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EXISTENCE OF ALMOST PERIODIC SOLUTIONS TO SOME THIRD-ORDER NONAUTONOMOUS DIFFERENTIAL EQUATIONS

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Abstract. In this paper using the well-known Schauder fixed point theorem we study and obtain the existence of almost periodic mild solutions to some classes of nonautonomous third-order differential equations on a separable infinite dimensional complex Hilbert space.

1. Introduction

Let $\mathbb{H}$ be a separable infinite dimensional complex Hilbert space. In the recent papers by Diagana [10, 11], the existence of almost periodic solutions to some second-order nonautonomous differential equations was obtained. For that, Diagana made extensive use of dichotomy tools and Schauder fixed point theorem.

In this paper using the well-known Schauder fixed point principle, we study the problem which consists of the existence of almost periodic solutions to the nonautonomous third-order differential equations

$$\frac{d}{dt}\Big[u'' + g(t, Bu(t))\Big] = w(t)Au(t) + f(t, Cu(t)), \quad t \in \mathbb{R} \tag{1.1}$$

where the following preliminary assumptions will be made:

- (**i**) $A : D(A) \subset \mathbb{H} \mapsto \mathbb{H}$ is a self-adjoint linear operator on $\mathbb{H}$ whose spectrum consists of isolated eigenvalues $0 < \lambda_1 < \lambda_2 < ... < \lambda_l \to \infty$ as $l \to \infty$ with each eigenvalue having a finite multiplicity γ_j equals to the multiplicity of the corresponding eigenspace;
- (**ii**) the algebraic sum of the (possibly unbounded) linear operators B and C defined by $B+C : D(B) \cap D(C) \subset \mathbb{H} \mapsto \mathbb{H}$ is assumed to be a nontrivial linear operator and the following holds

$$\mathbb{H}_\alpha := \big(\mathbb{H}, D(A)\big)_{\alpha,\infty} \subset D(B) \cap D(C), \tag{1.2}$$

  with $\mathbb{H}_\alpha := \big(\mathbb{H}, D(A)\big)_{\alpha,\infty}$ being the real interpolation space of order (α, ∞) between $\mathbb{H}$ and $D(A)$ [23];
- (**iii**) the function $w : \mathbb{R} \mapsto \mathbb{R}$ given by $w(t) = -\rho(t)$ for all $t \in \mathbb{R}$ is assumed to be almost periodic and further there exist two constants $\rho_0, \rho_1 > 0$ satisfying the following conditions

$$\rho_0 \le \rho(t) \le \rho_1 \quad \text{for all} \quad t \in \mathbb{R}; \quad \text{and} \tag{1.3}$$

- (**iv**) the functions $f, g : \mathbb{R} \times D(A) \mapsto \mathbb{H}$ are almost periodic in the first variable uniformly in the second variable.

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As an immediate application of one of the main results of the paper (Theorem 4.1), we study and obtain the existence of almost periodic solutions to the nonautonomous third-order differential equations

$$u''' + B(t)u' + A(t)u = h(t,u), \tag{1.4}$$

where $A(t) = q(t)A$, and $B(t) = p(t)B$ for each $t \in \mathbb{R}$, the functions $p, q : \mathbb{R} \mapsto \mathbb{R}$ are almost periodic, A, B are the same as in Eq. (1.1), and the function $h : \mathbb{R} \times \mathbb{H}_\alpha \mapsto \mathbb{H}$, $(t,u) \mapsto h(t,u)$ is almost periodic in $t \in \mathbb{R}$ uniformly in $u \in \mathbb{H}_\alpha$ (Theorem 5.1).

To deal with Eq. (1.1), we rewrite it as a nonautonomous first-order differential equation on $\mathbb{H} \times \mathbb{H} \times \mathbb{H}$ and next study the obtained first order differential equation with the help of the well-known Schauder fixed point principle. Indeed, assuming that $u \in D(A)$ is three times differentiable and setting

$$z := \begin{pmatrix} u \\ \\ u' \\ \\ u'' + g(t, Bu) \end{pmatrix},$$

then Eq. (1.1) can be rewritten on $\mathbb{X} := \mathbb{H} \times \mathbb{H} \times \mathbb{H}$ in the following form

$$\frac{dz}{dt} = \mathcal{A}(t)z + F(t, z(t)), \ t \in \mathbb{R}, \tag{1.5}$$

where $\mathcal{A}(t)$ is the family of 3×3-operator matrices defined by

$$\mathcal{A}(t) = \begin{pmatrix} 0 & I_{\mathbb{H}} & 0 \\ \\ 0 & 0 & I_{\mathbb{H}} \\ \\ w(t)A & 0 & 0 \end{pmatrix}, \qquad t \in \mathbb{R}, \tag{1.6}$$

where $I_{\mathbb{H}}$ is the identity operator of $\mathbb{H}$.

Clearly, domain $D(\mathcal{A}(t)) = D(A) \times \mathbb{H} \times \mathbb{H}$ is constant in $t \in \mathbb{R}$.

