A Massera Type Criterion for Almost Automorphy of Nonautonomous Boundary Differential Equations*

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Abstract

For abstract linear nonautonomous boundary differential equations with an almost automorphic forcing term, a Massera type criterion is established for the existence of an almost automorphic solution with the help of the spectrum of monodromy operator, which extends the classical theorem due to Massera on the existence of periodic solutions for linear periodic ordinary differential equations.

Keywords: Almost automorphy; Massera type criteria; boundary differential equation

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1 Introduction

A classical result of Massera in his landmark paper [1] says that a necessary and sufficient condition for an ω-periodic linear scalar ordinary differential equation to have an ω-periodic solution is that it has a bounded solution on the positive half line. Since then, there has been an increasing interest in extending this classical result to various classes of functions (such as anti-periodic functions [2], quasi-periodic functions [3], almost periodic functions [4, 5, 6], almost automorphic functions [7, 8]) and also to various classes of dynamical systems (such as ordinary differential equations [1], functional differential equations [9, 10], quasi-linear partial differential equations [11], dynamic equations on time scale [12]).

Recently, there has been an increasing interest in the almost automorphy of dynamical systems, which is first introduced by Bochner [13] and is more general than the almost periodicity and attracts more and more attention. One can see [14, 15] for a complete background on almost automorphic functions and see the important Memoirs [16] for almost automorphic dynamics. Many different kinds of criteria are established for the existence of almost automorphic solutions of various kinds of dynamical systems [14, 16, 17, 18, 19, 20, 21, 22, 23].

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Particularly, some Massera type criteria are derived for some neutral functional differential equations [7], evolution equations [8], and differential equation with piecewise constant arguments [24]. In this note, we will make an attempt to give an extension of the classical result of Massera to almost automorphic solutions of nonautonomous boundary differential equations (or sometimes, nonautonomous boundary Cauchy problems), which are an abstract formulation of partial differential equations with boundary conditions modeling natural phenomena such as retarded differential (difference) equations, dynamic population equations, and boundary control problems, and has been widely studied (see [25] and references cited therein).

The paper is organized as follows. Section 2 introduces some notations, assumptions and preliminary results on almost automorphic functions and nonautonomous boundary Cauchy problems. Section 3 investigates the almost automorphy of bounded solutions of a nonautonomous boundary differential equations in a Banach spaces and establishes a necessary and sufficient criterion of Massera type in term of spectral countability condition and boundedness of solutions for the existence of almost automorphic solution.

2 Preliminaries

We begin in this section by fixing some notations, assumptions and recalling a few basic results on almost automorphic functions and nonautonomous inhomogeneous boundary Cauchy problems.

2.1 Notations

Let $\mathbb{N}, \mathbb{Z}, \mathbb{R}$, and $\mathbb{C}$ stand for the set of natural numbers, integers, real numbers, and complex numbers, respectively. Let X, Y be two Banach spaces and $L(X, Y)$ denote the space of all bounded linear operators from X to Y. $C(\mathbb{R}, X)$ stands for the set of continuous functions from $\mathbb{R}$ to X with the supreme norm. $l^\infty(\mathbb{Z}, X)$ denotes the space of all bounded (two-sided) sequences in a Banach space X with supreme norm, i.e., $\|u\| := \sup_{n\in\mathbb{Z}} \|u(n)\|$ for $u = \{u(n)\}_{n\in\mathbb{Z}} \in l^\infty(\mathbb{Z}, X)$. c_0 denotes the Banach space of all numerical sequence $x = \{x_n\}_{n=1}^\infty$ satisfying $\lim_{n\to\infty} x_n = 0$, endowed with the supreme norm. For A being a linear operator on X, $D(A)$, $\sigma(A)$ and $\rho(A)$ stand for the domain, the spectrum and the resolvent set of A, respectively.

2.2 Almost automorphic functions and sequences

We recall the definition of almost automorphic functions and some of their properties.

Definition 2.1. (Bochner [13]) A function $f \in \mathrm{C}(\mathbb{R}, X)$ is said to be almost automorphic in Bochner's sense if for every sequence of real numbers $(\sigma_n)_{n\in\mathbb{N}}$, there exists a subsequence $(s_n)_{n\in\mathbb{N}} \subset (\sigma_n)_{n\in\mathbb{N}}$ such that $g(t) := \lim_{n\to\infty} f(t+s_n)$ is well defined for each $t \in \mathbb{R}$ and $\lim_{n\to\infty} g(t-s_n) = f(t)$ for each $t \in \mathbb{R}$.

The collection $AA(\mathbb{R}, X)$ of all almost automorphic X-valued functions is a Banach space under the supreme norm. In addition, if the convergence in the definition above is uniform in $t \in \mathbb{R}$, then $f \in AP(\mathbb{R}, X)$, the space of all almost periodic functions with values in X. A typical example [14] of almost automorphic function but not almost periodic reads

$$\varphi(t) = \cos\left(\frac{1}{2 + \sin\sqrt{2}t + \sin t}\right), \quad t \in \mathbb{R}.$$

Therefore, $AP(\mathbb{R}, X) \subset AA(\mathbb{R}, X)$ with strict inclusion.

