

Multiple solutions of nonlocal boundary value problems for fractional differential equations on the half-line *

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Abstract: In this paper, we study the existence of multiple solutions of nonlocal boundary value problems for fractional differential equations with integral boundary conditions on the half-line. Applying the fixed point theory and the upper and lower solutions method, some new results on the existence of at least three nonnegative solutions are obtained. An example is presented to illustrate the application of our main results.

Keywords: Fractional differential equations; Caputo derivative; Integral boundary condition; Lower and upper solutions; Half-line.

MSC: 34B15, 26A33.

1 Introduction

In this paper, we consider the following nonlocal boundary value problem for fractional differential equations with integral boundary condition on the half-line

$$\begin{cases} {}^C D^\alpha(p(t)u'(t)) + q(t)f(t, u(t)) = 0, & t > 0, \\ p(0)u'(0) = 0, \\ \lim_{t \rightarrow +\infty} u(t) = \int_0^{+\infty} g(t)u(t)dt, \end{cases} \quad (1.1)$$

where ${}^C D^\alpha$ is the standard Caputo derivative, $0 < \alpha < 1$ is a constant, f , g , p and q are given functions.

Boundary value problems (BVPs) of differential equation have received much attention in recent years due to their broad applications in applied mathematics and physics. There are many papers

*Supported by Natural Science Foundation of China (No. 11071164) and the Innovation Program of Shanghai Municipal Education Commission (No. 10ZZ93).

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concerning the existence of solutions, positive solutions or multiple solutions of two point BVPs, three point BVPs, m -point even nonlocal boundary conditions such as integral boundary conditions about the integer order differential equation. For details we can refer to [6, 10, 13, 16–21, 24, 26]. Boundary value problems on the half-line have been applied in unsteady flow of gas through a semi-infinite porous medium, the theory of drain flows, etc. In the paper [1], Agarwal and O'Regan gave infinite interval problems modeling phenomena which arise in the theory of plasma and electrical potential theory. In [6, 10, 11, 13, 19, 26], authors studied two-point or multipoint boundary value problems on the half-line by using different method. The papers [20, 21] studied the existence of positive solutions for second-order boundary value problems of differential equations system with integral boundary condition on the half-line.

It is well known that fractional order differential equations have been proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering, and they also have been of great interests, see [9, 15]. Recently, there are some papers which deal with the existence of the solutions of the boundary values problems for fractional differential equations on finite intervals. For details, see [2, 4, 5, 7, 8, 12, 14, 23, 25, 27, 28] and the references therein.

In [9] and [15], the basic theories for the fractional calculus and the fractional differential equations were discussed. In [5], Benchohra, Hamania and Ntouyas investigated the existence and uniqueness of solutions for problem:

$$\begin{cases} {}^C D^\alpha y(t) = f(t, y(t)), & t \in [0, T], 1 < \alpha \leq 2, \\ y(0) = g(y), y(T) = y_T. \end{cases}$$

By using Schauder's fixed point theorem combined with the diagonalization method, Arara and co-authors (see [4]) studied the existence of solutions for boundary value problems for fractional order differential equation of the form

$$\begin{cases} {}^C D^\alpha y(t) + f(t, y(t)) = 0, & t \in [0, \infty), 1 < \alpha \leq 2, \\ y(0) = y_0, y \text{ is bounded on } [0, \infty). \end{cases}$$

In [2], Ahmad and Nieto studied some existence results for a boundary value problem involving a nonlinear integrodifferential equation of fractional order $1 < q \leq 2$ with integral boundary conditions by using contraction mapping principle and Krasnoselskii's fixed point theorem.

However, researches for the multiple solutions of the fractional differential equations with non-local boundary condition on infinite intervals are few. In this paper, we aim to discuss the multiple solutions for fractional differential equations with integral boundary condition on the half-line. Applying the well-known Amann theorem and the upper and lower solutions method, we obtain a new result on the existence of at least three distinct nonnegative solutions under some conditions. An example is presented to illustrate the application of our main result.

2 Preliminaries

In this section, we introduce preliminary facts which are used throughout this paper. We denote that $\mathbb{R} = (-\infty, +\infty)$ and $\mathbb{R}^+ = [0, +\infty)$.

Definition 2.1 (See [9, 15]) Let $\alpha > 0$. The fractional (arbitrary) order integral of the function $y : \mathbb{R}^+ \rightarrow \mathbb{R}$ of order α is defined by

$$I^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds.$$

provided the integral exists, where Γ is the Gamma function.