The vector-valued function F appearing in Eq. (1.5) is defined on $\mathbb{R} \times \mathbb{X}_\alpha$ for some $\alpha \in (0,1)$ by

$$F(t, z(t)) = \begin{pmatrix} 0 \\ \\ -g(t, Bu) \\ \\ f(t, Cu) \end{pmatrix},$$

where $\mathbb{X}_\alpha$ is the real interpolation space of order (α, ∞) between $\mathbb{X}$ and $D(\mathcal{A}(t))$ which is explicitly given by

$$\mathbb{X}_\alpha = \big(\mathbb{X}, D(\mathcal{A}(t))\big)_{\alpha,\infty} = \big(\mathbb{H} \times \mathbb{H} \times \mathbb{H}, D(A) \times \mathbb{H} \times \mathbb{H}\big)_{\alpha,\infty} = \mathbb{H}_\alpha \times \mathbb{H} \times \mathbb{H}.$$

Clearly, if $p : \mathbb{R} \mapsto \mathbb{R}$ is differentiable, one can easily check that Eq. (1.4) is a special case of Eq. (1.1). Indeed, Eq. (1.4) can be rewritten as

$$\frac{d}{dt}\left[\frac{du}{dt} + \widetilde{g}(t, Bu(t))\right] = \widetilde{w}(t)Au(t) + \widetilde{f}(t, Bu(t)), \quad t \in \mathbb{R} \tag{1.7}$$

where $C = B$, $\widetilde{w}(t) = -q(t)$, $\widetilde{g}(t, Bu) = p(t)Bu$, and $\widetilde{f}(t, Bu) = h(t, u) + p'(t)Bu$ for all $t \in \mathbb{R}$.

Once we rewrite Eq. (1.7) in the form Eq. (1.5), its corresponding vector-valued function F which we denote by $\widetilde{F}$ is defined on $\mathbb{R} \times \mathbb{X}_\alpha$ for some $\alpha \in (0, 1)$ by

$$\widetilde{F}(t, z(t)) = \begin{pmatrix} 0 \\ \\ -p(t)Bu \\ \\ h(t, u) + p'(t)Bu \end{pmatrix}.$$

The stability, asymptotic behavior, boundedness and the existence of solutions to third-order differential equations have been widely studied in the literature [18, 19, 20, 27, 28, 30, 31]. However, to the best of our knowledge, the original problem which consists of the existence of almost periodic mild solutions to both Eq. (1.1) and then to Eq. (1.4), remains an untreated question which constitutes the main impetus of this paper. In order to study the above-mentioned issues, we will make extensive use of ideas and techniques utilized in [4, 10, 11, 17, 21], the exponential stability of the associated evolution family, and the Schauder fixed point theorem. For more on abstract second- and higher-order differential and related issues, see, e.g., [6, 7, 13, 16, 25, 32, 33, 34].

2. Preliminaries

Let $(\mathbb{X}, \|\cdot\|)$ be a Banach space. If L is a linear operator on the Banach space $\mathbb{X}$, then, $D(L)$, $\rho(L)$, $\sigma(L)$, $N(L)$, and $R(L)$ stand respectively for its domain, resolvent, spectrum, null-space or kernel; and range. If $L : D = D(L) \subset \mathbb{X} \mapsto \mathbb{X}$ is a linear operator, one sets $R(\lambda, L) := (\lambda I - L)^{-1}$ for all $\lambda \in \rho(A)$.

If $\mathbb{Y}, \mathbb{Z}$ are Banach spaces, then the space $B(\mathbb{Y}, \mathbb{Z})$ denotes the collection of all bounded linear operators from $\mathbb{Y}$ into $\mathbb{Z}$ equipped with its natural topology. This is simply denoted by $B(\mathbb{Y})$ when $\mathbb{Y} = \mathbb{Z}$. If P is a projectin, we set $Q = I - P$.

Let $B(\mathbb{R}, \mathbb{X})$ stand for the Banach space of all bounded continuous functions $\varphi : \mathbb{R} \mapsto \mathbb{X}$ when equipped with the sup norm defined by

$$\|\varphi\|_\infty := \sup_{t\in\mathbb{R}} \|\varphi(t)\|$$

for $\varphi \in BC(\mathbb{R}, \mathbb{X})$. Similarly, $B(\mathbb{R}, \mathbb{X}_\alpha)$ for $\alpha \in (0, 1)$, stands for the Banach space of all bounded continuous functions $\varphi : \mathbb{R} \mapsto \mathbb{X}_\alpha$ when equipped with the α-sup norm

$$\|\varphi\|_{\alpha,\infty} := \sup_{t\in\mathbb{R}} \|\varphi(t)\|_\alpha$$

for $\varphi \in BC(\mathbb{R}, \mathbb{X}_\alpha)$.

2.1. **Almost Periodic Functions.**

Definition 2.1. A continuous function $f : \mathbb{R} \mapsto \mathbb{X}$ is called (Bohr) almost periodic if for each $\varepsilon > 0$ there exists $l(\varepsilon) > 0$ such that every interval of length $l(\varepsilon)$ contains a number τ with the property that

$$\big\|f(t+\tau) - f(t)\big\| < \varepsilon \quad \text{for each} \quad t \in \mathbb{R}.$$

The number τ above is called an *ε-translation* number of f, and the collection of all such functions will be denoted $AP(\mathbb{X})$.

Definition 2.2. A continuous function $F : \mathbb{R} \times \mathbb{X} \mapsto \mathbb{X}$ is called (Bohr) almost periodic in $t \in \mathbb{R}$ uniformly in $x \in K$ where $K \subset \mathbb{X}$ is any compact subset if for each $\varepsilon > 0$ there exists $l(\varepsilon)$ such that every interval of length $l(\varepsilon)$ contains a number τ with the property that

$$\big\|F(t+\tau, y) - F(t, y)\big\| < \varepsilon \quad \text{for each} \quad t \in \mathbb{R}, \quad y \in K.$$

The proof of our main result requires the following composition theorems.