Lemma 2.1. [14] *Let $f, f_1, f_2 \in AA(\mathbb{R}, X)$, then*

- $f_1 + f_2 \in AA(\mathbb{R}, X)$.
- $\lambda f \in AA(\mathbb{R}, X)$ *for any* $\lambda \in \mathbb{R}$.
- $f_\alpha \in AA(\mathbb{R}, X)$, *where* $f_\alpha : \mathbb{R} \to X$ *is defined by* $f_\alpha(\cdot) := f(\cdot + \alpha)$.
- *the range* $\Re_f := \{f(t) : t \in \mathbb{R}\}$ *is relatively compact in* X, *thus* f *is bounded in norm.*
- *if* $f_n \in AA(\mathbb{R}, X)$ *and* $f_n \to f$ *uniformly on* $\mathbb{R}$, *then* $f \in AA(\mathbb{R}, X)$.

Similarly as for functions, we define below the almost automorphy of sequences.

Definition 2.2. [23] A sequence $u = (u(n))_{n\in\mathbb{Z}} \in l^\infty(\mathbb{Z}, X)$ is said to be almost automorphic if for every sequence of integers $(\kappa_n)_{n\in\mathbb{N}}$, there exists a subsequence $(k_n)_{n\in\mathbb{N}} \subset (\kappa_n)_{n\in\mathbb{N}}$ such that $v(p) = \lim_{n\to\infty} u(p + k_n)$ is well defined for each $p \in \mathbb{Z}$ and $u(p) = \lim_{n\to\infty} v(p - k_n)$ for each $p \in \mathbb{Z}$.

The set $aa(\mathbb{Z}, X)$ of all almost automorphic sequences in X forms a closed subspace of $l^\infty(\mathbb{Z}, X)$. It is well known that the range of an almost automorphic sequence is precompact. Also, if the convergence in Definition 2.2 is uniform in $p \in \mathbb{Z}$, then the almost automorphic sequence is almost periodic. Moreover, if u is an almost automorphic function defined on $\mathbb{R}$, then $u|_{\mathbb{Z}}$ is an almost automorphic sequence.

Consider the linear difference equation

$$u(n + 1) = Bu(n) + f(n), \qquad n \in \mathbb{Z}, \tag{2.1}$$

where B is a bounded linear operator.

Lemma 2.2. [23] *Let X be a Banach space and not contain any subspace being isomorphic to c_0. If $\sigma_\Gamma(B) := \sigma(B) \cap \{z \in \mathbb{C} : |z| = 1\}$ is countable and $f \in aa(\mathbb{Z}, X)$, then each bounded solution of (2.1) is almost automorphic.*

It is well known a uniformly convex Banach space or any finite-dimensional space does not contain any subspace isomorphic to c_0.

2.3 Boundary differential equations

Definition 2.3. [25, 26] A family of linear densely defined operators $(A(t), D(A(t)))_{t\in\mathbb{R}}$ is called a stable family if there are constants $M \geq 1$ and $\omega_0 \in \mathbb{R}$ such that $(\omega_0, \infty) \subset \rho(A(t))$ for all $t \in \mathbb{R}$ and

$$\left\| \prod_{i=1}^{n} R(\lambda, A(t_i)) \right\| \leq \frac{M}{(\lambda - \omega_0)^n}$$

for $\lambda > \omega_0$ and any finite sequence $(t_i)_{i=1}^k$ with $t_1 \leq t_2 \leq \cdots \leq t_k \in \mathbb{R}$ and $k \in \mathbb{N}$, where $R(\lambda, A(t_i)) = (\lambda - A(t_i))^{-1}$ is the resolvent of $A(t_i)$ at the point λ.

Definition 2.4. [25, 27] A family of linear bounded operators $\mathcal{U} := \{U(t,s) : t \geq s, t, s \in \mathbb{R}\}$ on a Banach space X is called a (strong continuous) evolution family if

(*i*) $U(t,s) = U(t,r)U(r,s)$ and $U(s,s) = I$ (I is the identity on X) for $t \geq r \geq s$ and $t, r, s \in \mathbb{R}$;

(*ii*) the mapping $\{(\tau, \sigma) \in \mathbb{R}^2 : \tau \geq \sigma\} \ni (t,s) \to U(t,s)$ is strongly continuous.

Moreover, the evolution family is said to be q-periodic if there exists a positive constant $q > 0$ such that $U(t+q, s+q) = U(t,s)$ for all $t \geq s$.

Consider the linear inhomogeneous nonautonomous boundary Cauchy problem

$$\begin{cases} u'(t) = A_m(t)u(t) + f(t), \quad t \in \mathbb{R}, \\ L(t)u(t) = g(t), \quad t \in \mathbb{R} \end{cases} \tag{2.2}$$

where the first equation is defined in a Banach space X called state space and the second equation is in a "boundary space" ∂X.

We now introduce the setting of our abstract boundary Cauchy problems. Let $X, D, \partial X$ be Banach spaces such that D is dense and continuously embedded in X. Consider the operators $A_m(t) \in L(D, X)$, $L(t) \in L(D, \partial X)$ for $t \in \mathbb{R}$, subject to the following hypotheses:

(H_1) $\mathbb{R} \ni t \to A_m(t)x$ is 1-periodic continuous differential for all $x \in D$.

(H_2) the family of operators $(A(t))_{t\in\mathbb{R}}, A(t) := A_m(t)|_{\ker L(t)}$ is stable with stability constants (M, ω_0).