Definition 2.2 (See [9, 15]) The Caputo fractional order derivative of the function y of order α is defined by

$${}^C D^\alpha y(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{y^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds,$$

provided the right side is pointwise defined on $(0, +\infty)$, where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α .

Throughout the paper, we suppose that the following hypotheses are satisfied:

(H1) $g \in L^1(\mathbb{R}^+)$, $g(t) \geq 0$, $t \in \mathbb{R}^+$, and $0 \leq \int_0^{+\infty} g(t) dt := \|g\|_1 < 1$.

(H2) $p(t) > 0$ for all $t \in \mathbb{R}^+$, $\int_0^{+\infty} \frac{1}{p(r)} dr$ exists and the function $k(s) = \int_s^{+\infty} \frac{(r-s)^{\alpha-1}}{p(r)} dr < +\infty$ is continuous on \mathbb{R}^+ .

It is obvious that $0 < 1 - \|g\|_1 \leq 1$ if (H1) holds. From (H2), we can get that $k(s) \geq 0$ and $\lim_{s \rightarrow +\infty} k(s) = 0$. So $k(s)$ is bounded, which implies that there exists a constant $K_0 > 0$ such that

$$0 \leq k(s) \leq K_0 = \sup_{s \in \mathbb{R}^+} k(s), \text{ for } s \in \mathbb{R}^+.$$

We define that

$$K(t, s) = \begin{cases} \int_t^{+\infty} \frac{(r-s)^{\alpha-1}}{p(r)} dr, & 0 \leq s < t, \\ \int_s^{+\infty} \frac{(r-s)^{\alpha-1}}{p(r)} dr, & 0 \leq t \leq s, \end{cases} \quad (2.1)$$

and

$$G(t, s) = \frac{1}{(1 - \|g\|_1)\Gamma(\alpha)} \left((1 - \|g\|_1)K(t, s) + \int_0^{+\infty} g(r)K(r, s) dr \right). \quad (2.2)$$

By (H1), (H2), (2.1) and (2.2), we can easily get that K and G satisfy the following lemma.

Lemma 2.1 Suppose (H1) and (H2) hold. Then

- (1) $K(t, s)$ is well defined and continuous, for all $(t, s) \in \mathbb{R}^+ \times \mathbb{R}^+$;
- (2) $0 \leq K(t, s) \leq K(s, s) = k(s) \leq K_0$, for all $(t, s) \in \mathbb{R}^+ \times \mathbb{R}^+$;
- (3) $G(t, s)$ is well defined, continuous, and $0 \leq G(t, s) \leq G(s, s) \leq \frac{1}{(1 - \|g\|_1)\Gamma(\alpha)} K_0$, for all $(t, s) \in \mathbb{R}^+ \times \mathbb{R}^+$;

(4) For any $s \in \mathbb{R}^+$, $\lim_{t \rightarrow +\infty} K(t, s) = 0$, and denote

$$G_\infty(s) := \lim_{t \rightarrow +\infty} G(t, s) = \frac{1}{(1 - \|g\|_1)\Gamma(\alpha)} \int_0^{+\infty} g(r)K(r, s)dr,$$

then $G_\infty(s)$ is continuous, and

$$G_\infty(s) \leq \frac{1}{(1 - \|g\|_1)\Gamma(\alpha)} K_0.$$

□

Lemma 2.2 Suppose that (H1) and (H2) hold, and $h \in L^1(\mathbb{R}^+)$. Then the following boundary value problem

$$\begin{cases} {}^C D^\alpha(p(t)u'(t)) + h(t) = 0, & t > 0, \\ p(0)u'(0) = 0, \\ u(\infty) = \int_0^{+\infty} g(t)u(t)dt, \end{cases} \quad (2.3)$$

has a unique solution

$$u(t) = \int_0^{+\infty} G(t, s)h(s)ds,$$

where $u(\infty) := \lim_{t \rightarrow +\infty} u(t)$.