Theorem 2.3. *Let $F : \mathbb{R} \times \mathbb{X} \mapsto \mathbb{X}$ be an almost periodic function. Suppose that there exists $L \geq 0$ such that*

$$\big\|F(t, u) - F(t, v)\big\| \leq L\big\|u - v\big\|$$

for all $u, v \in \mathbb{X}$ for all $t \in \mathbb{R}$. If $g \in AP(\mathbb{X})$, then $\Gamma : \mathbb{R} \to \mathbb{X}$ defined by $\Gamma(\cdot) := F(\cdot, g(\cdot))$ belongs to $AP(\mathbb{X})$.

Theorem 2.4. *Let $F : \mathbb{R} \times \mathbb{X} \mapsto \mathbb{X}$ be an almost periodic function. Suppose $F(t, u)$ is uniformly continuous on every bounded subset $K' \subset \mathbb{X}$ uniformly for $t \in \mathbb{R}$. If $g \in AP(\mathbb{X})$, then $\Gamma : \mathbb{R} \to \mathbb{X}$ defined by $\Gamma(\cdot) := F(\cdot, g(\cdot))$ belongs to $AP(\mathbb{X})$.*

For more on almost periodic functions and related issues, we refer the reader to Corduneanu [8] and Diagana [14].

2.2. **Evolution Families.** Note that this subsection is similar to that of Diagana [9]. For the sake of clarity, we reproduce it here.

Definition 2.5. A family of closed linear operators $A(t)$ for $t \in \mathbb{R}$ on $\mathbb{X}$ with domain $D(A(t))$ (possibly not densely defined) satisfy the so-called Acquistapace–Terreni conditions, that is, there exist constants $\omega \in \mathbb{R}$, $\theta \in \left(\frac{\pi}{2}, \pi\right)$, $K, L \geq 0$ and $\mu, \nu \in (0, 1]$ with $\mu + \nu > 1$ such that

$$S_\theta \cup \{0\} \subset \rho\left(A(t) - \omega\right) \ni \lambda, \qquad \left\|R\left(\lambda, A(t) - \omega\right)\right\| \leq \frac{K}{1 + |\lambda|} \tag{2.1}$$

and

$$\left\|\left(A(t) - \omega\right) R\left(\lambda, A(t) - \omega\right) \left[R\left(\omega, A(t)\right) - R\left(\omega, A(s)\right)\right]\right\| \leq L\, |t - s|^{\mu}\, |\lambda|^{-\nu} \tag{2.2}$$

for $t, s \in \mathbb{R}$, $\lambda \in S_\theta := \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| \leq \theta\}$.

Note that in the particular case when $A(t)$ has a constant domain $D = D(A(t))$, it is well-known [3, 26] that Eq. (2.2) can be replaced with the following: there exist constants L and $0 < \mu \leq 1$ such that

$$\left\|\left(A(t) - A(s)\right) R\left(\omega, A(r)\right)\right\| \leq L\, |t - s|^{\mu}, \quad s, t, r \in \mathbb{R}. \tag{2.3}$$

Among other things, it ensures that there exists a unique evolution family

$$U = \{U(t,s) : t, s \in \mathbb{R} \ \text{ such that } \ t \geq s\}$$

on $\mathbb{X}$ associated with $A(t)$ such that $U(t,s)\mathbb{X} \subseteq D(A(t))$ for all $t, s \in \mathbb{R}$ with $t \geq s$, and

(a) $U(t,s)U(s,r) = U(t,r)$ for $t, s \in \mathbb{R}$ such that $t \geq s \geq s$;

(b) $U(t,t) = I$ for $t \in \mathbb{R}$ where I is the identity operator of $\mathbb{X}$;

(c) $(t,s) \mapsto U(t,s) \in B(\mathbb{X})$ is continuous for $t > s$;

(d) $U(\cdot, s) \in C^1((s,\infty), B(\mathbb{X}))$, $\dfrac{\partial U}{\partial t}(t,s) = A(t)U(t,s)$ and

$$\left\| A(t)^k U(t,s) \right\| \leq K\,(t-s)^{-k}$$

for $0 < t - s \leq 1$, $k = 0, 1$; and

(e) $\partial_s^+ U(t,s)x = -U(t,s)A(s)x$ for $t > s$ and $x \in D(A(s))$ with $A(s)x \in \overline{D(A(s))}$.

It should also be mentioned that the above-mentioned proprieties were mainly established in [1, Theorem 2.3] and [36, Theorem 2.1], see also [2, 35]. In that case we say that $A(\cdot)$ generates the evolution family $U(\cdot,\cdot)$.

Definition 2.6. An evolution family $U = \{U(t,s) : t, s \in \mathbb{R} \ \text{ such that } \ t \geq s\}$ is said to have an *exponential dichotomy* if there are projections $P(t)$ $(t \in \mathbb{R})$ that are uniformly bounded and strongly continuous in t and constants $\delta > 0$ and $N \geq 1$ such that

(f) $U(t,s)P(s) = P(t)U(t,s)$;

(g) the restriction $U_Q(t,s) : Q(s)\mathbb{X} \to Q(t)\mathbb{X}$ of $U(t,s)$ is invertible (we then set $U_Q(s,t) := U_Q(t,s)^{-1}$); and

(h) $\|U(t,s)P(s)\| \leq Ne^{-\delta(t-s)}$ and $\|U_Q(s,t)Q(t)\| \leq Ne^{-\delta(t-s)}$ for $t \geq s$ and $t, s \in \mathbb{R}$.

According to [29], the following sufficient conditions are required for $A(t)$ to have exponential dichotomy.

