Electronic Journal of Qualitative Theory of Differential Equations
2011, No. **87**, 1-13; **http://www.math.u-szeged.hu/ejqtde/**

Periodic boundary value problems for Riemann–Liouville sequential fractional differential equations *

Zhongli Wei [a, b] Wei Dong [c]
[a] Department of Mathematics, Shandong Jianzhu University,
Jinan, Shandong, 250101, People's Republic of China
[b] School of Mathematics, Shandong University,
Jinan, Shandong, 250100, People's Republic of China
[c]Department of Mathematics, Hebei University of Engineering,
Handan, Hebei, 056021, China
E-mail address: jnwzl@yahoo.com.cn (Z. Wei).

Abstract

In this paper, we shall discuss the properties of the well-known Mittag–Leffler function, and consider the existence of solution of the periodic boundary value problem for a fractional differential equation involving a Riemann–Liouville sequential fractional derivative by means of the method of upper and lower solutions and Schauder fixed point theorem.

Key words Periodic boundary value problem; Fractional differential equation; Riemann–Liouville sequential fractional derivatives; Upper solution and lower solution.

MSC2010 26A33, 34A08, 34B15, 34B99.

1. Introduction

Let $J=[a,b]$ be a compact interval on the real axis $\mathbb{R}$, and y be a measurable Lebesgue function, that is, $y\in L_1(a,b)$. Let $x\in J$ and $\alpha\in\mathbb{R}$ $(0<\alpha\le 1)$. The Riemann-Liouville fractional integrals I^{α}_{a+} and derivative D^{α}_{a+} are defined by (see, for example, [1][2])

$$(I^{\alpha}_{a+}y)(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-s)^{\alpha-1}y(s)\mathrm{d}s \quad \text{and} \quad (D^{\alpha}_{a+}y)(x)=\frac{\mathrm{d}}{\mathrm{d}x}(I^{1-\alpha}_{a+}y)(x). \tag{1.1}$$

We will work here following the definition of *a sequential fractional derivative* presented by Miller and Ross [3]

$$\begin{cases} \mathscr{D}^{\alpha}_{a+}y &= D^{\alpha}_{a+}y \\ \mathscr{D}^{k\alpha}_{a+}y &= \mathscr{D}^{\alpha}_{a+}\mathscr{D}^{(k-1)\alpha}_{a+}y \quad (k=2,\ 3,\ldots). \end{cases} \tag{1.2}$$

There is a close connection between the sequential fractional derivatives and the non sequential Riemann–Liouville derivatives. For example, in the case $k=2$, $0<\alpha<1/2$ and the Riemann–Liouville derivatives, the relationship between $\mathscr{D}^{k\alpha}_{a+}y$ and $D^{k\alpha}_{a+}y$ is given by

$$(\mathscr{D}^{2\alpha}_{a+}y)(x)=\left(D^{2\alpha}_{a+}\left[y(t)-(I^{1-\alpha}_{a+}y)(a+)\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}\right]\right)(x). \tag{1.3}$$

*Research supported by the NNSF-China (10971046) and the NSF of Shandong Province (ZR2009AM004).

We shall consider the existence of solution of the periodic boundary value problem for a fractional differential equation involving a Riemann–Liouville sequential fractional derivative, by using the method of upper and lower solutions and Schauder fixed point theorem.

$$\begin{cases} (\mathscr{D}_{0+}^{2\alpha}y)(x) = f(x,\ y,\ \mathscr{D}_{0+}^{\alpha}y), \quad x \in (0,T], \\ x^{1-\alpha}y(x)|_{x=0} = x^{1-\alpha}y(x)|_{x=T}, \\ x^{1-\alpha}(\mathscr{D}_{0+}^{\alpha}y)(x)|_{x=0} = x^{1-\alpha}(\mathscr{D}_{0+}^{\alpha}y)(x)|_{x=T}, \end{cases} \tag{1.4}$$

where $0 < T < +\infty$, and $f \in C([0,T] \times \mathbb{R} \times \mathbb{R})$.

