Electronic Journal of Qualitative Theory of Differential Equations
2011, No. **64**, 1-11; **http://www.math.u-szeged.hu/ejqtde/**

Approximate Solutions for Fractional Differential Equation in the Unit Disk

Rabha W. Ibrahim

Institute of Mathematical Sciences

University Malaya, 50603

Kuala Lumpur, Malaysia

E-mail: rabhaibrahim@yahoo.com

Abstract.

In this paper we establish the existence solution approximately for differential equation of fractional order takes the form

$$z^{\alpha}D[D^{\alpha}u(z)] + (b-z)u'(z) - au(z) = 0, \quad a \neq 0,\ |b| < 1,\ 0 < \alpha < 1,$$

subject to the initial conditions $u(0) = u_0$ and $u'(0) = u_1$, in the unit disk $U := \{z \in \mathbb{C} : |z| < 1\}$. The uniqueness of this solution is discussed. The general analytic solutions are posed. Moreover, the Hyers-Ulam stability is studied. An example is illustrated.

Keywords: Approximate solution; Fractional operators; Fractional differential equation; Unit disk; Hyers-Ulam stability; Existence and uniqueness.

AMS Mathematics Subject Classification: 34A12.

1 Introduction.

Recently, fractional differential equations and inclusions have been of great interest. It is caused both by the intensive development of the theory of fractional calculus [9] itself and by the applications of such constructions in various sciences and topics such as physics, mechanics, chemistry, engineering, control systems, etc. [1,4,10,11,12,18]. Moreover, fractional differential equations in complex domain have been studied and established [5-7].

In this paper, we deal with the existence of Kummer differential equation of fractional order in the complex plane. The Kummer differential equation which is also called the confluent hypergeometric differential equation, appears frequently in practical problems and applications. These equations have proved useful in many branches of physics. They have been used in problems involving diffusion, for example, in isotope separation and protein molecular weight determinations in the ultracentrifuge. The solution of the equation for the velocity distribution of electrons in high frequency gas discharges may frequently be expressed in terms of these functions. The high frequency breakdown electric field may then be predicted theoretically for gases by the use of such solutions together with kinetic theory [3,8,17,19]. Furthermore, an immensely useful class of special functions (namely, the generalized hypergeometric function) played a rather crucial role in the theory of analytic and univalent functions. These latter developments in an area other than the so-called traditional areas of applications of generalized hypergeometric functions have naturally provided a new impetus for the study of such an important class of special functions. Finally, hypergeometric function and its generalizations employed to define certain families of linear operators which reduced in terms of (for example) fractional derivatives and fractional integrals, Hadamard product or convolution [13,14].

In [15], Srivastava and Owa, gave definitions for fractional operators (derivative and integral) in the complex z-plane $\mathbb{C}$ as follows:

Definition 1.1. The fractional derivative of order α is defined, for a function $f(z)$ by

$$D_z^\alpha f(z) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^\alpha} d\zeta; \quad 0 \le \alpha < 1,$$

where the function $f(z)$ is analytic in simply-connected region of the complex z-plane $\mathbb{C}$ containing the origin and the multiplicity of $(z-\zeta)^{-\alpha}$ is removed by requiring $log(z-\zeta)$ to be real when$(z-\zeta) > 0$.

Definition 1.2. The fractional integral of order α is defined, for a function $f(z)$, by

$$I_z^\alpha f(z) := \frac{1}{\Gamma(\alpha)} \int_0^z f(\zeta)(z-\zeta)^{\alpha-1} d\zeta; \quad \alpha > 0,$$

where the function $f(z)$ is analytic in simply-connected region of the complex z-plane ($\mathbb{C}$) containing the origin and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $log(z-\zeta)$ to be real when$(z-\zeta) > 0$.