(H_3) the operator $L(t) : D \to \partial X$ is surjective for $t \in \mathbb{R}$ and $t \to L(t)x$ is 1-periodic continuous differentiable for all $x \in D$.

(H_4) there exist constants $\gamma > 0$ and $\omega \in \mathbb{R}$ such that

$$\|L(t)x\|_{\partial X} \geq \gamma^{-1}(\lambda - \omega)\|x\|, \quad x \in \ker(\lambda - A_m(t)), \quad \lambda > \omega, \ t \in \mathbb{R}.$$

(H_5) there are positive constants c_1 and c_2 such that

$$c_1\|x\|_D \leq \|x\| + \|A_m(t)x\| \leq c_2\|x\|_D, \quad x \in D, \ t \in \mathbb{R}.$$

(H_6) $f : \mathbb{R} \to X$ and $g : \mathbb{R} \to \partial X$ are continuous.

Under the above assumptions, it follows that there exists a 1-periodic evolution family $\mathcal{U} := \{U(t,s) : t \geq s, t, s \in \mathbb{R}\}$ generated by $(A(t), D(A(t)))_{t\in\mathbb{R}}$ having exponential growth, that is,

$$\|U(t,s)\| \leq Me^{\omega_0(t-s)}, \qquad t \geq s.$$

We emphasize that in this paper, for the sake of simplicity of the notations we assume the 1-periodicity, and this does not mean any restriction on the period of the operators or evolution family. For the 1-periodic evolution family $(U(t,s))_{t\geq s}$, we have a family of 1-periodic monodromy operators $P(t) := U(t+1,t), t \in \mathbb{R}$, i.e., $P(t+1) = P(t), t \in \mathbb{R}$. Denote $P := P(0) = U(1,0)$, then $\sigma(P(t))\backslash\{0\} = \sigma(P)\backslash\{0\}$, i.e., the characteristic multipliers of the monodromy operators are independent of t, and the resolvent $R(\lambda, P(t))$ is strongly continuous for $\lambda \in \rho(P)$.

Other consequences of our assumptions will be needed in the sequel. For the proof, see [25] and the references cited therein.

Lemma 2.3. *[25] Under the above assumptions, we have the following properties for $t \in \mathbb{R}$ and $\lambda, \mu \in \rho(A(t))$:*

- $D = D(A(t)) \oplus \ker(\lambda - A_m(t))$;
- $L(t)|_{\ker(\lambda - A_m(t))}$ *is an isomorphism from* $\ker(\lambda - A_m(t))$ *onto* ∂X, *and its inverse* $L_{\lambda,t} := (L(t)|_{\ker(\lambda-A_m(t))})^{-1} : \partial X \to \ker(\lambda - A_m(t))$ *satisfies the estimate* $\|\lambda L_{\lambda,t}\| \leq \gamma$;
- $R(\lambda, A(t))L_{\mu,t} = R(\mu, A(t))L_{\lambda,t}$;
- $t \to L_{\lambda,t}$ *is 1-periodic and* $(\lambda - A_m(t))L_{\lambda,t} = L(t)R(\lambda, A(t)) = 0, L(t)L_{\lambda,t} = Id_{\partial X}$ *and* $L_{\lambda,t}L(t)$ *is the projection from* D *onto* $\ker(\lambda - A_m(t))$.

Next, we introduce the following definition of a mild solution to the inhomogeneous boundary Cauchy problem (2.2) given by the variation of constants formula.

Definition 2.5. A continuous function $u : \mathbb{R} \to X$ is called a mild solution of (2.2) if it satisfies the following equation

$$u(t) = U(t,s)u(s) + \int_s^t U(t,\tau)f(\tau)d\tau + \lim_{\lambda\to\infty} \int_s^t U(t,\tau)\lambda L_{\lambda,\tau}g(\tau)d\tau, \quad \text{for} \quad t \geq s, \ t,s \in \mathbb{R}.$$

In [25], the wellposedness of nonautonomous boudary Cauchy problems (2.2) is well studied and there is a unique mild solution of (2.2) under the above assumptions.

3 Massera type criteria for almost automorphy of (2.2)

In this section, we explore the almost automorphy of bounded solutions of (2.2) and establishes a necessary and sufficient criterion of Massera type for the existence of almost automorphic solution for (2.2).

Lemma 3.1. *Assume that $f(t) \in AA(\mathbb{R}, X), g(t) \in AA(\mathbb{R}, \partial X)$. Then*

$$h(n) := \int_n^{n+1} U(n+1,\tau)f(\tau)d\tau + \lim_{\lambda\to\infty} \int_n^{n+1} U(n+1,\tau)\lambda L_{\lambda,\tau}g(\tau)d\tau$$

is almost automorphic, i.e., $\{h(n)\}_{n\in\mathbb{Z}} \in aa(\mathbb{Z}, X)$.