(i) Let $(A(t), D(t))_{t\in\mathbb{R}}$ be generators of analytic semigroups on $\mathbb{X}$ of the same type. Suppose that $D(A(t)) \equiv D(A(0))$, $A(t)$ is invertible,

$$\sup_{t,s\in\mathbb{R}} \left\| A(t)A(s)^{-1} \right\| < \infty,$$

and

$$\left\| A(t)A(s)^{-1} - I \right\| \leq L_0\,|t-s|^{\mu}$$

for $t, s \in \mathbb{R}$ and constants $L_0 \geq 0$ and $0 < \mu \leq 1$.

(j) The semigroups $(e^{\tau A(t)})_{\tau\geq 0}$, $t \in \mathbb{R}$, are hyperbolic with projection P_t and constants $N, \delta > 0$. Moreover, let

$$\left\| A(t)e^{\tau A(t)} P_t \right\| \leq \psi(\tau)$$

and

$$\left\| A(t)e^{\tau A_Q(t)} Q_t \right\| \leq \psi(-\tau)$$

for $\tau > 0$ and a function ψ such that $\mathbb{R} \ni s \mapsto \varphi(s) := |s|^{\mu}\,\psi(s)$ is integrable with $L_0\,\|\varphi\|_{L^1(\mathbb{R})} < 1$.

2.3. **Estimates for $U(t,s)$.** It should be mentioned that this subsection is similar to that of Diagana [9]. For the sake of clarity, we reproduce all these notions here, too. This setting requires some estimates related to $U(t,s)$. For that, we make extensive use of the real interpolation spaces of order (α,∞) between $\mathbb{X}$ and $D(A(t))$, where $\alpha \in (0,1)$. We refer the reader to the excellent book of Lunardi [23] for proofs and further information on theses interpolation spaces.

Let A be a sectorial operator on $\mathbb{X}$ (for that, in Definition 2.5, replace $A(t)$ with A) and let $\alpha \in (0,1)$. Define the real interpolation space

$$\mathbb{X}_\alpha^A := \left\{ x \in \mathbb{X} : \|x\|_\alpha^A := \sup_{r>0} \|r^\alpha (A-\omega)R(r, A-\omega)x\| < \infty \right\},$$

which, by the way, is a Banach space when endowed with the norm $\|\cdot\|_\alpha^A$. For convenience we further write

$$\mathbb{X}_0^A := \mathbb{X},\ \|x\|_0^A := \|x\|,\ \mathbb{X}_1^A := D(A)$$

and

$$\|x\|_1^A := \|(\omega - A)x\|.$$

Moreover, let $\hat{\mathbb{X}}^A := \overline{D(A)}$ of $\mathbb{X}$. In particular, we have the following continuous embedding

$$D(A) \hookrightarrow \mathbb{X}_\beta^A \hookrightarrow D((\omega - A)^\alpha) \hookrightarrow \mathbb{X}_\alpha^A \hookrightarrow \hat{\mathbb{X}}^A \hookrightarrow \mathbb{X}, \tag{2.4}$$

for all $0 < \alpha < \beta < 1$, where the fractional powers are defined in the usual way.

In general, $D(A)$ is not dense in the spaces $\mathbb{X}_\alpha^A$ and $\mathbb{X}$. However, we have the following continuous injection

$$\mathbb{X}_\beta^A \hookrightarrow \overline{D(A)}^{\|\cdot\|_\alpha^A}$$

for $0 < \alpha < \beta < 1$.

Given the family of linear operators $A(t)$ for $t \in \mathbb{R}$, satisfying (H.1), we set

$$\mathbb{X}_\alpha^t := \mathbb{X}_\alpha^{A(t)}, \quad \hat{\mathbb{X}}^t := \hat{\mathbb{X}}^{A(t)}$$

for $0 \le \alpha \le 1$ and $t \in \mathbb{R}$, with the corresponding norms. Then the embedding in Eq. (2.4) holds with constants independent of $t \in \mathbb{R}$. These interpolation spaces are of class $\mathcal{J}_\alpha$ ([23, Definition 1.1.1]) and hence there is a constant $c(\alpha)$ such that

$$\|y\|_\alpha^t \le c(\alpha) \|y\|^{1-\alpha} \|A(t)y\|^\alpha, \quad y \in D(A(t)).$$

We have the following fundamental estimates for the evolution family $U(t,s)$.

Proposition 2.7. [4] *Suppose the evolution family U has exponential dichotomy. For $x \in \mathbb{X}$, $0 \le \alpha \le 1$ and $t > s$, the following hold:*

(i) There is a constant $c(\alpha)$, such that

$$\|U(t,s)P(s)x\|_\alpha^t \le c(\alpha) e^{-\frac{\delta}{2}(t-s)} (t-s)^{-\alpha} \|x\|. \tag{2.5}$$

(ii) There is a constant $m(\alpha)$, such that

$$\left\| \widetilde{U}_Q(s,t)Q(t)x \right\|_\alpha^s \le m(\alpha) e^{-\delta(t-s)} \|x\|. \tag{2.6}$$

Remark 2.8. Note that if an evolution family U is exponential stable, that is, there exists constants $N, \delta > 0$ such that $\|U(t,s)\| \le Ne^{-\delta(t-s)}$ for $t \ge s$, then its dichotomy projection $P(t) = I$ ($Q(t) = I - P(t) = 0$). In that case, Eq. (2.5) still holds and can be rewritten as follows: for all $x \in \mathbb{X}$,

$$\|U(t,s)x\|_\alpha^t \le c(\alpha)e^{-\frac{\delta}{2}(t-s)}(t-s)^{-\alpha}\|x\|. \tag{2.7}$$

We will need the following assumptions:

(H.1) The linear operators $\{A(t)\}_{t\in\mathbb{R}}$ whose domains are constant in t satisfy the Acquistapace–Terreni conditions.