Remark 1.1.

$$D_z^\alpha z^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} z^{\mu-\alpha}, \quad \mu > -1; \ 0 \le \alpha < 1$$

and

$$I_z^\alpha z^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} z^{\mu+\alpha}, \quad \mu > -1; \ \alpha > 0.$$

In this paper we study the existence of holomorphic solution at the origin of the fractional pantograph equation in the unit disk takes the form

$$z^\alpha D[D^\alpha u(z)] + (b-z)u'(z) - au(z) = 0, \quad D = \frac{d}{dz}, \tag{1}$$

$$(a \neq 0, \ \ |b| < 1, \ \ 0 < \alpha < 1)$$

subject to the initial conditions

$$u(0) = u_0 \quad \text{and} \quad u'(0) = u_1,$$

in the unit disk U such that $u : U \to \mathbb{C}$.

2 Existence and Uniqueness

We have the following main results:

Theorem 2.1. Consider the initial-value problem (1). If one of the following cases is hold:

- $a = -j$ for $j \in \mathbb{N} \cup \{0\}$,
- $a \neq -j$ and $\alpha \to 0$,
- $a \neq -j$ and $\alpha \to 1$,

then the problem (1) has a holomorphic solution $u(z)$, $z \in U$. Moreover, if

$$b = -\frac{(m+1-\alpha)\Gamma(m+1)}{(m+1)\Gamma(m+2-\alpha)} \tag{2}$$

for some non negative integer m then (1) has no holomorphic solution U.

Proof. Let u be a solution to the initial- value problem (1) such that u is a holomorphic at $z = 0$ then u can be represented as a power series of the form

$$u(z) = \sum_{n=0}^{\infty} b_n z^n, \ \ (z \in U).$$

Substituting this expression into the equation (1), we obtain

$$b_n = b_0 \prod_{k=0}^{n-1} \frac{(a+k)}{\frac{\Gamma(k+2)(k+1-\alpha)}{\Gamma(k+2-\alpha)} + (k+1)b}.$$

For $u_0 \neq 0$ and $u_1 \neq 0$ we have $b_0 \neq 0$. If $a = -j$ for non negative integer j, then it is clear that the solution is a polynomial which is converge in U.
Now let $a \neq -j$ then we have the ratio

$$\left|\frac{b_{n+1}}{b_n}\right| = \left|\frac{(a+n)}{\frac{\Gamma(n+2)(n+1-\alpha)}{\Gamma(n+2-\alpha)} + (n+1)b}\right|.$$

For $a \neq -j$ and $\alpha \to 0$, we obtain that

$$\Big|\frac{b_{n+1}}{b_n}\Big| = |\frac{1}{1+b}|, \quad |b|<1, \text{ as } n \to \infty$$

this poses that there is a converge solution for the problem (1) in the unit disk. Also for $a \neq -j$ and $\alpha \to 1$, we have

$$\Big|\frac{b_{n+1}}{b_n}\Big| \to 0, \quad \text{as } n \to \infty,$$

thus the problem (1) has a solution. Finally, if b satisfies the relation (2), this implies that b_n is undefined and hence the problem (1) has no solution in the unit disk.

Theorem 2.2. Assume that $\frac{|a|+(1-|b|)\gamma}{(1-\alpha)\Gamma(\alpha+1)} < 1$, where $\gamma > 0$ satisfies $\|u'-v'\| \leq \gamma\|u-v\|$ then problem (1) has a unique solution.

Proof. Let u be a solution to the initial- value problem (1) such that u is a holomorphic at $z = 0$. The differential equation (1) can be converted into the equation:

$$D_z^\alpha u(z) = H(u(z), u'(z)), \quad (z \in U), \tag{3}$$

where

$$H(u(z), u'(z)) = \int_0^z \eta^{-\alpha}[au(\eta) - (b-\eta)u'(\eta)]d\eta.$$

We establish the existence of a local solution by showing that the integral equation

$$u(z) = \frac{1}{\Gamma(\alpha)}\int_0^z H(u(\zeta), u'(\zeta))(z-\zeta)^{\alpha-1}d\zeta + u_0 + zu_1, \quad (z \in U). \tag{4}$$

is a contraction mapping. Let $\mathcal{H}(\overline{U})$ denote the Banach space of functions in

$$\overline{U} := \{z \in U : |z| \leq r, \ 0 < r < 1\}$$

equipped with the norm $\|.\|$, $\|u\| = \sup_{z\in\overline{U}}|u|$, and let $\mathcal{P} : \mathcal{H}(\overline{U}) \to \mathcal{H}(\overline{U})$ be the operator defined by

$$\mathcal{P}u = \frac{1}{\Gamma(\alpha)}\int_0^z H(u(\zeta), u'(\zeta))(z-\zeta)^{\alpha-1}d\zeta + u_0 + zu_1.$$

