this section is as follows.

Theorem 2.1. *Suppose that $\alpha(r)$ is a nonnegative and nondecreasing function of r satisfying*

$$(2) \quad \liminf_{r \rightarrow \infty} \frac{\alpha(r)}{\log r} > 1,$$

furthermore, suppose that $a_j(z) \in F_\alpha, j = 0, 1, \dots, k-1$, then all solutions of (1) belong to F_{e^α} .

Proof. Since $a_j(z) \in F_\alpha$ for $j = 0, 1, \dots, k-1$, we have

$$\int_{\mathbb{C}} |a_j(z)|^2 e^{-\alpha(|z|)} dx dy = 2\pi \int_0^\infty |a_j(z)|^2 e^{-\alpha(r)} r dr < \infty.$$

Thus

$$\lim_{r \rightarrow \infty} |a_j(z)|^2 e^{-\alpha(r)} r = 0,$$

which yields

$$|a_j(z)|^2 \leq A \frac{e^{\alpha(r)}}{r}$$

for sufficiently large $|z| = r > 0$, or

$$(3) \quad |a_j(z)| \leq A \frac{e^{\alpha(r)/2}}{\sqrt{r}}, r > r_0.$$

If $f(z)$ is a solution of (1), from Lemma 2.1, we have

$$|f(re^{i\theta})| \leq A \exp \left\{ n_c \int_{r_0}^r \max_{j=0, \dots, k-1} |a_j(te^{i\theta})|^{1/(k-j)} dt \right\}$$

where A is some positive constant depends on the values of the derivatives of f and the values of $a_j(z_\theta)$ at z_θ and n_c is defined in Lemma 2.1, combination with (3) yields

$$\begin{aligned} |f(re^{i\theta})| &\leq \exp \left\{ A \int_{r_0}^r \frac{e^{\alpha(t)/2}}{\sqrt{t}} dt \right\} \\ &\leq A \exp \{ Ar^{1/2} e^{\alpha(r)/2} \}. \end{aligned}$$

By (2), we have

$$\alpha(r) > (1 + \varepsilon) \log r$$

for any $\frac{1}{2} > \varepsilon > 0$, then

$$2Ar^{1/2} e^{\alpha(r)/2} - e^{\alpha(r)} < e^{\alpha(r)/2} (2Ar^{1/2} - r^{1+\varepsilon}).$$

Thus

$$\begin{aligned} \|f\|_{e^\alpha}^2 &= \int_{\mathbb{C}} |f(z)|^2 e^{-e^\alpha(|z|)} dx dy \\ &\leq A \int_{r_0}^\infty \exp \{ 2Ar^{1/2} e^{\alpha(r)/2} - e^{\alpha(r)} \} r dr < \infty, \end{aligned}$$

proving that all of solutions of (1) belong to F_{e^α} . □

Remark 2.1. *Although we are unable to show the sharpness of the constant 1 in (4), we remark it is necessary. Actually, if (4) does not hold, we may suppose*

$$\limsup_{r \rightarrow \infty} \frac{\alpha(r)}{\log r} < 1.$$

Taking $\alpha(r) \sim \log r^{\eta_0}$ where $0 < \eta_0 < \frac{1}{2}$, for example, here the symbol \sim denotes that $\alpha(r)$ and $\log r^{\eta_0}$ have the same growth as r tends to infinity, then

$$2Ar^{1/2}e^{\alpha(r)/2} - e^{\alpha(r)} = e^{\alpha(r)/2}(2Ar^{1/2} - e^{\alpha(r)/2}) \sim r^{\frac{\eta_0+1}{2}},$$

and

$$\int_{r_0}^{\infty} \exp \{2Ar^{1/2}e^{\alpha(r)/2} - e^{\alpha(r)}\} r dr = \infty.$$

By the proof of Theorem 2.1, we know that $f \notin F_{e^\alpha}$ holds in this case.

3. INVERSE PROBLEM

To study the inverse problem, we need some background knowledge and some lemmas.

We present the following elementary result on inequality in [4] for later use.

Lemma 3.1. *Let $a_k \geq 0$ for $k = 1, \dots, n$. Then*

$$\left(\sum_{k=1}^n a_k\right)^p \leq n^{p-1} \left(\sum_{k=1}^n a_k^p\right)$$

for $1 \leq p < \infty$.

We also need the growth estimates of meromorphic functions in [5].