Proof. By (2.3), we have

$$p(t)u'(t) = p(0)u'(0) - I^\alpha h(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds.$$

We get $u'(t) = -\frac{1}{\Gamma(\alpha)p(t)} \int_0^t (t-s)^{\alpha-1} h(s)ds$, and

$$\begin{aligned} u(t) &= u(0) - \int_0^t \left(\frac{1}{\Gamma(\alpha)p(r)} \int_0^r (r-s)^{\alpha-1} h(s)ds \right) dr \\ &= u(0) - \frac{1}{\Gamma(\alpha)} \int_0^t dr \int_0^r \frac{(r-s)^{\alpha-1} h(s)}{p(r)} ds \\ &= u(0) - \frac{1}{\Gamma(\alpha)} \int_0^t ds \int_s^t \frac{(r-s)^{\alpha-1} h(s)}{p(r)} dr. \end{aligned} \quad (2.4)$$

So

$$u(\infty) = u(0) - \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} ds \int_s^{+\infty} \frac{(r-s)^{\alpha-1} h(s)}{p(r)} dr = \int_0^{+\infty} g(t)u(t)dt.$$

Then

$$u(0) = \int_0^{+\infty} g(r)u(r)dr + \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} ds \int_s^{+\infty} \frac{(r-s)^{\alpha-1} h(s)}{p(r)} dr. \quad (2.5)$$

Substituting (2.5) into (2.4), we have

$$\begin{aligned} u(t) &= \int_0^{+\infty} g(r)u(r)dr + \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} ds \int_s^{+\infty} \frac{(r-s)^{\alpha-1} h(s)}{p(r)} dr - \frac{1}{\Gamma(\alpha)} \int_0^t ds \int_s^t \frac{(r-s)^{\alpha-1} h(s)}{p(r)} dr \\ &= \int_0^{+\infty} g(r)u(r)dr + \frac{1}{\Gamma(\alpha)} \int_0^t ds \int_t^{+\infty} \frac{(r-s)^{\alpha-1} h(s)}{p(r)} dr + \frac{1}{\Gamma(\alpha)} \int_t^{+\infty} ds \int_s^{+\infty} \frac{(r-s)^{\alpha-1} h(s)}{p(r)} dr \\ &= \int_0^{+\infty} g(r)u(r)dr + \frac{1}{\Gamma(\alpha)} \int_0^t \left(\int_t^{+\infty} \frac{(r-s)^{\alpha-1}}{p(r)} dr \right) h(s)ds + \frac{1}{\Gamma(\alpha)} \int_t^{+\infty} \left(\int_s^{+\infty} \frac{(r-s)^{\alpha-1}}{p(r)} dr \right) h(s)ds. \end{aligned}$$

By (2.1), we get that

$$u(t) = \int_0^{+\infty} g(r)u(r)dr + \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} K(t, s)h(s)ds. \quad (2.6)$$

So

$$g(t)u(t) = g(t) \int_0^{+\infty} g(r)u(r)dr + \frac{g(t)}{\Gamma(\alpha)} \int_0^{+\infty} K(t, s)h(s)ds.$$

$$\begin{aligned} \int_0^{+\infty} g(t)u(t)dt &= \int_0^{+\infty} g(t)dt \cdot \int_0^{+\infty} g(r)u(r)dr + \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} \left(g(r) \int_0^{+\infty} K(t, s)h(s)ds \right) dr \\ &= \int_0^{+\infty} g(t)dt \cdot \int_0^{+\infty} g(r)u(r)dr + \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} ds \int_0^{+\infty} g(r)K(r, s)h(s)dr \\ &= \int_0^{+\infty} g(t)dt \cdot \int_0^{+\infty} g(r)u(r)dr + \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} \left(\int_0^{+\infty} K(r, s)g(r)dr \right) h(s)ds. \end{aligned}$$

Noting that $0 < 1 - \|g\|_1 = 1 - \int_0^{+\infty} g(t)dt \leq 1$, we have

$$\int_0^{+\infty} g(r)u(r)dr = \frac{1}{(1 - \|g\|_1)\Gamma(\alpha)} \int_0^{+\infty} \left(\int_0^{+\infty} K(r, s)g(r)dr \right) h(s)ds. \quad (2.7)$$