For any $0 < r < 1$ the function H is holomorphic in u. Now for any $u, v \in \mathcal{H}(\overline{U})$,

$$\begin{aligned}
\|\mathcal{P}u - \mathcal{P}v\| &= \|\frac{1}{\Gamma(\alpha)}\int_0^z H(u(\zeta), u'(\zeta))(z-\zeta)^{\alpha-1}d\zeta - \frac{1}{\Gamma(\alpha)}\int_0^z H(v(\zeta), v(\zeta))(z-\zeta)^{\alpha-1}d\zeta\| \\
&\leq \frac{|z|^\alpha}{\Gamma(\alpha+1)}\|H(u(\zeta), u(\beta\zeta)) - H(v(\zeta), v(\beta\zeta))\| \\
&\leq \frac{sup_{z\in\overline{U}}|z|\Big(|a| + \gamma(|b-z|)\Big)\Big|u(z) - v(z)\Big|}{(1-\alpha)\Gamma(\alpha+1)}.
\end{aligned}$$

Let

$$M := max\frac{\Big(|a| + \gamma(1 - |b|)\Big)}{(1-\alpha)\Gamma(\alpha+1)}$$

then for sufficiently $0 < r < 1$ we pose

$$\|\mathcal{P}u - \mathcal{P}v\| \le rM\|u - v\|.$$

Hence there is a constant $0 < \rho < 1$ such that

$$\|\mathcal{P}u - \mathcal{P}v\| \le \rho\|u - v\|.$$

Therefore the initial-value problem (1) has a unique solution in $\mathcal{H}(\overline{U})$.

3 General Solution

In this section, we will determine the general solution of the inhomogeneous fractional differential equation

$$z^\alpha D[D^\alpha u(z)] + (b - z)u'(z) - au(z) = \sum_{n=0}^{\infty} a_n z^n, \quad D = \frac{d}{dz}, \tag{5}$$

where neither a nor b is a non-positive integer and the coefficients a_n of the power series are given such that the radius of convergence is $\sigma > 0$. A power series solution of

$$z^\alpha D[D^\alpha u(z)] + (b - z)u'(z) - au(z) = 0 \tag{6}$$

is given by

$$w(z) = \sum_{n=0}^{\infty} \frac{(a)_n}{[\gamma]_n} z^n, \quad z \in U,$$

where $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & n = 0 \\ a(a+1)...(a+n-1), & n = \{1, 2, ...\}. \end{cases}$$

And

$$[\gamma]_n := \prod_{k=0}^{n-1} \gamma_k = \prod_{k=0}^{n-1} \Big[\frac{\Gamma(k+2)(k+1-\alpha)}{\Gamma(k+2-\alpha)} + (k+1)b\Big].$$

Here we assume that $b_0 = 1$. For some values of a and b, the power series $w(z)$ converges for all values $z \in U$.

Theorem 3.1. Let a, b be real constants such that $a \notin \mathbb{Z}^-$ and $\frac{ab}{\gamma_0} = 1$. Assume that the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$, $z \in U$ is $0 < \sigma < 1$ and that there exists a positive real number μ satisfies $\frac{1+\mu}{|1+b|} \le 1$ and

$$\Big|\frac{[\gamma]_n a_n}{(a)_{n+1}}\Big| \le \mu \Big|\sum_{j=0}^{n-1} \frac{a_j [\gamma]_j}{(a)_{j+1}}\Big| \tag{7}$$

for all sufficiently large integer n. Then every solution u of the non-homogenous equation (5) can be represented by

$$u(z) = u_h(z) + \sum_{n=1}^{\infty}\sum_{j=0}^{n-1} \frac{(a)_n[\gamma]_j a_j}{[\gamma]_n (a)_{j+1}} z^n,$$

where u_h is a solution for (6).