Lemma 3.2. *Let $f(z)$ be a transcendental meromorphic function, furthermore, let $\beta > 1$ be a positive constant. Then there exist a set $E \subset [0, 2\pi)$ that has linear measure zero, a constant $A > 0$ that depends only on β , and a constant $r_0 = r_0(\theta) > 1$ such that*

$$\left| \frac{f^{(m)}(re^{i\theta})}{f(re^{i\theta})} \right| \leq A \left[T(\beta r, f) \frac{\log^\beta r}{r} \log T(\beta r, f) \right]^m, \quad m \in \mathbb{N},$$

where $r > r_0$ and $\theta \in [0, 2\pi) \setminus E$.

Recall that the order reduction procedure is as follows(see [8]): if $\{f_1, \dots, f_k\}$ is a solution base of (1) in $|z| < r$, then the first order reduction of (1) results in

$$(4) \quad \nu_1^{(k-1)} + a_{1,k-2}(z)\nu_1^{(k-2)} + \dots + a_{1,0}(z)\nu_1 = 0,$$

where

$$(5) \quad a_{1,j}(z) = a_{j+1}(z) + \sum_{m=1}^{k-j-1} \binom{j+1+m}{m} a_{j+1+m}(z) \frac{f_1^{(m)}(z)}{f_1(z)}$$

for $j = 0, \dots, k-2$ and the meromorphic functions

$$(6) \quad \nu_{1,j}(z) = \frac{d}{dz} \left(\frac{f_{j+1}(z)}{f_1(z)} \right)$$

for $j = 0, \dots, k-1$ are linearly independent solutions of (5) in $|z| < r$.

We have the following relations between the solutions and its reductions.

Lemma 3.3. *Suppose $\alpha(r)$ is a nonnegative and nondecreasing continuous function of r satisfying*

$$(7) \quad \int_{r_0}^{\infty} e^{(\tau_0\alpha(\beta_0r) - \alpha(\beta r))} r dr < \infty,$$

for $r > r_0$, where τ_0, β_0 and β are some positive constants satisfying $\tau_0 > 4k$, $\beta_0 > 1$, and $\beta > \beta_0$. Let $\{f_1, \dots, f_k\}$ be a solution base of equation (1). If $f_j \in F_{e^{\alpha(r)}}$, $j = 1, 2, \dots$, then for any nonnegative integer $m \leq 2$ and $l = 1, 2, \dots, k$,

$$(8) \quad \int_{\mathbb{C}} \left| \frac{f_j^{(l)}(z)}{f_j(z)} \right|^{2m} e^{-\alpha(\beta|z|)} dx dy < \infty,$$

$$(9) \quad \int_{\mathbb{C}} \left| \frac{v_{1,j}^{(l)}(z)}{v_{1,j}(z)} \right|^{2m} e^{-\alpha(\beta|z|)} dx dy < \infty,$$

and

$$(10) \quad \int_{\mathbb{C}} \left| \frac{v_{k,j}^{(l)}(z)}{v_{k,j}(z)} \right|^{2m} e^{-\alpha(\beta|z|)} dx dy < \infty.$$

Proof. Since $f_j(z) \in F_{e^{\alpha}}$ for $j = 1, \dots, k$, we have

$$\int_{\mathbb{C}} |f_j(z)|^2 e^{-e^{\alpha(|z|)}} dx dy = 2\pi \int_0^{\infty} |f_j(z)|^2 e^{-e^{\alpha(r)}} r dr < \infty.$$

Thus

$$\lim_{r \rightarrow \infty} |f_j(z)|^2 e^{-e^{\alpha(r)}} r = 0,$$

which yields

$$|f_j(z)|^2 \leq A \frac{e^{e^{\alpha(r)}}}{r}$$

for sufficiently large $|z| = r > 0$, or

$$\log M(r, f_j) \leq \frac{1 + o(1)}{2} e^{\alpha(r)}.$$

Thus,

$$(11) \quad (T(\beta_0 r, f_j))^m \leq A e^{m\alpha(\beta_0 r)}$$

for any nonnegative integer $m \leq 2$ and $\beta_0 > 1$. By Lemma 3.2, for given positive constant $\beta_0 > 1$, there exist some set $E \subset [0, 2\pi)$ of measure zero and some constant $r_1 = \sup_{\theta_0 \in [0, 2\pi) \setminus E} r_0(\theta_0) > 0$ such that

$$(12) \quad \left| \frac{f_j^{(m)}(z)}{f_j(z)} \right| \leq A |T(\beta_0 r, f_j)| \frac{\log^\beta r}{r} \log |T(\beta_0 r, f_j)|^m$$