Proof. Since $f(t) \in AA(\mathbb{R}, X), g(t) \in AA(\mathbb{R}, \partial X)$, then for any sequence $\{s_k\}$ with $s_k \in \mathbb{Z}$, there exists a subsequence $\{n_k\} \subset \{s_k\}$ and functions f_1, g_1 such that

$$\lim_{k\to\infty} f(t+n_k) = f_1(t), \qquad \lim_{k\to\infty} f_1(t-n_k) = f(t), \quad t \in \mathbb{R}.$$

$$\lim_{k\to\infty} g(t+n_k) = g_1(t), \qquad \lim_{k\to\infty} g_1(t-n_k) = g(t), \quad t \in \mathbb{R}.$$

Define

$$h_1(n) := \int_n^{n+1} U(n+1,\tau) f_1(\tau) d\tau + \lim_{\lambda\to\infty} \int_n^{n+1} U(n+1,\tau)\lambda L_{\lambda,\tau} g_1(\tau) d\tau, \ n \in \mathbb{Z}. \tag{3.1}$$

Using the fact that $L_{\lambda,t}$ and $(U(t,s))_{t\geq s}$ are 1-periodic, by Lemma 2.3, we have

$$\begin{aligned}
\|h(n+n_k) - h_1(n)\| &= \left\| \int_0^1 U(n+n_k+1, \tau+n_k+n) f(\tau+n_k+n) d\tau - \int_0^1 U(n+1,\tau+n) f_1(\tau+n) d\tau \right\| \\
&+ \left\| \lim_{\lambda\to\infty} \int_0^1 U(n+n_k+1,\tau+n_k+n)\lambda L_{\lambda,\tau+n_k+n} g(\tau+n_k+n) d\tau \right. \\
&- \left. \lim_{\lambda\to\infty} \int_0^1 U(n+1,\tau+n)\lambda L_{\lambda,\tau+n} g_1(\tau+n) d\tau \right\| \\
&= \left\| \int_0^1 U(1,\tau) f(\tau+n_k+n) d\tau - \int_0^1 U(1,\tau) f_1(\tau+n) d\tau \right\| \\
&+ \left\| \lim_{\lambda\to\infty} \int_0^1 U(1,\tau)\lambda L_{\lambda,\tau} g(\tau+n_k+n) d\tau - \lim_{\lambda\to\infty} \int_0^1 U(1,\tau)\lambda L_{\lambda,\tau} g_1(\tau+n) d\tau \right\| \\
&\leq \int_0^1 \|U(1,\tau)\| \|f(\tau+n_k+n) - f_1(\tau+n)\| d\tau \\
&+ \gamma \int_0^1 \|U(1,\tau)\| \|g(\tau+n_k+n) - g_1(\tau+n)\| d\tau,
\end{aligned}$$

then by Lebesgue's Dominated Convergence Theorem, one has

$$\lim_{k\to\infty} h(n+n_k) = h_1(n), \quad n \in \mathbb{Z}.$$

Similarly, we have $\lim_{k\to\infty} h_1(n-n_k) = h(n), n \in \mathbb{Z}$. Therefore, $\{h(n)\}_{n\in\mathbb{Z}} \in aa(\mathbb{Z}, X)$. ☐

Lemma 3.2. *Assume that $f(t) \in AA(\mathbb{R}, X), g(t) \in AA(\mathbb{R}, \partial X)$ and u is a bounded mild solution of (2.2). Then $u(t) \in AA(\mathbb{R}, X)$ if and only if $u(n) \in aa(\mathbb{Z}, X)$.*

Proof. Necessity: if $u(t)$ is almost automorphic function, then $\{u(n)\}_{n\in\mathbb{Z}}$ is almost automorphic sequence.

Sufficiency: For any sequence $\{s_k\}$, we first assume that $s_k \in \mathbb{Z}$. By the almost automorphy of f, g and u, there exists a subsequence $\{n_k\} \subset \{s_k\}$ and functions $f_1(t), g_1(t), t \in \mathbb{R}$ and $\{v(n)\}, n \in \mathbb{Z}$ such that

$$\begin{array}{lll} \lim\limits_{k\to\infty} u(n+n_k) = v(n), & \lim\limits_{k\to\infty} v(n-n_k) = u(n), & n \in \mathbb{Z}, \\ \lim\limits_{k\to\infty} f(t+n_k) = f_1(t), & \lim\limits_{k\to\infty} f_1(t-n_k) = f(t), & t \in \mathbb{R}, \\ \lim\limits_{k\to\infty} g(t+n_k) = g_1(t), & \lim\limits_{k\to\infty} g_1(t-n_k) = g(t), & t \in \mathbb{R}. \end{array} \tag{3.2}$$

Define

$$v(\eta) := U(\eta, [t])v([t]) + \int_{[t]}^{\eta} U(\eta,\tau)f_1(\tau)d\tau + \lim_{\lambda\to\infty} \int_{[t]}^{\eta} U(\eta,\tau)\lambda L_{\lambda,\tau} g_1(\tau)d\tau, \quad \eta \in [[t],[t]+1),$$

where $[\cdot]$ is the integer part function. Then in this way, the function $v(t)$ is well-defined on $\mathbb{R}$. We claim that

$$\lim_{k\to\infty} u(t+n_k) = v(t).$$

In fact, $u(t+n_k) - v(t) := I_1(k) + J_1(k) + F_1(k)$, where

$$\begin{aligned} I_1(k) &= U(t+n_k, [t]+n_k)u([t]+n_k) - U(t,[t])v([t]), \\ J_1(k) &= \int_{[t]+n_k}^{t+n_k} U(t+n_k,\tau)f(\tau)d\tau - \int_{[t]}^{t} U(t,\tau)f_1(\tau)d\tau, \\ F_1(k) &= \lim_{\lambda\to\infty} \int_{[t]+n_k}^{t+n_k} U(t+n_k,\tau)\lambda L_{\lambda,\tau} g(\tau)d\tau - \lim_{\lambda\to\infty} \int_{[t]}^{t} U(t,\tau)\lambda L_{\lambda,\tau} g_1(\tau)d\tau. \end{aligned}$$