Remark 1.1 In the special case: $\alpha = 1$, problem (1.4) becomes the periodic boundary value problem for a second ordinary differential equation

$$\begin{cases} y''(x) = f(x,\ y,\ y'), \quad x \in (0,T], \\ y(0) = y(T), \quad y'(0) = y'(T). \end{cases} \tag{1.5}$$

Differential equations of fractional order occur more frequently in different research areas and engineering, such as physics, chemistry, control of dynamical systems etc. Recently, many researchers paid attention to existence result of solution of the initial value problem for fractional differential equations, such as [4–7]. Some recent contributions to the theory of fractional differential equations can be seen in [8-12].

In [4], the existence and uniqueness of solution of the following initial value problem for a fractional differential equation

$$\begin{cases} D_0^{\alpha}u(t) = f(t,u(t)), \quad t \in (0,T], \\ t^{1-\alpha}u(t)|_{t=0} = u_0. \end{cases}$$

was discussed by using the method of upper and lower solutions and its associated monotone iterative.

In [5], the global existence results for an initial value problem associated to a large class of fractional differential equations

$$\begin{cases} D_0^{\alpha}(u-u_0)(t) = f(t,u(t)), \quad t > 0, \\ u(0) = u_0. \end{cases}$$

was presented by means of a comparison result and the fixed point theory.

In [7], the authors considered the existence of minimal and maximal solutions and uniqueness of solution of the initial value problem for a fractional differential equation involving a Riemann–Liouville sequential fractional derivative, by using the method of upper and lower solutions and its associated monotone iterative method.

$$\begin{cases} (\mathscr{D}_{0+}^{2\alpha}y)(x) = f(x,\ y,\ \mathscr{D}_{0+}^{\alpha}y), \quad x \in (0,T], \\ x^{1-\alpha}y(x)|_{x=0} = y_0, \quad x^{1-\alpha}(\mathscr{D}_{0+}^{\alpha}y)(x)|_{x=0} = y_1, \end{cases}$$

where $0 < T < +\infty$, and $f \in C([0,T] \times \mathbb{R} \times \mathbb{R})$.

While for the existence of solution of the periodic boundary value problem (1.4) for a fractional differential equation a involving Riemann–Liouville sequential fractional derivative has not been given up to now, the research proceeds slowly and appears some new difficulties in obtaining comparison results.

Now, in this paper, we shall discuss the properties of the well-known Mittag–Leffler function, and consider the existence of solution of the periodic boundary value problem (1.4) for a fractional differential equation involving Riemann–Liouville sequential fractional derivative by using the method of upper and lower solutions and Schauder fixed point theorem.

Let

$$\begin{aligned} &C([0,T]) = \left\{ y \ : \ y(x) \ \text{ is continuous on } [0,T],\ \|y\|_C = \max_{t\in[0,1]} |y(t)| \right\} \\ &C_{1-\alpha}([0,T]) = \{y \ \in C(0,T]:\ x^{1-\alpha}y(x) \in C([0,T]),\ \|y\|_{C_{1-\alpha}} = \|x^{1-\alpha}y\|_C\} \\ &C^{\alpha}_{1-\alpha}([0,T]) = \{y \ \in C_{1-\alpha}([0,T]):\ x^{1-\alpha}(\mathscr{D}^{\alpha}_{0+}y)(x) \in C([0,T])\}. \end{aligned}$$

Definition 1.1. We call a function $y(x)$ a classical solution of problem (1.4), if: (i) $y(x) \in C^{\alpha}_{1-\alpha}([0,T])$ and its fractional integral $(I^{1-\alpha}y(t))(x)$, $(I^{1-\alpha}\mathscr{D}^{\alpha}_{0+}y(t))(x)$ are continuously differentiable for $(0,T]$; (ii) $y(x)$ satisfies problem (1.4).