for $\arg z \in \{\theta_0 : \theta_0 \in [0, 2\pi) \setminus E\}$ and $|z| \geq r_1$. Combine (7) with (11) and (12), we have

$$\int_{\mathbb{C}} \left| \frac{f_j^{(l)}(z)}{f_j(z)} \right|^{2m} e^{-\alpha(\beta|z|)} dx dy \leq A \int_{\mathbb{C}} e^{(\tau_0 \alpha(\beta_0 r) - \alpha(\beta r))} r dr < \infty,$$

for any nonnegative integer $m \leq 2$, $\beta > \beta_0$ and $l = 1, 2, \dots, k$, which is (8). The same reasoning yields (9) and (10).

We also need the following result on reduction.

Lemma 3.4. *Suppose $\alpha(r)$ is a nonnegative and nondecreasing continuous function of r satisfying (7) for some $\tau_0 > 4k$ and $\beta > \beta_0 > 1$ furthermore, assume that for any nonnegative integer $m \leq 2$ and $\beta > \beta_0$*

$$(13) \quad \int_{\mathbb{C}} |a_{1,j}(z)|^{2m} e^{-\alpha(\beta|z|)} dx dy < \infty,$$

where $a_{1,j}(z)$ is defined in (5), then

$$(14) \quad \int_{\mathbb{C}} |a_j(z)|^{2m} e^{-\alpha(\beta|z|)} dx dy < \infty,$$

where $a_j(z)$ ($j = 0, \dots, k-1$) are coefficients of (1).

Proof. Note that $a_k(z) \equiv 1$, for $j = 1, 2, \dots, k$

$$(15) \quad a_{1,k-2}(z) = a_{k-1}(z) + k \frac{f_1'(z)}{f_1(z)}.$$

Let $f_1(z)$ denote any transcendental entire solution of (1). For any nonnegative integer $m \leq 2$, by Lemma 3.2, there exist some set $E \subset$

$[0, 2\pi)$ of measure zero and some constant $r_1 = \sup_{\theta_0 \in [0, 2\pi) \setminus E} r_0(\theta_0) > 0$ such that

$$(16) \quad \left| \frac{f_1^{(m)}(z)}{f_1(z)} \right| \leq A |T(\beta_0 r, f_1) \frac{\log^\beta r}{r} \log T(\beta_0 r, f_1)|^m$$

for $\arg z \in \{\theta_0 : \theta_0 \in [0, 2\pi) \setminus E\}$ and $|z| \geq r_1, \beta_0 > 1$. Combination of (15) and (16) yields

$$(17) \quad |a_{k-1}(z)| \leq |a_{1,k-2}(z)| + |T(\beta_0 r, f_1) \frac{\log^\beta r}{r} \log T(\beta_0 r, f_1)|^m$$

for $|z| \geq r_1, \arg z = \theta \in [0, 2\pi) \setminus E$.

From the proof of Lemma 3.3, we know that for $f_1 \in F_{e^\alpha}$,

$$(18) \quad (T(\beta_0 r, f_j))^m \leq A e^{m\alpha(\beta_0 r)}.$$

Square both sides, multiply both sides of (17) by $e^{-\alpha(\beta|z|)}$, then integrate over the annulus, combine with Lemma 3.1, (18) and the fact (7), we have

$$\begin{aligned} & \int_{\mathbb{C}} |a_{k-1}(z)|^{2m} e^{-\alpha(\beta|z|)} dx dy \\ &= \int_0^{2\pi} d\theta \int_0^\infty |a_{k-1}(re^{i\theta})|^{2m} e^{-\alpha(\beta r)} r dr \\ &\leq \int_0^{2\pi} d\theta \int_0^{r_0} |a_{k-1}(re^{i\theta})|^{2m} e^{-\alpha(\beta r)} r dr \\ &+ A \int_0^{2\pi} d\theta \int_{r_0}^\infty |a_{1,k-2}(re^{i\theta})|^{2m} e^{-\alpha(\beta r)} r dr \\ &+ A \int_{r_0}^\infty |T(\beta_0 r, f_1) \frac{\log^\beta r}{r} \log T(\beta_0 r, f_1)|^{2m} e^{-\alpha(\beta r)} r dr \\ &\leq A \int_{r_0}^\infty |a_{1,k-2}(re^{i\theta})|^{2m} e^{-\alpha(\beta r)} dx \\ &+ A \int_{r_0}^\infty e^{(\tau_0 \alpha(\beta_0 r) - \alpha(\beta r))} r dr \\ &< \infty, \end{aligned}$$

where A is some positive constant .