Then by (3.2), one has

$$\begin{aligned} \|I_1(k)\| &= \|U(t,[t])u([t]+n_k) - U(t,[t])v([t])\| \\ &\le \|U(t,[t])\|\|u([t]+n_k) - v([t])\| \to 0, \quad k \to \infty. \end{aligned}$$

and

$$\begin{aligned} \|J_1(k)\| &\le \left\| \int_{[t]}^{t} U(t+n_k,\tau+n_k)f(\tau+n_k)d\tau - \int_{[t]}^{t} U(t,\tau)f_1(\tau)d\tau \right\| \\ &\le \int_{[t]}^{t} \|U(t,\tau)\|\|f(\tau+n_k)d\tau - f_1(\tau)\|d\tau \to 0, \quad k \to \infty. \end{aligned}$$

Using the fact that $L_{\lambda,t}$ and $(U(t,s))_{t\ge s}$ are 1-periodic, we have

$$\|F_1(k)\| = \left\| \lim_{\lambda\to\infty} \int_{[t]}^{t} U(t+n_k,\tau+n_k)\lambda L_{\lambda,\tau+n_k} g(\tau+n_k)d\tau - \lim_{\lambda\to\infty} \int_{[t]}^{t} U(t,\tau)\lambda L_{\lambda,\tau} g_1(\tau)d\tau \right\|$$

$$= \left\| \lim_{\lambda\to\infty} \int_{[t]}^{t} U(t,\tau)\lambda L_{\lambda,\tau} g(\tau+n_k)d\tau - \lim_{\lambda\to\infty} \int_{[t]}^{t} U(t,\tau)\lambda L_{\lambda,\tau} g_1(\tau)d\tau \right\|$$

$$\le \lim_{\lambda\to\infty} \int_{[t]}^{t} \|U(t,\tau)\| \|\lambda L_{\lambda,\tau}\| \|g(\tau+n_k) - g_1(\tau)\| d\tau$$

$$\le \gamma \int_{[t]}^{t} \|U(t,\tau)\| \|g(\tau+n_k) - g_1(\tau)\| d\tau.$$

By Lebesgue's dominated convergence theorem, $\lim_{k\to\infty} F_1(k) = 0$. Hence, $\lim_{k\to\infty} u(t+n_k) = v(t)$. Similarly, we have

$$\lim_{k\to\infty} v(t-n_k) = u(t).$$

Now, we assume that $s_k \in \mathbb{R}$. Note that $s_k - [s_k] \in [0,1)$, we choose a subsequence $\{n_k\} \subset \{[s_k]\}$ and a sequence $\{t_k\} \subset \{s_k - [s_k]\}$ such that $\lim_{k\to\infty} t_k = t_0 \in [0,1]$ and (3.2) holds. We conclude that

$$\lim_{k\to\infty} u(t+t_k+n_k) = \lim_{k\to\infty} u(t+t_0+n_k). \tag{3.3}$$

The proof of this claim is divided into two cases.

Case 1. $t+t_0 > [t+t_0]$. Then, for some sufficiently large k, one has $[t+t_k] = [t+t_0]$. Set

$$u(t+t_k+n_k) - u(t+t_0+n_k) := I_2(k) + J_2(k) + F_2(k),$$

where $I_2(k), J_2(k), F_2(k)$ are defined below. By the 1-periodicity of $(U(t,s))_{t\ge s}$ and the fact that $[t+t_k] = [t+t_0]$, we have

$$\begin{aligned} \|I_2(k)\| &:= \|U(t+t_k+n_k, [t+t_k]+n_k)u([t+t_k]+n_k) - U(t+t_0+n_k, [t+t_0]+n_k)u([t+t_0]+n_k)\| \\ &= \|U(t+t_k, [t+t_0])u([t+t_0]+n_k) - U(t+t_0, [t+t_0])u([t+t_0]+n_k)\|. \end{aligned}$$

The strong continuity of $(U(t,s))_{t\ge s}$ implies that $\lim_{k\to\infty} I_2(k) = 0$.

$$\begin{aligned} \|J_2(k)\| &:= \left\| \int_{[t+t_k]+n_k}^{t+t_k+n_k} U(t+t_k+n_k,\tau)f(\tau)d\tau - \int_{[t+t_0]+n_k}^{t+t_0+n_k} U(t+t_0+n_k,\tau)f(\tau)d\tau \right\| \\ &= \left\| \int_{[t+t_k]}^{t+t_k} U(t+t_k+n_k,\tau+n_k)f(\tau+n_k)d\tau - \int_{[t+t_0]}^{t+t_0} U(t+t_0+n_k,\tau+n_k)f(\tau+n_k)d\tau \right\| \\ &= \left\| \int_{[t+t_0]}^{t+t_k} U(t+t_k,\tau)f(\tau+n_k)d\tau - \int_{[t+t_0]}^{t+t_0} U(t+t_0,\tau)f(\tau+n_k)d\tau \right\|, \end{aligned}$$