Proof. Let $u_p(z) = u(z) - u_h(z)$. We show that $u_p(z)$ satisfies the equation (5). Since

$$u_p'(z) = \sum_{n=1}^{\infty}\sum_{j=0}^{n-1} n\frac{(a)_n[\gamma]_j a_j}{[\gamma]_n (a)_{j+1}} z^{n-1} = \sum_{n=0}^{\infty}\sum_{j=0}^{n} (n+1)\frac{(a)_{n+1}[\gamma]_j a_j}{[\gamma]_{n+1} (a)_{j+1}} z^n,$$

$$z^\alpha D[D^\alpha u_p(z)] = \sum_{n=1}^{\infty}\sum_{j=0}^{n} (n-\alpha)\frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}\frac{(a)_n[\gamma]_j a_j}{[\gamma]_n (a)_{j+1}} z^{n-1},$$

we obtain

$$\begin{aligned} z^\alpha D[D^\alpha u_p(z)] + (b-z)u_p'(z) - a u_p(z) &= \frac{ab}{\gamma_0}a_0 \\ &+ \sum_{n=1}^{\infty}\sum_{j=0}^{n} \frac{(a)_{n+1}[\gamma]_j a_j}{[\gamma]_{n+1} (a)_{j+1}}[\frac{\Gamma(n+2)(n+1-\alpha)}{\Gamma(n+2-\alpha)} + nb]z^n \\ &- \sum_{n=1}^{\infty}\sum_{j=0}^{n-1} \frac{(a)_n[\gamma]_j a_j}{[\gamma]_n (a)_{j+1}}[n+a]z^n \\ &= a_0 + \sum_{n=1}^{\infty} a_n z^n. \end{aligned}$$

Next we prove that the power expression of $u_p(z)$ converges for all $z \in U$. By applying the ratio test to the power series expression of $u_p(z)$ and using the assumption (7) we have

$$\sum_{n=1}^{\infty}\sum_{j=0}^{n-1} \frac{(a)_n[\gamma]_j a_j}{[\gamma]_n (a)_{j+1}} z^n = \sum_{n=1}^{\infty} c_n z^n$$

and for $\alpha \to 0$ we pose

$$\begin{aligned} \lim_{n\to\infty} |\frac{c_{n+1}}{c_n}| &\le \lim_{n\to\infty} \left|\frac{a+n}{\frac{\Gamma(n+2)(n+1-\alpha)}{\Gamma(n+2-\alpha)} + (n+1)b}\right| \Big[1 + \left|\frac{[\gamma]_n a_n}{(a)_{n+1}}\right| \Big|\sum_{j=0}^{n-1} \frac{[\gamma]_j a_j}{(a)_{j+1}}\Big|^{-1}\Big] \\ &\le \frac{1+\mu}{|1+b|} \le 1. \end{aligned}$$

Therefore the power series of $u_p(z)$ converges for all $z \in U$ and consequently those of $u_p'(z)$ and $D^\alpha u_p(z)$. Hence the power series expression for $z^\alpha D[D^\alpha u(z)] + (b-z)u'(z) - au(z)$ has the same convergence region (the maximum open disk where the relevant power series converges) as that of $u_p(z)$. This completes the proof.

4 Hyers-Ulam Stability

Assume that X and Y are a topological vector space and a normed space, respectively, and that I is an open subset of X. If for any function $f : I \to Y$ satisfying the differential inequality

$$\|a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_0(x)y(x) + h(x)\| \leq \varepsilon$$

for all $x \in I$ and for some $\varepsilon \geq 0$, there exists a solution $f_0 : I \to Y$ of the differential equation

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_0(x)y(x) + h(x) = 0$$

such that $\|f(x) - f_0(x)\| \leq K(\varepsilon)$ then we say that the above differential equation satisfies the Hyers-Ulam stability (or the local Hyers-Ulam stability if the domain I is not the whole space X) (see [2]).
In the following theorem, we shall prove a local Hyers-Ulam stability of the equation (1) under additional conditions.