Let $U = \{U(t,s) : t, s \in \mathbb{R} \text{ such that } t \ge s\}$ denote the evolution family associated with the family of linear operators $A(t)$.

(H.2) The evolution family $U(t,s)$ is compact for $t > s$ and is exponential stable, that is, there exists constants $N, \delta > 0$ such that $\|U(t,s)\| \le Ne^{-\delta(t-s)}$ for $t \ge s$.

Remark 2.9. Under assumption (H.2), it can be shown that for each given $t \in \mathbb{R}$ and $\tau > 0$, the family $\{U(\cdot,s) : s \in (-\infty, t-\tau)\}$ is equicontinuous in t for the uniform operator topology.

3. Existence of Almost Periodic Mild Solutions

Let α, β are real numbers such that $0 < \alpha < \beta < 1$. The bound of the injection $\mathbb{X}_\beta \hookrightarrow \mathbb{X}_\alpha$ will be denoted by c, i.e.,

$$\|u(t)\|_\alpha \le c\|u(t)\|_\beta$$

for all $u \in \mathbb{X}_\beta$.

Consider the nonautonomous differential

$$u'(t) = A(t)u(t) + F(t, u(t)), \quad t \in \mathbb{R}, \tag{3.1}$$

where $F : \mathbb{R} \times \mathbb{X}_\alpha \mapsto \mathbb{X}$ is jointly continuous.

The rest of Section is slightly is similar to the one given Diagana [9]. However, for the sake of clarity, we reproduce it here.

Definition 3.1. Under assumption (H.1), a continuous function $u : \mathbb{R} \mapsto \mathbb{X}_\alpha$ is said to be a mild solution to Eq. (3.1) provided that

$$u(t) = U(t,s)u(s) + \int_s^t U(t,\tau)F(\tau, u(\tau))d\tau \tag{3.2}$$

for each $\forall t \ge s,\ t, s \in \mathbb{R}$.

Let us indicate that if $F : \mathbb{R} \times \mathbb{X}_\alpha \mapsto \mathbb{X}$ is a jointly continuous bounded function, then u satisfying

$$u(t) = \int_{-\infty}^t U(t,s)F(s, u(s))ds. \tag{3.3}$$

for all $t \in \mathbb{R}$, is a mild solution to Eq. (3.1).

This setting requires the following additional assumptions:

(H.3) $R(\omega, A(\cdot)) \in AP(B(\mathbb{X}_\alpha, \mathbb{X}))$ where ω is the constant appearing in Definition 2.5.

(H.4) The function $F : \mathbb{R} \times \mathbb{X}_\alpha \mapsto \mathbb{X}$ is almost periodic in the first variable uniformly in the second one. For each bounded subset $K \subset \mathbb{X}_\alpha$, $F(\mathbb{R}, K)$ is bounded. Moreover, the function $u \mapsto F(t, u)$ is uniformly continuous on any bounded subset K of $\mathbb{X}_\alpha$ for each $t \in \mathbb{R}$. Finally, we suppose that there exists $L > 0$ such that

$$\sup_{t\in\mathbb{R},\ \ \|u\|_\alpha \le L} \|F(t,u)\| \le \frac{L}{e(\beta)},$$

where $e(\beta) := cc(\beta)\delta^\beta \Gamma(1-\beta)$.

(H.5) Let $(u_n)_{n\in\mathbb{N}} \subset AP(\mathbb{X}_\alpha)$ be uniformly bounded and uniformly convergent in every compact subset of $\mathbb{R}$. Then $F(\cdot, u_n(\cdot))$ is relatively compact in $BC(\mathbb{R}, \mathbb{X}_\alpha)$.

Set

$$(Su)(t) = \int_{-\infty}^{t} U(t,s)P(s)F(s,u(s))ds.$$

We need the following Lemma.

Lemma 3.2. [9, Diagana] *Under assumptions* (H.1)–(H.3)*, the mapping* $S : BC(\mathbb{R}, \mathbb{X}_\alpha) \mapsto BC(\mathbb{R}, \mathbb{X}_\alpha)$ *is well-defined and continuous.*

Proof. First of all, $S(BC(\mathbb{R}, \mathbb{X}_\alpha)) \subset BC(\mathbb{R}, \mathbb{X}_\alpha)$. Indeed, setting $g(t) := F(t, u(t))$ and using Proposition 2.7, we obtain

$$\begin{aligned}
\|Su(t)\|_\alpha &\le c\|Su(t)\|_\beta \\
&\le c\int_{-\infty}^{t} \|U(t,s)g(s)\|_\beta ds \\
&\le cc(\beta)\int_{-\infty}^{t} e^{-\frac{\delta}{2}(t-s)}(t-s)^{-\beta}\|g(s)\|ds \\
&\le cc(\beta)\|g\|_\infty \int_{0}^{+\infty} e^{-\sigma}\left(\frac{2\sigma}{\delta}\right)^{-\beta}\frac{2d\sigma}{\delta} \\
&\le cc(\beta)\delta^\beta\Gamma(1-\beta)\|g\|_\infty,
\end{aligned}$$

and hence $\|Su(t)\|_\alpha \le e(\beta)\|g\|_\infty$ for all $t \in \mathbb{R}$.

To complete the proof, we need to show that that S is continuous. For that consider an arbitrary sequence of functions $u_n \in BC(\mathbb{R}, \mathbb{X}_\alpha)$ which converges uniformly to some $u \in BC(\mathbb{R}, \mathbb{X}_\alpha)$, that is, $\|u_n - u\|_{\alpha,\infty} \to 0$ as $n \to \infty$.