From the the strong continuity of $(U(t,s))_{t\ge s}$ and the precompactness of the range of f, it follows that $\lim_{k\to\infty} J_2(k) = 0$.

$$\|F_2(k)\| := \left\| \lim_{\lambda\to\infty} \int_{[t+t_k]+n_k}^{t+t_k+n_k} U(t+t_k+n_k,\tau)\lambda L_{\lambda,\tau} g(\tau)d\tau - \lim_{\lambda\to\infty} \int_{[t+t_0]+n_k}^{t+t_0+n_k} U(t+t_0+n_k,\tau)\lambda L_{\lambda,\tau} g(\tau)d\tau \right\|$$

$$= \left\| \lim_{\lambda\to\infty} \int_{[t+t_k]}^{t+t_k} U(t+t_k,\tau)\lambda L_{\lambda,\tau+n_k} g(\tau+n_k)d\tau - \lim_{\lambda\to\infty} \int_{[t+t_0]}^{t+t_0} U(t+t_0,\tau)\lambda L_{\lambda,\tau+n_k} g(\tau+n_k)d\tau \right\|$$

$$= \left\| \lim_{\lambda\to\infty} \int_{[t+t_0]}^{t+t_k} U(t+t_k,\tau)\lambda L_{\lambda,\tau} g(\tau+n_k)d\tau - \lim_{\lambda\to\infty} \int_{[t+t_0]}^{t+t_0} U(t+t_0,\tau)\lambda L_{\lambda,\tau} g(\tau+n_k)d\tau \right\|,$$

then $\lim_{k\to\infty} F_2(k) = 0$. Therefore, (3.3) holds.

Case 2. $t+t_0 = [t+t_0]$, i.e., $t+t_0$ is an integer. If $t+t_k \geq t+t_0$, then $[t+t_k] = t+t_0$. The rest is exactly the same as those for Case 1. If $t+t_k < t+t_0$, then $[t+t_k] = t+t_0-1$. Set

$$u(t+t_k+n_k) - u(t+t_0+n_k) := I_3(k) + J_3(k) + F_3(k),$$

where $I_3(k), J_3(k), F_3(k)$ are defined below.

$$\begin{aligned} \|I_3(k)\| &:= \|U(t+t_k+n_k,[t+t_k]+n_k)u([t+t_k]+n_k) \\ &\quad - U(t+t_0+n_k,t+t_0-1+n_k)u(t+t_0-1+n_k)\| \\ &= \|U(t+t_k,t+t_0-1)u(t+t_0-1+n_k) - U(t+t_0,t+t_0-1)u(t+t_0-1+n_k)\|. \end{aligned}$$

For $J_3(k)$ and $F_3(k)$, using the fact that $L_{\lambda,t}$ and $(U(t,s))_{t\geq s}$ are 1-periodic, one has

$$\begin{aligned} \|J_3(k)\| &:= \left\| \int_{[t+t_k]+n_k}^{t+t_k+n_k} U(t+t_k+n_k,\tau)f(\tau)d\tau - \int_{t+t_0-1+n_k}^{t+t_0+n_k} U(t+t_0+n_k,\tau)f(\tau)d\tau \right\| \\ &= \left\| \int_{[t+t_k]}^{t+t_k} U(t+t_k+n_k,\tau+n_k)f(\tau+n_k)d\tau - \int_{t+t_0-1}^{t+t_0} U(t+t_0+n_k,\tau+n_k)f(\tau+n_k)d\tau \right\| \\ &= \left\| \int_{t+t_0-1}^{t+t_k} U(t+t_k,\tau)f(\tau+n_k)d\tau - \int_{t+t_0-1}^{t+t_0} U(t+t_0,\tau)f(\tau+n_k)d\tau \right\|, \end{aligned}$$

and

$$\begin{aligned} \|F_3(k)\| &:= \left\| \lim_{\lambda\to\infty} \int_{[t+t_k]+n_k}^{t+t_k+n_k} U(t+t_k+n_k,\tau)\lambda L_{\lambda,\tau} g(\tau)d\tau - \lim_{\lambda\to\infty} \int_{t+t_0-1+n_k}^{t+t_0+n_k} U(t+t_0+n_k,\tau)\lambda L_{\lambda,\tau} g(\tau)d\tau \right\| \\ &= \left\| \lim_{\lambda\to\infty} \int_{[t+t_k]}^{t+t_k} U(t+t_k,\tau)\lambda L_{\lambda,\tau+n_k} g(\tau+n_k)d\tau - \lim_{\lambda\to\infty} \int_{t+t_0-1}^{t+t_0} U(t+t_0,\tau)\lambda L_{\lambda,\tau+n_k} g(\tau+n_k)d\tau \right\| \\ &= \left\| \lim_{\lambda\to\infty} \int_{t+t_0-1}^{t+t_k} U(t+t_k,\tau)\lambda L_{\lambda,\tau} g(\tau+n_k)d\tau - \lim_{\lambda\to\infty} \int_{t+t_0-1}^{t+t_0} U(t+t_0,\tau)\lambda L_{\lambda,\tau} g(\tau+n_k)d\tau \right\|. \end{aligned}$$