Theorem 4.1. Let a, b be real constants such that $a \notin \mathbb{Z}^-$. Suppose a function $u : U \to \mathbb{C}$ is represented by the power series $\sum_{n=0}^{\infty} b_n z^n$, $z \in U$. Assume that there exist nonnegative constants $\mu \neq 0$ and ν satisfying the condition

$$\Big|\frac{[\gamma]_n a_n}{(a)_{n+1}}\Big| \leq \mu \Big|\sum_{j=0}^{n-1} \frac{a_j[\gamma]_j}{(a)_{j+1}}\Big| \leq \nu \Big|\frac{(n+1)[\gamma]_n a_n}{(a)_{n+1}}\Big|, \quad \forall\, n \in \mathbb{N}_0 \tag{8}$$

and there is a constant $L \geq 0$ such that

$$\sum_{n=0}^{\infty} |a_n z^n| \leq L \Big|\sum_{n=0}^{\infty} a_n z^n\Big|, \quad z \in U,$$

where

$$a_n = \Big[\frac{(n+1+\alpha)\Gamma(n+1)}{\Gamma(n+1-\alpha)} + b(n+1)\Big] b_{n+1} - (n+a)b_n. \tag{9}$$

If u satisfies the fractional differential inequality

$$\Big|z^\alpha D[D^\alpha u(z)] + (b-z)u'(z) - au(z)\Big| \leq \varepsilon \tag{10}$$

for all $z \in U$ and for some $\varepsilon \geq 0$ then there exists a solution $u_h : U \to \mathbb{C}$ of the equation (1) such that

$$\Big|u(z) - u_h(z)\Big| \leq \begin{cases} \frac{\nu}{\mu} L\varepsilon, & a > 1 \\ \frac{\nu}{\mu} L\varepsilon \Big[\sum_{n=0}^{n_0-1} \Big| |\frac{n+1}{n+a}| - |\frac{n+2}{n+1+a}| \Big| + \Big(\frac{n_0+1}{n_0+a}\Big)\Big], & a \leq 1, \end{cases}$$

where $n_0 = max\{0, \lceil -a \rceil\}$.

Proof. By the definition of a_n, we have

$$\begin{aligned}
&|z^\alpha D[D^\alpha u(z)] + (b-z)u'(z) - au(z)| \\
&\qquad = \Big|\sum_{n=0}^{\infty}\Big[\Big(\frac{(n+1+\alpha)\Gamma(n+1)}{\Gamma(n+1-\alpha)} + b(n+1)\Big)b_{n+1} - (n+a)b_n\Big]z^n\Big| \\
&\qquad = |\sum_{n=0}^{\infty} a_n z^n| \\
&\qquad \le \varepsilon.
\end{aligned}$$

Consequently yields

$$\sum_{n=0}^{\infty}|a_n z^n| \le L|\sum_{n=0}^{\infty} a_n z^n| \le L\varepsilon, \quad z \in U. \tag{11}$$

In view of Theorem 3.1, we have

$$u(z) = u_h(z) + \sum_{n=1}^{\infty}\sum_{j=0}^{n-1}\frac{(a)_n[\gamma]_j a_j}{[\gamma]_n (a)_{j+1}} z^n, \quad z \in U.$$