For problem (1.4), we have the following definitions of upper and lower solutions.

Definition 1.2. A function $p \in C^{\alpha}_{1-\alpha}([0,T])$ is called a lower solution of problem (1.4), if it satisfies

$$\begin{cases} (\mathscr{D}^{2\alpha}_{0+}p)(x) \le f(x,\ p,\ \mathscr{D}^{\alpha}_{0+}p), \quad x \in (0,T], \\ x^{1-\alpha}p(x)|_{x=0} = x^{1-\alpha}p(x)|_{x=T}, \\ x^{1-\alpha}(\mathscr{D}^{\alpha}_{0+}p)(x)|_{x=0} = x^{1-\alpha}(\mathscr{D}^{\alpha}_{0+}p)(x)|_{x=T}. \end{cases} \tag{1.6}$$

Analogously, a function $q \in C^{\alpha}_{1-\alpha}([0,T])$ is called an upper solution of problem (1.4), if it satisfies

$$\begin{cases} (\mathscr{D}^{2\alpha}_{0+}q)(x) \ge f(x,\ q,\ \mathscr{D}^{\alpha}_{0+}q), \quad x \in (0,T], \\ x^{1-\alpha}q(x)|_{x=0} = x^{1-\alpha}q(x)|_{x=T}, \\ x^{1-\alpha}(\mathscr{D}^{\alpha}_{0+}q)(x)|_{x=0} = x^{1-\alpha}(\mathscr{D}^{\alpha}_{0+}q)(x)|_{x=T}. \end{cases} \tag{1.7}$$

In what follows, we assume that

$$\begin{cases} p(x) \le q(x), \quad x \in (0,T]: \ \ x^{1-\alpha}p(x)|_{x=0} \le x^{1-\alpha}q(x)|_{x=0}, \\ x^{1-\alpha}(\mathscr{D}^{\alpha}_{0+}p)(x)|_{x=0} \le x^{1-\alpha}(\mathscr{D}^{\alpha}_{0+}q)(x)|_{x=0}, \end{cases} \tag{1.8}$$

and define that the ordered interval in space $C^{\alpha}_{1-\alpha}([0,T])$

$$\begin{aligned} [p,q] = \ \ &\Big\{u \in C^{\alpha}_{1-\alpha}([0,T]):\ \ p(t) \le u(t) \le q(t),\ t \in (0,T], \\ &t^{1-\alpha}p(t)|_{t=0} \le t^{1-\alpha}u(t)|_{t=0} \le t^{1-\alpha}q(t)|_{t=0}, \\ &t^{1-\alpha}(\mathscr{D}^{\alpha}_{0+}p)(t)|_{t=0} \le t^{1-\alpha}(\mathscr{D}^{\alpha}_{0+}u)(t)|_{t=0} \le t^{1-\alpha}(\mathscr{D}^{\alpha}_{0+}q)(t)|_{t=0}\Big\}. \end{aligned} \tag{1.9}$$

The following is an existence result of the solution for the linear periodic boundary value problem for a fractional differential equation and a property of Riemann–Liouville fractional calculus, which are important for us to obtain existence of solutions for problem (1.4).

Lemma 1.1 (see [1]) . Suppose that $u \in C_{1-\alpha}([0,T])$, then the linear initial value problem

$$\begin{cases} \mathscr{D}_{0+}^{\alpha} u(x) + Mu(x) = \sigma(x), \quad x \in (0,T], \\ x^{1-\alpha} u(x)|_{x=0} = u_0, \end{cases} \tag{1.10}$$

where $M \in \mathbb{R}$ is a constant and $\sigma \in C_{1-\alpha}[0,T]$, has the following integral representation of solution

$$u(x) = \Gamma(\alpha) u_0 e_\alpha(-M, x) + [e_\alpha(-M,t) * \sigma(t)](x), \tag{1.11}$$

where

$$(g * f)(x) = \int_0^x g(x-t) f(t) \mathrm{d}t, \tag{1.12}$$

$$e_\alpha(\lambda, z) = z^{\alpha-1} E_{\alpha,\alpha}(\lambda z^\alpha) = z^{\alpha-1} \sum_{k=0}^{\infty} \lambda^k \frac{z^{\alpha k}}{\Gamma((k+1)\alpha)}, \tag{1.13}$$