From the strong continuity of $(U(t,s))_{t\geq s}$, it follows that $\lim_{k\to\infty} I_3(k) = \lim_{k\to\infty} J_3(k) = \lim_{k\to\infty} F_3(k) = 0$. Therefore, (3.3) is valid. Therefore,

$$\lim_{k\to\infty} u(t+t_0+(s_k-t_0)) = \lim_{k\to\infty} u(t+t_k+n_k) = \lim_{k\to\infty} u(t+t_0+n_k) = v(t+t_0).$$

Similarly, we have

$$\lim_{k\to\infty} v(t+t_0-(s_k-t_0)) = \lim_{k\to\infty} v(t+t_0-n_k) = u(t+t_0).$$

The proof is complete. □

Theorem 3.1. *Assume that $f(t) \in AA(\mathbb{R}, X), g(t) \in AA(\mathbb{R}, \partial X)$, and X dose not contain any subspace isomorphic to c_0 and $\sigma_\Gamma(P)$ is countable. Then (2.2) has an almost automorphic mild solution on $\mathbb{R}$ if and only if it admits a bounded mild solution on $\mathbb{R}$.*

Proof. We only need to prove the sufficiency. Consider the difference equation

$$u(n+1) = U(n+1,n)u(n) + \int_n^{n+1} U(n+1,\tau)f(\tau)d\tau + \lim_{\lambda\to\infty}\int_n^{n+1} U(n+1,\tau)\lambda L_{\lambda,\tau} g(\tau)d\tau. \quad (3.4)$$

From the 1-periodicity of $(U(t,s))_{t\geq s}$, (3.4) rewrites

$$u(n+1) = Pu(n) + h(n), \quad n \in \mathbb{Z}, \quad (3.5)$$

where

$$P = U(1,0), \quad h(n) = \int_n^{n+1} U(n+1,\tau)f(\tau)d\tau + \lim_{\lambda\to\infty}\int_n^{n+1} U(n+1,\tau)\lambda L_{\lambda,\tau} g(\tau)d\tau, \quad n \in \mathbb{Z}.$$

By Lemma 3.1, $h(n) \in aa(\mathbb{Z}, X)$. Since $\{u(n)\}_{n\in\mathbb{Z}}$ is a bounded solution of (3.5), X dose not contain any subspace isomorphic to c_0, and $\sigma_\Gamma(P)$ is countable, by Lemma 2.2, $u(n) \in aa(\mathbb{Z}, X)$. Then, by Lemma 3.2, $u(t) \in AA(\mathbb{R}, X)$. □

4 Application

In this section, we provide example to illustrate our main results.

Example 4.1. Consider the boundary differential equation

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,x) = \delta(t)\dfrac{\partial^2 v}{\partial x^2}(t,x) + \beta(t)v(t,x) + f(t), & t \geq 0, x \in [0,1], \\ \dfrac{\partial v}{\partial x}(t,0) = g_1(t); \quad \dfrac{\partial v}{\partial x}(t,1) = g_2(t), & t \geq 0, \end{cases} \quad (4.1)$$

where $\beta(\cdot), \delta(\cdot)$ are strictly positive 1-periodic functions in $C^1([0,1], \mathbb{R}^+)$, the functions $f : \mathbb{R}^+ \to \mathbb{R}, g_1 : \mathbb{R}^+ \to \mathbb{R}$ and $g_2 : \mathbb{R}^+ \to \mathbb{R}$ are almost automorphic functions.

We take $X := L^1(0,1)$ the Banach space of integrable functions in $[0,1]$ endowed with the norm

$$\|h\| = \int_0^1 |h(x)|dx,$$

and let D be the subspace of X given by $D := W^{2,1}(0,1) = \{h \in L^1(0,1) : h', h'' \in L^1(0,1)\}$ endowed with the norm

$$\|h\|_D = \|h\| + \|h'\| + \|h''\|.$$

Then $(D, \|\cdot\|_D)$ is a Banach space continuously embedded and dense in $(X, \|\cdot\|)$.

For $t \geq 0$, let $A_m(t) : D \subset X \to X$ be the family of operators defined by

$$A_m(t)h = \delta(t)h'' + \beta(t)h \quad \text{for} \ \ h \in D.$$

Let $L : D \to \mathbb{R}^2$ be the operator defined by

$$Lh = (h'(0), h'(1))^T \quad \text{for} \ \ h \in D.$$

Consider the almost automorphic functions $f(t)$ and $g(t) = (g_1(t), g_2(t))^T, t \geq 0$, then the boundary differential equation (4.1) take the abstract form (2.2). The assumptions (H_1)-(H_6) are satisfied, see [28].

In [28], the evolution family generated by $(A(t))_{t\geq 0}$ is given by

$$U(t,s) = \exp\left(\int_s^t \beta(\tau)d\tau\right) T_\Delta\left(\int_s^t \delta(\tau)d\tau\right) \quad \text{for} \ \ t \geq s \geq 0,$$

where $(T_\Delta(t))_{t\geq 0}$ denotes the semigroup generated by the Laplacian Δ with Neuman boundary conditions on $L^1(0,1)$, then $P := U(1,0) = \exp\left(\int_0^1 \beta(\tau)d\tau\right) T_\Delta\left(\int_0^1 \delta(\tau)d\tau\right)$. By using the spectral mapping theorem and the spectrum of the Laplacian operator, one has

$$\sigma(P) = \left\{\exp\left(\int_0^1 \beta(t)dt - n^2\pi^2 \int_0^1 \delta(t)dt\right), n \in \mathbb{N}\right\}.$$

for more details, see [28]. By Theorem 3.1, we claim that if $\sigma_\Gamma(P) := \sigma(P) \cap \{z \in \mathbb{C} : |z| = 1\}$ is countable, then (4.1) has an almost automorphic mild solution on $\mathbb{R}^+$ if and only if it admits a bounded mild solution on $\mathbb{R}^+$.