Suppose that the assertion is proved for $j = k - 1, \dots, k - l, l \in \{1, \dots, k - 2\}$. From

$$a_{1,k-(l+1)}(z) = a_{k-(l+2)}(z) + \sum_{m=1}^{l+1} \binom{k-l-1+m}{m} a_{k-(l+1)+m}(z) \frac{f_1^{(m)}(z)}{f_1(z)},$$

we have

$$|a_{k-(l+1)}(z)| \leq |a_{1,k-(l+2)}(z)| + A \left| \frac{f_1^{(l+1)}(z)}{f_1(z)} \right| + A \sum_{m=1}^l |a_{k-(l+1)+m}(z)| \left| \frac{f_1^{(m)}(z)}{f_1(z)} \right|.$$

Thus, by Lemma 3.1, there exists a positive constant A such that

$$|a_{k-(l+1)}(z)|^2 \leq A(|a_{1,k-(l+2)}(z)|^2 + \left| \frac{f_1^{(l+1)}(z)}{f_1(z)} \right|^2 + \sum_{m=1}^l \left(\left| a_{k-(l+1)+m}(z) \frac{f_1^{(m)}(z)}{f_1(z)} \right|^2 \right)).$$

Multiply both sides of (17) by $e^{-\alpha(\beta|z|)}$, the Cauchy-Schwartz inequality yields,

$$\begin{aligned} & \int_{\mathbb{C}} |a_{k-(l+1)}(z)|^{2m} e^{-\alpha(\beta|z|)} dx dy \\ & \leq A \left(\int_{\mathbb{C}} |a_{1,k-(l+2)}(z)|^2 e^{-\alpha(\beta|z|)} dx dy + \int_{\mathbb{C}} \left| \frac{f_1^{(l+1)}(z)}{f_1(z)} \right|^2 e^{-\alpha(\beta|z|)} dx dy \right. \\ & \quad \left. + \sum_{m=1}^l \left(\int_{\mathbb{C}} |a_{k-(l+1)+m}(z)|^2 e^{-\alpha(\beta|z|)} dx dy \right)^{1/2} \right. \\ & \quad \left. \times \left(\int_{\mathbb{C}} \left| \frac{f_1^{(m)}(z)}{f_1(z)} \right|^4 e^{-\alpha(\beta|z|)} dx dy \right)^{1/2} \right). \end{aligned}$$

Thus, by (7), (16) and (18), we know that (14) holds for $j=1, \dots, k$.

For $a_0(z)$, from

$$a_0(z) = -\frac{f_1^{(k)}}{f_1} - a_{k-1} \frac{f_1^{(k-1)}}{f_1} - \dots - a_1 \frac{f_1'}{f_1},$$

the conclusion follows from Lemma 3.3. □

Our result on the inverse problem is as follows.

Theorem 3.1. *Suppose that $\alpha(r)$ is a nonnegative and nondecreasing continuous function of r satisfying (7) for some $\tau_0 > 4k$ and $\beta > \beta_0 > 1$. Let $a_j(z)$ ($j = 0, \dots, k-1$) denote the coefficients of (1). If all of the solutions of (1) belong to $F_{e^{\alpha(r)}}$, then for any nonnegative integer $m \leq 2$ and $\beta > \beta_0 > 1$, $(a_j(z))^m$ ($j = 0, \dots, k-1$) belong to $F_{\alpha(\beta r)}$.*

Proof. When $k = 1$, the equation (1) has the following form

$$f' + a_0(z)f = 0.$$

Let f be a nontrivial entire function solution of (1). By (8) in Lemma 3.3, it is obvious that for any nonnegative integer $m \leq 2k$ and $\beta > 1$, $(a_0(z))^m \in F_{\alpha(\beta r)}$.

Suppose that $k \geq 2$. After $k - 1$ order reduction steps, we obtain the differential equation

$$\nu'_{k-1} + a_{k-1,0}(z)\nu_{k-1} = 0.$$

Thus

$$a_{k-1,0} = -\frac{\nu'_{k-1}(z)}{\nu_{k-1}(z)},$$

where $\nu_{k-1}(z)$ is the meromorphic function defined in (6). Combine (9) in Lemma 3.3 with Lemma 3.4, we conclude $a_{k-1}(z) \in F_{\alpha(\beta r)}$, proving Theorem 3.1

□

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