Now

$$\begin{aligned}
&\Big\| \int_{-\infty}^{t} U(t,s)P(s)[F(s,u_n(s)) - F(s,u(s))]\, ds\Big\|_\alpha \\
&\quad \le c(\alpha)\int_{-\infty}^{t} (t-s)^{-\alpha}e^{-\frac{\delta}{2}(t-s)}\|F(s,u_n(s)) - F(s,u(s))\|\, ds.
\end{aligned}$$

Now, using the continuity of F and the Lebesgue Dominated Convergence Theorem we conclude that

$$\Big\| \int_{-\infty}^{t} U(t,s)P(s)[F(s,u_n(s)) - F(s,u(s))]\, ds \Big\|_{\alpha} \to 0 \quad \text{as} \quad n \to \infty \,,$$

and hence $\big\| Su_n - Su \big\|_{\alpha,\infty} \to 0$ as $n \to \infty$.

□

Lemma 3.3. [9, Diagana] *Let $B_\alpha = \{u \in AP(\mathbb{X}_\alpha) : \|u\|_\alpha \le L\}$. Under assumptions* (H.1)-(H.2)-(H.4)*, then the functions in $S(B_\alpha)$ are equicontinuous on $\mathbb{R}$.*

Theorem 3.4. [9, Diagana] *Suppose assumptions* (H.1)—(H.5) *hold, then Eq. (3.1) has at least one almost periodic mild solution*

Proof. First of all, note that using the proof of Lemma 3.2 one can easily show that $S(B_\alpha) \subset B_\alpha$. In view of Lemma 3.2 and Lemma 3.3, it remains to show that $V = \{Su(t) : u \in B_\alpha\}$ is a relatively compact subset of $\mathbb{X}_\alpha$ for each $t \in \mathbb{R}$. For that, fix $t \in \mathbb{R}$ and consider an arbitrary $\varepsilon > 0$.

We have

$$\begin{aligned} (S_\varepsilon u)(t) &:= \int_{-\infty}^{t-\varepsilon} U(t,s)F(s,u(s))ds, \; u \in B_\alpha \\ &= U(t,t-\varepsilon)\int_{-\infty}^{t-\varepsilon} U(t-\varepsilon,s)F(s,u(s))ds, \; u \in B_\alpha \\ &= U(t,t-\varepsilon)(Su)(t-\varepsilon), \; u \in B_\alpha \end{aligned}$$

and hence $V_\varepsilon := \{S_\varepsilon u(t) : u \in B_\alpha\}$ is relatively compact in $\mathbb{X}_\alpha$ as $U(t,t-\varepsilon)$ is compact by assumption.

Now

$$\begin{aligned} \|Su(t) - U(t,t-\varepsilon)\int_{-\infty}^{t-\varepsilon} U(t-\varepsilon,s)F(s,u(s))ds\|_\alpha &\le c\|Su(t) - U(t,t-\varepsilon)\int_{-\infty}^{t-\varepsilon} U(t-\varepsilon,s)F(s,u(s))ds\|_\beta \\ &\le c\int_{t-\varepsilon}^{t} \|U(t,s)F(s,u(s))\|_\beta ds \\ &\le cc(\beta)\int_{t-\varepsilon}^{t} e^{-\frac{\delta}{2}(t-s)}(t-s)^{-\beta}\, \|F(s,u(s))\|\, ds \\ &\le \frac{cc(\beta)L}{e(\beta)}\int_0^\varepsilon e^{-\frac{\delta}{2}\sigma}\sigma^{-\beta}d\sigma \\ &\le \frac{cc(\beta)L}{e(\beta)}\int_0^\varepsilon \sigma^{-\beta}d\sigma \\ &= \frac{cc(\beta)L}{(1-\beta)e(\beta)}\varepsilon^{1-\beta}. \end{aligned}$$

The rest of the proof follows slightly along the same lines as in [22]. Indeed, using the facts that B_α is a closed convex subset of $AP(\mathbb{X}_\alpha)$ and that $S(B_\alpha) \subset B_\alpha$, one can easily see that $\overline{co}\, S(B_\alpha) \subset B_\alpha$. Consequently, the following inclusions hold

$$S(\overline{co}\, S(B_\alpha)) \subset S(B_\alpha) \subset \overline{co}\; S(B_\alpha).$$

Moreover, one can easily check that $\{u(t) : u \in \overline{co}\, S(B_\alpha)\}$ is relatively compact in $\mathbb{X}_\alpha$ for each fixed $t \in \mathbb{R}$ and that functions in $\overline{co}\, S(B_\alpha)$ are equicontinuous on $\mathbb{R}$. By the well-known Arzela-Ascoli theorem, the restriction of $\overline{co}\, S(B_\alpha)$ to any compact subset I of $\mathbb{R}$ is relatively compact in $C(I, \mathbb{X}_\alpha)$. In view of the above, it follows that $S : \overline{co}\, S(B_\alpha) \mapsto \overline{co}\, S(B_\alpha)$ is continuous and compact. Using the Schauder fixed point it follows that S has a fixed-point, which obviously is an almost periodic mild solution to Eq. (3.1).

□

4. Almost Periodic Solutions to Eq. (1.1)

Fix once and for all an infinite dimensional separable complex Hilbert space $\mathbb{H}$ equipped with a norm and inner product denoted respectively by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$.

Let $\{e_j^k\}$ be a (complete) orthonormal sequence of eigenvectors associated with the eigenvalues $\{\lambda_j\}_{j\geq 1}$. Therefore, for each

$$u \in D(A) := \Big\{ u \in \mathbb{H} : \quad \sum_{j=1}^{\infty} \lambda_j^2 \big\|E_j u\big\|^2 < \infty \Big\},$$

$$Au = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} \langle u, e_j^k \rangle e_j^k = \sum_{j=1}^{\infty} \lambda_j E_j u$$

where $E_j u = \displaystyle\sum_{k=1}^{\gamma_j} \langle u, e_j^k \rangle e_j^k$.