Substituting (2.7) into (2.6), we have

$$\begin{aligned} u(t) &= \frac{1}{(1 - \|g\|_1)\Gamma(\alpha)} \int_0^{+\infty} \left(\int_0^{+\infty} K(r, s)g(r)dr \right) h(s)ds + \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} K(t, s)h(s)ds \\ &= \frac{1}{(1 - \|g\|_1)\Gamma(\alpha)} \int_0^{+\infty} \left(\int_0^{+\infty} K(r, s)g(r)dr + (1 - \|g\|_1)K(t, s) \right) h(s)ds \\ &= \int_0^{+\infty} G(t, s)h(s)ds, \end{aligned}$$

where

$$G(t, s) = \frac{1}{(1 - \|g\|_1)\Gamma(\alpha)} \left(\int_0^{+\infty} K(r, s)g(r)dr + (1 - \|g\|_1)K(t, s) \right).$$

□

Lemma 2.3 Suppose (H1) and (H2) hold, if $u = u(t)$ satisfies

$$\begin{cases} {}^C D^\alpha(p(t)u'(t)) \leq 0, & t \in (0, +\infty), \\ p(0)u'(0) \leq 0, \\ u(\infty) - \int_0^{+\infty} g(r)u(r)dr \geq 0. \end{cases} \quad (2.8)$$

Then $u(t) \geq 0$ for $t \in \mathbb{R}^+$.

Proof. Let ${}^C D^\alpha(p(t)u'(t)) = -h(t) \leq 0$, $p(0)u'(0) = k_0 \leq 0$ and $u(\infty) - \int_0^{+\infty} g(r)u(r)dr = k_1 \geq 0$. We consider the following boundary value problem

$$\begin{cases} {}^C D^\alpha(p(t)u'(t)) = -h(t), & t \in (0, +\infty), \\ p(0)u'(0) = k_0, \\ u(\infty) - \int_0^{+\infty} g(r)u(r)dr = k_1. \end{cases} \quad (2.9)$$

Similar to the proof of Lemma 2.2, we can obtain that the BVP (2.9) has a unique solution.

$$u(t) = \frac{k_1}{(1 - \|g\|_1)} - \frac{k_0}{(1 - \|g\|_1)} \left(\int_0^{+\infty} ds \int_s^{+\infty} \frac{g(s)}{p(r)} dr + (1 - \|g\|_1) \int_t^{+\infty} \frac{dr}{p(r)} \right) + \int_0^{+\infty} G(t, s) h(s) ds. \quad (2.10)$$

Since $k_0 \leq 0$, $k_1 \geq 0$ and $h(t) \geq 0$ for $t > 0$, it is easy to show

$$u(t) \geq 0, \text{ for } t \in \mathbb{R}^+$$

from (2.10), (H1) and (H2). □

Let E be a Banach space, $P \subset E$ be a cone in E . A cone P is called solid if it contains interior points, i.e., $\overset{\circ}{P} \neq \emptyset$. Every cone P in E defines a partial ordering in E given by $x \preceq y$ iff $y - x \in P$. If $x \preceq y$ and $x \neq y$, we write $x \precneq y$; if a cone P is solid and $y - x \in \overset{\circ}{P}$, we write $x \ll y$. A cone P is said to be normal if there exists a constant $N > 0$ such that $0 \preceq x \preceq y$ implies $\|x\| \leq N\|y\|$. If P is normal, then every order interval $[x, y] = \{z \in E | x \preceq z \preceq y\}$ is bounded.

The following Lemma 2.4 is the well-known Amann three-solution theorem (see [3, 22]), which will be used in the later proof of our main results about the multiple solutions of the boundary value problem.

Lemma 2.4 *Let E be a Banach space, and P be a normal solid cone. Suppose that there exist $\alpha_1, \beta_1, \alpha_2, \beta_2 \in E$ with*

$$\alpha_1 \precneq \beta_1 \precneq \alpha_2 \precneq \beta_2,$$

and $T : [\alpha_1, \beta_2] \rightarrow E$ is a completely continuous strongly increasing operator such that

$$\alpha_1 \preceq T\alpha_1, \quad T\beta_1 \precneq \beta_1, \quad \alpha_2 \precneq T\alpha_2, \quad T\beta_2 \preceq \beta_2.$$

Then the operator T has at least three fixed points x_1, x_2, x_3 such that

$$\alpha_1 \preceq x_1 \ll \beta_1, \quad \alpha_2 \ll x_2 \preceq \beta_2, \quad \alpha_2 \not\prec x_3 \not\prec \beta_1.$$

3 Multiple solutions of the boundary value problem

Definition 3.1. $u = u(t)$ is called an upper (lower) solution of boundary value problem (1.1), if it satisfies

$$\begin{cases} {}^C D^\alpha(p(t)u'(t)) + q(t)f(t, u(t)) \leq 0 (\geq 0), & t > 0, \\ p(0)u'(0) \leq 0 (\geq 0), \\ u(\infty) - \int_0^{+\infty} g(t)u(t)dt \geq 0 (\leq 0). \end{cases}$$

In order to obtain the results, we suppose the following conditions hold:

(H3) $q \in L^1(\mathbb{R}^+)$, $q(t)$ is nonnegative on \mathbb{R}^+ and $q > 0$ a.e. on \mathbb{R}^+ .