$E_{\alpha,\alpha}(x) = \sum\limits_{k=0}^{\infty} \dfrac{x^k}{\Gamma((k+1)\alpha)}$ is Mittag–Leffler function (see [1], [13]).

Remark 1.2 Obviously, $\mathscr{D}_{0+}^{\alpha} e_\alpha(\lambda, z) = \lambda e_\alpha(\lambda, z)$. For $\alpha = 1$, initial problem (1.10) is $u'(x) + Mu(x) = \sigma(x), \ \ u(0) = u_0$ and the solution given by (1.11) is valid (it is the classical solution using the variation of constants formula).

Lemma 1.2 . Suppose that $u \in C_{1-\alpha}([0,T])$, then the linear periodic boundary value problem

$$\begin{cases} \mathscr{D}_{0+}^{\alpha} u(x) + Mu(x) = \sigma(x), \quad x \in (0,T], \\ x^{1-\alpha} u(x)|_{x=0} = u_0 = x^{1-\alpha} u(x)|_{x=T}, \end{cases} \tag{1.14}$$

where $M \in \mathbb{R}$ is a constant and $\sigma \in C_{1-\alpha}[0,T]$, has the following integral representation of solution

$$u(x) = \Gamma(\alpha) \frac{(e_\alpha(-M,t) * \sigma(t))(T)}{T^{\alpha-1} - \Gamma(\alpha) e_\alpha(-M,T)} e_\alpha(-M,x) + [e_\alpha(-M,t) * \sigma(t)](x). \tag{1.15}$$

Proof By Lemma 1.1, we have that the linear initial value problem (1.14) has the integral representation of solution (1.11). By the condition of periodic boundary value problem (1.14), we have

$$u_0 = \frac{(e_\alpha(-M,t) * \sigma(t))(T)}{T^{\alpha-1} - \Gamma(\alpha) e_\alpha(-M,T)}. \tag{1.16}$$

Substituting (1.16) into (1.11), we obtain (1.15). The proof of Lemma 1.2 is completed. □

Lemma 1.3 . Suppose that $u \in C_{1-\alpha}^{\alpha}([0,T])$, then the linear periodic boundary value problem

$$\begin{cases} (\mathscr{D}_{0+}^{2\alpha} u)(x) + N \mathscr{D}_{0+}^{\alpha} u(x) + Mu(x) = \sigma(x), \quad x \in (0,T], \\ x^{1-\alpha} u(x)|_{x=0} = u_0 = x^{1-\alpha} u(x)|_{x=T}, \\ x^{1-\alpha} (\mathscr{D}_{0+}^{\alpha} u)(x)|_{x=0} = u_1 = x^{1-\alpha} (\mathscr{D}_{0+}^{\alpha} u)(x)|_{x=T}, \end{cases} \tag{1.17}$$

where $N > 0,\ M \in \mathbb{R},\ \ N^2 > 4M$ are constants and $\sigma \in C_{1-\alpha}[0,T]$, has the following representation of solution

$$u(x) = \Gamma(\alpha)u_0 e_\alpha(\lambda_2, x) + \Gamma(\alpha)\bar{y}_0[e_\alpha(\lambda_2, t) * e_\alpha(\lambda_1, t)](x) + [e_\alpha(\lambda_2, t) * e_\alpha(\lambda_1, t) * \sigma(t)](x). \quad (1.18)$$