In this section, we let $\mathbb{X} = \mathbb{H}^3 = \mathbb{H} \times \mathbb{H} \times \mathbb{H}$, which is a Hilbert space when equipped with its Euclidean topology. To study the existence of almost periodic solutions to Eq. (1.1), in addition to the previous assumptions, we suppose that the following additional assumption holds:

(H.6) The linear operator $B, C : \mathbb{H}_\alpha \mapsto \mathbb{H}$ are bounded. Let $K > 0$ be their bound, that is,

$$\big\|Bu\big\| \leq K\big\|u\big\|_\alpha \quad \text{and} \quad \big\|Cu\big\| \leq K\big\|u\big\|_\alpha$$

for all $u \in \mathbb{H}_\alpha$.

Theorem 4.1. *Under previous assumptions and if (H.4)—(H.6) hold, then Eq. (1.1) has at least one almost periodic solution $u \in \mathbb{H}_\alpha$.*

Proof. For all $z := \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in D = D(\mathcal{A}(t)) = D(A) \times \mathbb{H} \times \mathbb{H}$, we obtain the following

$$\mathcal{A}(t)z = \sum_{n=1}^{\infty} \mathcal{A}_n(t) P_n z,$$

where $I_{\mathbb{H}}$ is the identity operator of $\mathbb{H}$,

$$P_n := \begin{pmatrix} E_n & 0 & 0 \\ 0 & E_n & 0 \\ 0 & 0 & E_n \end{pmatrix}, \; n \geq 1,$$

and

$$\mathcal{A}_n(t) := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ w(t)\lambda_n & 0 & 0 \end{pmatrix}, \; n \geq 1, \quad t \in \mathbb{R}.$$

Clearly, the characteristic equation for $\mathcal{A}_n(t)$ is given by

$$-\lambda^3 + \lambda_n w(t) = -\lambda^3 - \lambda_n \rho(t) = 0, \tag{4.1}$$

from which we obtain its eigenvalues given by

$$\lambda_1^n(t) = -\sqrt[3]{\lambda_n \rho(t)}, \quad \lambda_2^n(t) = \sqrt[3]{\lambda_n \rho(t)}\, e^{i\frac{2\pi}{3}}, \quad \text{and} \quad \lambda_3^n(t) = \sqrt[3]{\lambda_n \rho(t)} e^{-i\frac{2\pi}{3}}$$

and therefore $\sigma(\mathcal{A}_n(t)) = \left\{\lambda_1^n(t), \lambda_2^n(t), \lambda_3^n(t)\right\}$.

In view of the above it follows that there exists $\theta \in \left(\frac{\pi}{2}, \pi\right)$ such that

$$S_\theta \cup \{0\} \subset \rho\left(\mathcal{A}(t)\right).$$

More precisely, any θ of the form $\theta = \frac{\pi}{2} + \varepsilon$ with $\varepsilon \in (0, \frac{\pi}{6})$ would be fine.

It is also clear that λ_1^n, λ_2^n, λ_3^n are distinct and each of them is of multiplicity one, then $\mathcal{A}_n(t)$ is diagonalizable. Further, it is not difficult to see that $\mathcal{A}_n(t) = K_n^{-1}(t) J_n(t) K_n(t)$, where $J_n(t), K_n(t)$ and $K_n^{-1}(t)$ are respectively given by

$$J_n(t) = \begin{pmatrix} \lambda_1^n(t) & 0 & 0 \\ 0 & \lambda_2^n(t) & 0 \\ 0 & 0 & \lambda_3^n(t) \end{pmatrix}, \quad K_n(t) = \begin{pmatrix} 1 & 1 & 1 \\ \lambda_1^n(t) & \lambda_2^n(t) & \lambda_3^n(t) \\ [\lambda_1^n(t)]^2 & [\lambda_2^n(t)]^2 & [\lambda_3^n(t)]^2 \end{pmatrix}.$$

For $\lambda \in S_\theta$ and $z \in \mathbb{X}$, one has

$$\begin{aligned} R(\lambda, \mathcal{A}(t))z &= \sum_{n=1}^{\infty} (\lambda - \mathcal{A}_n(t))^{-1} P_n z \\ &= \sum_{n=1}^{\infty} K_n(t)(\lambda - J_n(t))^{-1} K_n^{-1}(t) P_n z. \end{aligned}$$

Hence,

$$\left\|R(\lambda,\mathcal{A}(t))z\right\|^2 \leq \sum_{n=1}^{\infty}\left\|K_n(t)(\lambda-J_n(t))^{-1}K_n^{-1}(t)\right\|^2\left\|P_nz\right\|^2$$
$$\leq \sum_{n=1}^{\infty}\left\|K_n(t)\right\|^2\left\|(\lambda-J_n(t))^{-1}\right\|^2\ \left\|K_n^{-1}(t)\right\|^2\left\|P_nz\right\|^2.$$

It is not hard to see that there exists there exists $K'>0$ such

$$\left\|R(\lambda,\mathcal{A}(t))\right\| \leq \frac{K'}{1+|\lambda|}$$

for all $\lambda\in S_\theta$ and $t\in\mathbb{R}$.