(H4) $f : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a Carathéodory function, that is to say, $f(\cdot, u)$ is measurable for any $u \in \mathbb{R}^+$ and $f(t, \cdot)$ is continuous for almost every $t \in \mathbb{R}^+$. $f(t, u)$ is bounded for $t \in \mathbb{R}^+$ when u is bounded, and

$$f(t, u_1) < f(t, u_2) \text{ with } u_1 < u_2 \in \mathbb{R}^+, \text{ for almost every } t \in \mathbb{R}^+.$$

Let $E = \{u \in C(\mathbb{R}^+) \mid \lim_{t \rightarrow +\infty} u(t) \text{ exists}\}$ be endowed with the norm $\|u\| := \sup_{t \in \mathbb{R}^+} |u(t)|$, then E is a Banach space. We define the cone $P \subset E$ by

$$P := \{u \in E \mid u(t) \geq 0, t \in \mathbb{R}^+\}.$$

Obviously, P is a normal solid cone in E , and $u \preceq v \in E$ if and only if $u(t) \leq v(t)$ for $t \in \mathbb{R}^+$. $u \not\preceq v \in E$ if and only if $u(t) \leq v(t)$ and $u(t) \neq v(t)$, which implies that there exists an interval $[a_0, b_0] \subset \mathbb{R}^+$ such that $u(t) < v(t)$ for $t \in [a_0, b_0]$.

Lemma 3.1 (See [11]) *Let E be defined as before and $D \subset E$. Then D is relatively compact in E if the following conditions hold:*

- (a) D is uniformly bounded in E ;
- (b) the functions from D are equicontinuous on any compact interval of $[0, +\infty)$;
- (c) the functions from D are equiconvergent, that is, for any given $\varepsilon > 0$, there exists a $R_0 = R(\varepsilon) > 0$ such that $|u(t) - u(+\infty)| < \varepsilon$, for any $t > R_0$, $u \in D$.

Now, we define an operator $T : P \longrightarrow E$ by

$$(Tu)(t) = \int_0^{+\infty} G(t, s)q(s)f(s, u(s))ds.$$

Lemma 3.2 *Suppose that (H1)–(H4) hold. Then the operator $T : P \longrightarrow P$, and it is completely continuous.*

Proof. First of all, let us show the operator T is well defined, and $T : P \longrightarrow P$.

For any fixed $u \in P$, it implies that u is bounded, by Lemma 2.1, (H3) and (H4), and we can get $(Tu)(t) \geq 0$ for $t \in \mathbb{R}^+$. And there exists a constant $f_{M_0} > 0$ such that $0 \leq f(t, u(t)) \leq f_{M_0}$ for any $t \in \mathbb{R}^+$. Then

$$0 \leq (Tu)(t) = \int_0^{+\infty} G(t, s)q(s)f(s, u(s))ds \leq \frac{K_0 f_{M_0}}{(1 - \|g\|_1)\Gamma(\alpha)} \int_0^{+\infty} q(s)ds < +\infty.$$

And

$$\lim_{t \rightarrow +\infty} (Tu)(t) = \lim_{t \rightarrow +\infty} \int_0^{+\infty} G(t, s)q(s)f(s, u(s))ds = \frac{1}{(1 - \|g\|_1)\Gamma(\alpha)} \int_0^{+\infty} G_\infty(s)q(s)f(s, u(s))ds < +\infty.$$

Thus, $T : P \longrightarrow P$ is well defined.

Secondly, we show that T is continuous.