where

$$\lambda_2 = \frac{-N - \sqrt{N^2 - 4M}}{2} < \lambda_1 = \frac{-N + \sqrt{N^2 - 4M}}{2}, \quad (1.19)$$

$$y(x) = \Gamma(\alpha)\bar{y}_0 e_\alpha(\lambda_1, x) + [e_\alpha(\lambda_1, t) * \sigma(t)](x), \quad x \in (0,T], \quad (1.20)$$

$$\bar{y}_0 = u_1 - \lambda_2 u_0 = \frac{(e_\alpha(\lambda_1, t) * \sigma(t))(T)}{T^{\alpha-1} - \Gamma(\alpha)e_\alpha(\lambda_1, T)}, \quad (1.21)$$

$$u_0 = \frac{(e_\alpha(\lambda_2, t) * y(t))(T)}{T^{\alpha-1} - \Gamma(\alpha)e_\alpha(\lambda_2, T)}. \quad (1.22)$$

Proof Let

$$(\mathscr{D}^\alpha_{0+} - \lambda_2)u(x) = y(x), \qquad x \in (0,T].$$

Then the problem (1.17) is equivalent to

$$\begin{cases} (\mathscr{D}^\alpha_{0+} - \lambda_1)y(x) = \sigma(x), & x \in (0,T], \\ x^{1-\alpha}y(x)|_{x=0} = \bar{y}_0 = u_1 - \lambda_2 u_0 = x^{1-\alpha}y(x)|_{x=T}, \end{cases} \quad (1.23)$$

and

$$\begin{cases} (\mathscr{D}^\alpha_{0+} - \lambda_2)u(x) = y(x), & x \in (0,T], \\ x^{1-\alpha}u(x)|_{x=0} = u_0 = x^{1-\alpha}u(x)|_{x=T}. \end{cases} \quad (1.24)$$

By the Lemma 1.2, we have that the linear periodic boundary value problems (1.23) and (1.24) have the following representation of solutions

$$y(x) = \Gamma(\alpha)\bar{y}_0 e_\alpha(\lambda_1, x) + [e_\alpha(\lambda_1, t) * \sigma(t)](x), \quad (1.25)$$

$$u(x) = \Gamma(\alpha)u_0 e_\alpha(\lambda_2, x) + [e_\alpha(\lambda_2, t) * y(t)](x), \quad (1.26)$$

where $\bar{y}_0,\ u_0$ are given by (1.21) and (1.22). Substituting (1.25) into (1.26), we obtain (1.18). The proof of Lemma 1.3 is completed. □

Lemma 1.4 (see [7]) .

$$[e_\alpha(\lambda_2, t) * e_\alpha(\lambda_1, t)](x) = [e_\alpha(\lambda_1, t) * e_\alpha(\lambda_2, t)](x) = \frac{1}{\lambda_1 - \lambda_2}\Big[e_\alpha(\lambda_1, t) - e_\alpha(\lambda_2, t)\Big](x), \quad x \in \mathbb{R}. \quad (1.27)$$

This paper is organized as follows. In Section 2 we give some preliminaries, including a property of Mittag–Leffler function which will be used in our main result, a comparison result. The main results are established in Section 3.

2. A property of Mittag–Leffler function and some Lemmas

In the following, we shall use the definition and properties of the Γ function which listed as follows (see [14]).

$$\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t}\,\mathrm{d}t, \tag{2.1}$$

$$\frac{1}{\Gamma(\alpha)} = \lim_{n\to\infty} \frac{1}{(n+1)^\alpha}\alpha(1+\alpha)\left(1+\frac{\alpha}{2}\right)\cdots\left(1+\frac{\alpha}{n}\right), \tag{2.2}$$

$$\frac{1}{\Gamma(1+\alpha)} = \lim_{n\to\infty} \frac{1}{(n+1)^\alpha}(1+\alpha)\left(1+\frac{\alpha}{2}\right)\cdots\left(1+\frac{\alpha}{n}\right), \tag{2.3}$$