References

[1] J.L. Massera, The existence of periodic solutions of systems of differential equations, Duke Math. J. 17 (1950), 457–475.

[2] Y.Q. Chen, On Massera's theorem for anti-periodic solution, Adv. Math. Sci. Appl. 9 (1999), 125–128.

[3] R. Ortega, M. Tarallo, Massera's theorem for quasi-periodic differential equations, Topol. Methods Nonlinear Anal. 19 (2002), 39–62.

[4] Y. Hino, T. Naito, N. Van Minh, J.S. Shin, Almost Periodic Solutions of Differential Equations in Banach Spaces, Taylor and Francis Group, London, New York, 2002.

[5] Q. Liu, N. Van Minh, G.M. N'Guérékata, R. Yuan, Massera type theorems for abstract functional differential equations, Funkcial. Ekvac. 51 (2008), 329–350.

[6] S. Murakami, T. Naito, N. Van Minh, Massera's theorem for almost periodic solutions of functional differential equations, J. Math. Soc. Japan 56 (2004), 247–268.

[7] K. Ezzinbi, G.M. N'Guérékata, Massera type theorem for almost automorphic solutions of functional differential equations of neutral type, J. Math. Anal. Appl. 316 (2006), 707–721.

[8] J. Liu, G.M. N'Guérékata, N. Van Minh, A Massera type theorem for almost automorphic solutions of differential equations, J. Math. Anal. Appl. 299 (2004), 587–599.

[9] R. Benkhalti, K. Ezzinbi, A Massera type criterion for some partial functional differential equations, Dynam. Systems Appl. 9 (2000), 221–228.

[10] M. Fan, K. Wang, Periodic solutions of linear neutral functional differential equations with infinite delay, Acta Math. Sinica (Chin. Ser.) 43 (2000), 695–702.

[11] K. Wang, M. Fan, M.Y. Li, A Massera theorem for quasi-linear partial differential equations of first order, Rocky Mountain J. Math. 36 (2006), 1715–1727.

[12] X.L. Liu, W.T. Li, Periodic solutions for dynamic equations on time scales, Nonlinear Anal. 67 (2007), 1457–1463.

[13] S. Bochner, A new approach to almost periodicity, Proc. Natl. Acad. Sci. USA 48 (1962), 2039–2043.

[14] G.M. N'Guérékata, Almost Automorphic Functions and Almost Periodic Functions in Abstract Spaces, Kluwer Academic/Plenum Publishers, New York, Berlin, Moscow, 2001.

[15] G.M. N'Guérékata, Topics in Almost Automorphy, Springer, New York, 2005.

[16] W. Shen, Y. Yi, Almost Automorphic and Almost Periodic Dynamics in Skew-Product Semiflows, Mem. Amer. Math. Soc. No. 647, Vol. 136 (1998).

[17] S. Boulite, L. Maniar, G. M. N'Guérékata, Almost automorphic solutions for semilinear boundary differential equations, Proc. Amer. Math. Soc. 134 (2006), 3613–3624.

[18] M. Baroun, L. Maniar, G. M. N'Guérékata, Almost periodic and almost automorphic solutions semilinear parabolic boundary differential equations, Nonlinear Anal. 69 (2008), 2114–2124.

[19] Y.K. Chang, Z.H. Zhao, G. M. N'Guérékata, A new composition theorem for square-mean almost automorphic functions and applications to stochastic differential equations, Nonlinear Anal. 74 (2011), 2210–2219.

[20] T. Diagana, G. M. N'Guérékata, Almost automorphic solutions to some classes of partial evolution equations, Appl. Math. Lett. 20 (2007), 462–466.

[21] H.S. Ding, J. Liang, T.J. Xiao, Some properties of Stepanov-like almost automorphic functions and applications to abstract evolution equations, Appl. Anal. 88 (2009), 1079–1091.

[22] Y. Hino, S. Murakami, Almost automorphic solutions for abstract functional differential equations, J. Math. Anal. Appl. 286 (2003), 741–752.

[23] N. Van Minh, T. Naito, G.M. N'Guérékata, A spectral countability condition for almost automorphy of solutions of differential equations, Proc. Amer. Math. Soc. 134 (2006), 3257–3266.

[24] N. Van Minh, T. Dat, On the almost automorphy of bounded solutions of differential equations with piecewise constant argument, J. Math. Anal. Appl. 326 (2007), 165–178.

[25] S. Boulite, L. Maniar, M. Moussi, Wellposedness and asymptotic behaviour of non-autonomous boundary evolution equations, Forum Math. 18 (2006), 611–638.

[26] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, Berlin, 1983.

[27] K.J. Engel, R. Nagel, One-Parameter Semigroups for Linear Evolution Equations, in: Gradute Text in Mathematics, Springer-Verlag, Berlin, 2001.

[28] T. Akrid, L. Maniar, A. Ouhinou, Periodic solutions for some nonautonomous boundary evolutions equations, Nonlinear Anal. 71 (2009), 1000–1011.

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