Let $\{u_n\} \subset P$, $u \in P$, and $u_n \rightarrow u_0$ as $n \rightarrow \infty$. So, there exists a constant $f_{M_1} > 0$, such that $0 \leq f(t, u_n(t)), f(t, u_0(t)) \leq f_{M_1}$ for any $t \in \mathbb{R}^+$. By (H3), (H4) and Lemma 2.1, we can see

$$\begin{aligned} \|Tu_n - Tu_0\| &= \sup_{t \in \mathbb{R}^+} |(Tu_n)(t) - (Tu_0)(t)| \leq \int_0^{+\infty} G(s, s)q(s)|f(s, u_n(s)) - f(s, u_0(s))|ds, \\ G(s, s)q(s)|f(s, u_n(s)) - f(s, u_0(s))| &\leq \frac{2K_0f_{M_1}}{(1 - \|g\|_1)\Gamma(\alpha)}q(s) \end{aligned}$$

and

$$\lim_{n \rightarrow \infty} (f(s, u_n(s)) - f(s, u_0(s))) = 0, \text{ a.e. } s \in \mathbb{R}^+.$$

According to the Lebesgue's dominated convergence theorem, we can show

$$\lim_{n \rightarrow \infty} \int_0^{+\infty} G(s, s)q(s)|f(s, u_n(s)) - f(s, u_0(s))|ds = 0.$$

Therefore, the operator T is continuous.

Finally, we will prove that the operator T maps bounded sets into relatively compact sets.

For the bounded set $\Omega \subset P$, there exists a constant $M_2 > 0$, such that $\|u\| \leq M_2$ for any $u \in \Omega$. Thus there exists a constant $f_{M_2} > 0$, such that $0 \leq f(t, u(t)) \leq f_{M_2}$ for any $t \in \mathbb{R}^+$. And

$$0 \leq (Tu)(t) = \int_0^{+\infty} G(t, s)q(s)f(s, u(s))ds \leq \frac{K_0f_{M_2}}{(1 - \|g\|_1)\Gamma(\alpha)} \int_0^{+\infty} q(s)ds < +\infty.$$

Thus, the set $T(\Omega)$ is uniformly bounded.

For any $[a, b] \subset [0, +\infty)$ and any $t_1, t_2 \in [a, b]$, by Lemma 2.1, we have $G(t_1, s) - G(t_2, s) \rightarrow 0$, as $t_1 \rightarrow t_2$ for any $s \in \mathbb{R}^+$. And

$$0 \leq |G(t_1, s) - G(t_2, s)|q(s)f(s, u(s)) \leq 2G(s, s)q(s)f_{M_2} \leq \frac{2K_0f_{M_2}}{(1 - \|g\|_1)\Gamma(\alpha)}q(s).$$

Then

$$\begin{aligned} |(Tu)(t_1) - (Tu)(t_2)| &= \left| \int_0^{+\infty} G(t_1, s)q(s)f(s, u(s))ds - \int_0^{+\infty} G(t_2, s)q(s)f(s, u(s))ds \right| \\ &\leq \int_0^{+\infty} |G(t_1, s) - G(t_2, s)|q(s)f(s, u(s))ds \\ &\leq \int_0^{+\infty} |G(t_1, s) - G(t_2, s)|q(s)f_{M_2}ds \\ &\rightarrow 0, \text{ as } t_1 \rightarrow t_2. \end{aligned} \tag{3.1}$$

That is, Tu from $T(\Omega)$ is equicontinuous on any compact interval of $[0, +\infty)$.

By Lemma 2.1, we have

$$\begin{aligned} |(Tu)(t) - (Tu)(\infty)| &= \left| \int_0^{+\infty} (G(t, s) - G_\infty(s))q(s)f(s, u(s))ds \right| \\ &\leq \int_0^{+\infty} |G(t, s) - G_\infty(s)|q(s)f_{M_2}ds \\ &\rightarrow 0, \text{ as } t \rightarrow +\infty. \end{aligned}$$

Then Tu from $T(\Omega)$ is equiconvergent.

Using Lemma 3.1, we can obtain that the set $T(\Omega)$ is a relatively compact set. Hence, the operator T maps bounded sets into relatively compact sets.

Therefore, we can get that the operator T is completely continuous. □

Theorem 3.3 *Suppose that (H1)–(H4) hold, and there exist two lower solutions x_1, x_2 and two upper solutions y_1, y_2 of boundary value problem (1.1) such that x_2, y_1 are not the solutions of the boundary value problem (1.1) with*

$$x_1 \not\leq y_1 \not\leq x_2 \not\leq y_2.$$

Then the boundary value problem (1.1) has at least three distinct nonnegative solutions u_1, u_2, u_3 which satisfy that for $t \in \mathbb{R}^+$

$$x_1(t) \leq u_1(t) < y_1(t), \quad x_2(t) < u_2(t) \leq y_2(t), \quad x_2(t) \not\leq u_3(t) \not\leq y_1(t).$$

Proof. It is obvious that the boundary value problem (1.1) has nonnegative solutions if and only if the operator T has fixed points on P .