Hence by using Abel′s formula (see [16]), we can estimate

$$\begin{aligned}
|u(z) - u_h(z)| &= \Big|\sum_{n=0}^{\infty}\sum_{j=0}^{n-1}\frac{(a)_n[\gamma]_j a_j}{[\gamma]_n (a)_{j+1}} z^n\Big| \\
&\le \sum_{n=0}^{\infty}\Big|\frac{a_n(n+1)z^n}{n+a}\Big|\Big|\frac{(a)_{n+1}}{(n+1)a_n[\gamma]_n}\Big|\Big|\sum_{j=0}^{n-1}\frac{[\gamma]_j a_j}{(a)_{j+1}}\Big| \\
&\le \frac{\nu}{\mu}\lim_{m\to\infty}\sum_{n=0}^{m}\Big|\frac{(n+1)a_n z^n}{n+a}\Big| \\
&= \frac{\nu}{\mu}\lim_{m\to\infty}\Big(\sum_{j=0}^{m}|a_j z^j|\Big)|\frac{m+2}{m+1+a}| \\
&+ \sum_{n=0}^{m}\Big(\sum_{j=0}^{n}|a_j z^j|\Big)\Big(|\frac{n+1}{n+a}| - |\frac{n+2}{n+1+a}|\Big).
\end{aligned}$$

Assume that $n_0 = max\{0, \lceil -a\rceil\}$ where $\lceil -a\rceil$ is the ceiling of $-a$. It is clear that

$$\text{if } a > 1, \text{ then } \frac{n+1}{n+a} < \frac{n+2}{n+1+a} \quad \text{for } n \ge 0;$$
$$\text{if } a \le 1, \text{ then } \frac{n+1}{n+a} \ge \frac{n+2}{n+1+a} \quad \text{for } n \ge n_0.$$

Thus we have

$$|u(z)-u_h(z)| \le \begin{cases} \frac{\nu}{\mu}\lim_{n\to\infty}\Big[L\varepsilon|\frac{n+2}{n+1+a}| + \sum_{n=0}^{m} L\varepsilon(\frac{n+2}{n+1+a}-\frac{n+1}{n+a})\Big], & a>1 \\ \\ \frac{\nu}{\mu}\lim_{n\to\infty}\Big[L\varepsilon|\frac{n+2}{n+1+a}| + \sum_{n=0}^{n_0-1} L\varepsilon\Big||\frac{n+1}{n+a}|-|\frac{n+2}{n+1+a}|\Big|\Big], & a\le 1 \\ +\sum_{n=n_0}^{m} L\varepsilon\Big(\frac{n+1}{n+a}-\frac{n+2}{n+1+a}\Big) \end{cases}$$

$$\le \begin{cases} \frac{\nu}{\mu}L\varepsilon, & a>1 \\ \\ \frac{\nu}{\mu}L\varepsilon\Big[\sum_{n=0}^{n_0-1}\Big||\frac{n+1}{n+a}|-|\frac{n+2}{n+1+a}|\Big| + \Big(\frac{n_0+1}{n_0+a}\Big)\Big], & a\le 1 \end{cases}$$

for all $z\in U$. This completes the proof.

5 An example

For fix $\alpha=0.5$, $a=b=1$, $\varepsilon>0$, $b_0=0$ and $b_n=\frac{\varepsilon}{s(n+1)}$, $n\ge 1$. Define a function $u(z)=\sum_{n=0}^{\infty} b_n z^n$, $z\in U$. Setting

$$a_n = \Big[\frac{(n+3/2)\Gamma(n+1)}{(n+2)\Gamma(n+0.5)} + \frac{n+1}{n+2} - 1\Big]\frac{\varepsilon}{s}, \quad n\in\mathbb{N} \tag{12}$$

and using (9), we pose $a_0 \simeq \frac{\varepsilon}{s}$. It is clear that a_n are positive for all n, therefore,

$$|\sum_{n=0}^{\infty} a_n z^n| \ge a_0 \simeq \frac{\varepsilon}{s}. \tag{13}$$

Assume that the reduce of the solution satisfies $|z|\le r<1$, then relation (12) implies

$$\begin{aligned} \sum_{n=0}^{\infty}|a_n z^n| &\le \sum_{n=0}^{\infty} a_n r^n = a_0 + \sum_{n=1}^{\infty} a_n r^n \\ &= \sum_{n=1}^{\infty}\Big[\frac{(n+3/2)\Gamma(n+1)}{(n+2)\Gamma(n+0.5)} + \frac{n+1}{n+2}\Big]\frac{\varepsilon}{s} r^n \\ &\le \sum_{n=1}^{\infty}\frac{2\varepsilon}{s} r^n \\ &= \varepsilon \end{aligned} \tag{14}$$