Clearly, the domain $D=D(\mathcal{A}(t))$ is constant in t. Moreover, $\mathcal{A}(t)$ is invertible with

$$\mathcal{A}(t)^{-1} \quad = \quad \begin{pmatrix} 0 & 0 & w(t)^{-1}A^{-1} \\ & & \\ I_{\mathbb{H}} & 0 & 0 \\ & & \\ 0 & I_{\mathbb{H}} & 0 \end{pmatrix}, \quad t\in\mathbb{R}.$$

Therefore, for $t,s,r\in\mathbb{R}$, one has

$$\Big(\mathcal{A}(t)-\mathcal{A}(s)\Big)\mathcal{A}(r)^{-1}$$
$$= \begin{pmatrix} 0 & 0 & 0 \\ & & \\ 0 & 0 & 0 \\ & & \\ 0 & 0 & w(r)^{-1}(w(t)-w(s))I_{\mathbb{H}} \end{pmatrix},$$

and hence assuming that there exist $M_0\geq 0$ and $\mu\in(0,1]$ such that

$$\Big|w(t)-w(s)\Big| \leq M_0\Big|t-s\Big|^{\mu} \tag{4.2}$$

it follows that there exists $M'>0$ such that

$$\Big\|(\mathcal{A}(t)-\mathcal{A}(s))\mathcal{A}(r)^{-1}z\Big\| \leq M'\Big|t-s\Big|^{\mu}\Big\|z\Big\|.$$

Therefore, the family of linear operators $\Big\{\mathcal{A}(t)\Big\}_{t\in\mathbb{R}}$ satisfy Acquistapace–Terreni conditions.

Now, for every $t\in\mathbb{R}$, the family of linear operators $\mathcal{A}(t)$ generate an analytic semigroup $(e^{\tau\mathcal{A}(t)})_{\tau\geq 0}$ on $\mathbb{X}$ given by

$$e^{\tau\mathcal{A}(t)}z = \sum_{n=0}^{\infty}K_n(t)^{-1}P_ne^{\tau J_n}P_nK_n(t)P_nz,\ z\in\mathbb{X}.$$

On the other hand, we have

$$\left\|e^{\tau\mathcal{A}(t)}z\right\| \quad = \quad \sum_{n=0}^{\infty}\left\|K_n(t)^{-1}P_n\right\|\left\|e^{\tau J_n}P_n\right\|\left\|K_n(t)P_n\right\|\left\|P_nz\right\|,$$

with for each $z = \begin{pmatrix} z_1 \\ z_2 \\ z_2 \end{pmatrix}$

$$\begin{aligned} \left\|e^{\tau J_n} P_n z\right\|^2 &= \left\| \begin{pmatrix} e^{\lambda_1^n(t)\tau} E_n & 0 & 0 \\ 0 & e^{\lambda_2^n(t)\tau} E_n & 0 \\ 0 & 0 & e^{\lambda_3^n(t)\tau} E_n & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_2 \end{pmatrix} \right\|^2 \\ &\leq \left\|e^{\lambda_1^n(t)\tau} E_n z_1\right\|^2 + \left\|e^{\lambda_2^n(t)\tau} E_n z_2\right\|^2 + \left\|e^{\lambda_3^n(t)\tau} E_n z_2\right\|^2 \\ &\leq e^{2\Re e(\lambda_2^n(t))\tau} \left\|z\right\|^2. \end{aligned}$$

Clearly, using Eq. (1.3) it follows that

$$\begin{aligned} \Re e(\lambda_2^n(t)) &= \sqrt[3]{\lambda_n \rho(t)} \cos(\frac{2\pi}{3}) \\ &= -\frac{\sqrt[3]{\lambda_n \rho(t)}}{2} \\ &\leq -\frac{\sqrt[3]{\lambda_1 \rho_0}}{2} \end{aligned}$$

Setting, $\delta = \sqrt[3]{\lambda_1 \rho_0} > 0$ it follows that there exists $C_0 > 0$ such

(4.3) $$\left\|e^{\tau \mathcal{A}(t)}\right\| \leq C_0 e^{-\delta\tau}, \quad \tau \geq 0.$$

Arguing as in [4] it follows that the evolution family $(U(t,s))_{t \geq s}$ is exponentially stable and hence (H.2) holds.

Using the fact that $t \mapsto w(t)$ and $t \mapsto w(t)^{-1}$ are almost periodic it follows that $t \mapsto \mathcal{A}(t)^{-1}$ is almost periodic with respect to operator topology. Using Theorem 3.4 it follows that Eq. (1.1) has at least one almost periodic mild solution.

□

5. Existence of Almost Periodic Mild Solutions to Eq. (1.4)

Suppose $\widetilde{F}$ satisfies (H.4)-(H.5) and that the following assumptions hold:

(h_1) The function $q : \mathbb{R} \mapsto \mathbb{C}$ is given such that $q(t) = \widetilde{\rho}(t)$ for all $t \in \mathbb{R}$ is almost periodic and there exist $\widetilde{\rho}_0, \widetilde{\rho}_1 > 0$ such that

$$\widetilde{\rho}_0 \leq \widetilde{\rho}(t) \leq \widetilde{\rho}_1$$

for all $t \in \mathbb{R}$.

(h_2) There exist $L_0 > 0$ and $\mu \in (0,1]$ such that

$$\left|q(t) - q(s)\right| \leq L_0 \left|t - s\right|^{\mu}$$

for all $s, t \in \mathbb{R}$.

(h_3) The function $p : \mathbb{R} \mapsto \mathbb{C}$ is uniformly continuous, almost periodic, and differentiable.

The proof of the next theorem is now clear.

Theorem 5.1. *Under previous assumptions and if* (h_1)-(h_2)-(h_3)-(H.6) *hold, then Eq. (1.4) has at least one almost periodic solution* $u \in \mathbb{H}_\alpha$.

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