It follows from Lemma 3.2 that $T : [x_1, y_2] \rightarrow P$ is completely continuous.

Let us prove that T is a strongly increasing operator.

For any $w_1, w_2 \in P$, with $w_1 \not\leq w_2$, that is to say that $w_1(t) \leq w_2(t)$ for all $t \in \mathbb{R}^+$, and there exists $[a_0, b_0] \subset \mathbb{R}^+$ such that $w_1(t) < w_2(t)$ for any $t \in [a_0, b_0]$.

Hence, for any $t \in \mathbb{R}^+$, using the conditions (H3) and (H4) we have

$$\begin{aligned} (Tw_2)(t) - (Tw_1)(t) &= \int_0^{+\infty} G(t, s)q(s)(f(s, w_2(s)) - f(s, w_1(s)))ds \\ &\geq \int_{a_0}^{b_0} G(t, s)q(s)(f(s, w_2(s)) - f(s, w_1(s)))ds \\ &> 0. \end{aligned}$$

We can get that

$$0 < (Tw_2)(t) - (Tw_1)(t) \in \mathring{P}.$$

Hence, we conclude that T is a strongly increasing operator.

Let us now prove that $x_1 \not\leq Tx_1$.

We denote $x = Tx_1 - x_1$.

Noting that x_1 is the lower solution of boundary value problems (1.1) and applying the definition

of the operator T , we have

$$\begin{aligned} {}^C D^\alpha(p(t)x'(t)) &= {}^C D^\alpha(p(t)((Tx_1)'(t) - x_1'(t))) \\ &= {}^C D^\alpha(p(t)(Tx_1)'(t)) - {}^C D^\alpha(p(t)x_1'(t)) \\ &= -q(t)f(t, x_1(t)) - {}^C D^\alpha(p(t)x_1'(t)) \\ &\leq 0, \end{aligned}$$

$$x'(0) = (Tx_1)'(0) - x_1'(0) \leq 0,$$

and

$$x(\infty) - \int_0^{+\infty} g(t)x(t)dt = (Tx_1)(\infty) - x_1(\infty) - \int_0^{+\infty} g(t)((Tx_1)(t) - x_1(t))dt \geq 0.$$

It follows from Lemma 2.3 that

$$x(t) = (Tx_1)(t) - x_1(t) \geq 0, \text{ for } t \in \mathbb{R}^+.$$

Then

$$x_1 \preceq Tx_1.$$

Similarly, we can get that

$$x_2 \preceq Tx_2.$$

Since x_2 is an lower solution of (1.1) and not a solution of (1.1), we have $(Tx_2) \neq x_2$. Thus

$$x_2 \not\preceq Tx_2.$$

Using the same method, we can also get that

$$Ty_1 \not\preceq y_1, \quad Ty_2 \preceq y_2.$$

Using the Lemma 2.4, we obtain T has at least three fixed points $u_1, u_2, u_3 \in [x_1, y_2]$.

Moreover, $u_1, u_2, u_3 \in P$ and

$$x_1 \preceq u_1 \ll y_1, \quad x_2 \ll u_2 \preceq y_2, \quad x_2 \not\preceq u_3 \not\preceq y_1.$$

Hence, the boundary value problem (1.1) has at least three distinct nonnegative solutions $u_1, u_2, u_3 \in [x_1, y_2]$ and we see for $t \in \mathbb{R}^+$

$$x_1(t) \leq u_1(t) < y_1(t), \quad x_2(t) < u_2(t) \leq y_2(t), \quad x_2(t) \not\leq u_3(t) \not\leq y_1(t).$$

□

4 Illustration

To illustrate our main results, we present an example.