where $s := 2\sum_{n=1}^{\infty} r^n$. Hence from (13) and (14), we obtain (11)

$$\sum_{n=0}^{\infty}|a_n z^n| \le \frac{2}{1-r}|\sum_{n=0}^{\infty} a_n z^n|.$$

It is clear that (8) is satisfied, thus in virtu of Theorem 4.1, there exists a solution $u_h : U \to \mathbb{C}$ of the equation (1) such that

$$|u(z) - u_h(z)| \le \frac{2\varepsilon}{1-r}, \quad z \in U.$$

References

[1] R. Caponetto, G. Dongola, L. Fortuna, I. Petras: Fractional Order Systems: Modeling and Control Applications. World Scientific, Singapore, 2010.

[2] S. Czerwik, Functional Equations and Inequalities in Several Variables, World Scientific, River Edge, NJ, USA, 2002.

[3] K. Dholakia, Optics: Against the spread of the light, Nature 45 (2008) 413.

[4] R. W. Ibrahim, On the existence for diffeo-integral inclusion of Sobolev-type of fractional order with applications, ANZIAM, J. 52 (2010) 1-21.

[5] R. W. Ibrahim, M. Darus, Subordination and superordination for univalent solutions for fractional differential equations, J. Math. Anal. Appl. 345 (2008) 871-879.

[6] R. W. Ibrahim, M. Darus, Subordination and superordination for analytic functions involving fractional integral operator, Complex Variables and Elliptic Equations, , Volume 53, Issue 11, (2008) 1021- 1031.

[7] R. W. Ibrahim, Existence and uniqueness of holomorphic solutions for fractional Cauchy problem, J. Math. Anal. Appl. 380 (2011) 232-240.

[8] S. K. Khan, Boundary layer viscoelastic fluid flow over an exponentially stretching sheet, Int. J. of Applied Mechanics and Engineering, vol.11, No.2,(2006) 321-335.

[9] A. A. Kilbas , H. M. Srivastava, J. J. Trujillo, Theory and applications of fractional differential equations, North-Holland, Mathematics Studies, Elsevier 2006.

[10] R. Magin: Fractional Calculus in Bioengineering. Begell House Inc., Redding, 2006.

[11] F. Mainardi: Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models. Imperial College Press, London, 2010.

[12] I. Petras, I. Podlubny, P. O'Leary, L. Dorcak, B. Vinagre: Analogue Realization of Fractional Order Controllers. FBERG, Technical University of Kosice, Kosice, 2002.

[13] T.N. Shanmugam, Hypergeometric functions in the geometric function theory, Applied Mathematics and Computation, Vol. 187, Issue 1, 1 (2007) 433-444.

[14] H. Silverman, Thomas Rosy, and S. Kavitha, On certain Sufficient condition involving Gaussian hypergeometric functions, International Journal of Mathematics and Mathematical Sciences Volume 2009, Article ID 989603, 15 pages.

[15] H. M. Srivastava, S. Owa, Univalent Functions, Fractional Calculus, and Their Applications, Halsted Press, John Wiley and Sons, New York, Chichester, Brisban, and Torento, 1989.

[16] W. R. Wade, An Introduction to Analysis, PrenticeHall, Upper Saddle River, NJ, USA, 2nd edition, 2000.

[17] H. Wang, L. Shi, B. Lukyanchuk, C. Sheppard and C. T. Chong,Creation of a needle of longitudinally polarized light in vacuum using binary optics, Nat. Photon. 2(2008) 501-505.

[18] G. M. Zaslavsky: Hamiltonian Chaos and Fractional Dynamics, Oxford University Press, 2005.

[19] Q. Zhan, Properties of circularly polarized vortex beams, Opt. Lett. 31(2006) 867-869.

(Received January 25, 2011)