Example 4.1. Consider the following integral boundary value problem

$$\begin{cases} {}^C D^{\frac{1}{2}}(p(t)u'(t)) + q(t)f(t, u) = 0, & t > 0, \\ p(0)u'(0) = 0, \\ u(\infty) = \int_0^{+\infty} e^{-2t}u(t)dt, \end{cases} \quad (4.1)$$

where

$$p(t) = e^t, \quad f(t, u) = \frac{200}{\sqrt{\pi}} \begin{cases} u^2, & 0 \leq u < 1, \\ \sqrt{u}, & u \geq 1. \end{cases}$$

We take $\alpha = \frac{1}{2}$, $q(t) = te^{-t}$, $g(t) = e^{-2t}$. It is easy to show that

$$g \in L^1(\mathbb{R}^+), \quad 0 \leq \|g\|_1 = \int_0^{+\infty} g(t)dt = \frac{1}{2} < 1, \quad \text{and } 1 - \|g\|_1 = 1 - \int_0^{+\infty} g(t)dt = \frac{1}{2},$$

then (H1) holds.

$$p(t) = e^t > 0, \quad k(s) = \int_s^{+\infty} \frac{(r-s)^{\alpha-1}}{p(r)} dr = \sqrt{\pi}e^{-s}, \quad s \in \mathbb{R}^+,$$

that is (H2) holds.

$$q(t) = te^{-t}, \quad \int_0^{+\infty} q(t)dt = \int_0^{+\infty} te^{-t}dt = 1 < +\infty,$$

it implies that (H3) holds.

We can easily verify the condition (H4) holds.

In view of Lemma 2.2, for any $h \in L^1(\mathbb{R}^+)$, $u(t) = \int_0^{+\infty} G(t, s)h(s)ds$ satisfies the boundary conditions of (4.1).

Now, let $h(t) = \frac{200t}{\sqrt{\pi}}e^{-t}$. It is obvious $h \in L^1(\mathbb{R}^+)$.

For $t \in \mathbb{R}^+$, we take

$$x_1(t) = 0, \quad x_2(t) = \frac{1}{24^2} \int_0^{+\infty} G(t, s)h(s)ds,$$

and

$$y_1(t) = \frac{1}{53^2} \int_0^{+\infty} G(t, s)h(s)ds, \quad y_2(t) = 53 \int_0^{+\infty} G(t, s)h(s)ds.$$

Then $x_1, x_2, y_1, y_2 \in P$.

It is easy to see $0 = x_1(t) < y_1(t) < x_2(t) < y_2(t)$ for $t \in \mathbb{R}^+$, that is $x_1 \precneq y_1 \precneq x_2 \precneq y_2$.

Moreover, we have

$$p(0)x'_i(0) = 0, \quad p(0)y'_i(0) = 0, \quad x_i(\infty) = \int_0^{+\infty} e^{-2t}x_i(t)dt, \quad y_i(\infty) = \int_0^{+\infty} e^{-2t}y_i(t)dt, \quad i = 1, 2.$$

Through calculation, we can get that $24 < \int_0^{+\infty} G(t, s)h(s)ds < 53$, and we can easily verify

$$\begin{aligned} {}^C D^{\frac{1}{2}}(p(t)x_1'(t)) + q(t)f(t, x_1(t)) &= 0, \\ {}^C D^{\frac{1}{2}}(p(t)x_2'(t)) + q(t)f(t, x_2(t)) &> {}^C D^{\frac{1}{2}}(p(t)x_2'(t)) + \frac{1}{24^2}h(t) = 0, \\ {}^C D^{\frac{1}{2}}(p(t)y_1'(t)) + q(t)f(t, y_1(t)) &< {}^C D^{\frac{1}{2}}(p(t)y_1'(t)) + \frac{1}{53^2}h(t) = 0 \end{aligned}$$

and

$${}^C D^{\frac{1}{2}}(p(t)y_2'(t)) + q(t)f(t, y_2(t)) \leq {}^C D^{\frac{1}{2}}(p(t)y_2'(t)) + 53h(t) = 0.$$

Therefore, $x_1(t), x_2(t)$ are lower solutions of BVP (4.1), and $y_1(t), y_2(t)$ are upper solutions of BVP (4.1).

It follows from Theorem 3.3 that the boundary value problem (4.1) has at least three distinct nonnegative solutions $u_1, u_2, u_3 \in [x_1, y_2]$. Moreover, for $t \in \mathbb{R}^+$

$$x_1(t) \leq u_1(t) < y_1(t), \quad x_2(t) < u_2(t) \leq y_2(t), \quad x_2(t) \not\leq u_3(t) \not\leq y_1(t).$$

□

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(Received February 20, 2011)