Let

$$\psi_n(\alpha) = \alpha(1+\alpha)\left(1+\frac{\alpha}{2}\right)\cdots\left(1+\frac{\alpha}{n}\right). \tag{2.4}$$

Then

$$\frac{1}{\Gamma(\alpha)} = \lim_{n\to\infty} \frac{1}{(n+1)^\alpha}\psi_n(\alpha). \tag{2.5}$$

Lemma 2.1 (see [7]) For $0 < \alpha \le 1$, there exist positive constants

$$b_n^0 > 0,\ b_n^1 > 0,\ \cdots,\ b_n^n > 0, \quad \text{such that} \quad \psi_n(k\alpha) = \sum_{i=0}^{n} b_n^i C_{k+i}^{i+1}. \tag{2.6}$$

Hence, we have

$$(k-1)\psi_n(k\alpha) = \sum_{i=0}^{n} (i+2) b_n^i C_{k+i}^{i+2}, \tag{2.7}$$

$$(1+k\alpha)\left(1+\frac{k\alpha}{2}\right)\cdots\left(1+\frac{k\alpha}{n}\right) = \frac{1}{\alpha}\sum_{i=0}^{n}\frac{1}{i+1} b_n^i C_{k+i}^{i}. \tag{2.8}$$

Note

$$\begin{cases} F(x) = E_{\alpha,\alpha}(x) = \sum\limits_{k=0}^{\infty} \dfrac{x^k}{\Gamma((k+1)\alpha)}, \quad g(x) = \sum\limits_{k=1}^{\infty} \dfrac{kx^{k-1}}{\Gamma((k+1)\alpha)}, \\ h(x) = \sum\limits_{k=0}^{\infty} \dfrac{x^k}{\Gamma(k\alpha+1)}. \end{cases} \tag{2.9}$$

Lemma 2.2 (see [7]) For $0 < \alpha \le 1$, we have

$$F(x) > 0, \quad g(x) > 0, \quad h(x) > 0,\ \forall x \in \mathbb{R} = (-\infty,\ +\infty). \tag{2.10}$$

Lemma 2.3 For $0 < \alpha \le 1$, we have

$$\begin{aligned} &0 < F(x) < F(0) = \frac{1}{\Gamma(\alpha)} < F(y), \quad \text{for } x < 0 < y \quad \text{and} \\ &\lim_{x\to+\infty} F(x) = +\infty, \quad \lim_{x\to-\infty} F(x) = 0. \end{aligned} \tag{2.11}$$

Proof By means of $F'(x) = g(x) > 0, \ \ \forall x \in \mathbb{R}$, we have

$$0 < F(x) < F(0) = \frac{1}{\Gamma(\alpha)} < F(y), \quad \text{for} \ \ x < 0 < y.$$

And

$$F(x) = F(0) + \int_0^x g(t)\ \mathrm{d}t \geq \frac{1}{\Gamma(\alpha)} + g(0)x = \frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(2\alpha)}x, \quad \forall x > 0.$$

Hence, $\lim\limits_{x\to+\infty} F(x) = +\infty$.

$$\because \ \ \alpha h'(x) = F(x), \ \ F(x) > 0, \ h(x) > 0, \ \ \text{for} \ \ x < 0,$$

$$(-x)F(x) = \int_x^0 F(x)\mathrm{d}t \leq \int_x^0 F(t)\mathrm{d}t = \alpha(h(0) - h(x)) < \alpha h(0) = \alpha, \ \ \text{for} \ \ x < 0.$$

Therefore, $0 < F(x) < \dfrac{\alpha}{-x}$ for $x < 0$, and $\lim\limits_{x\to-\infty} F(x) = 0$. The proof of Lemma 2.3 is completed. □

The following results will play a very important role in this paper.

Lemma 2.4. (a comparison result) If $w \in C_{1-\alpha}([0, T])$ and satisfies the relations

$$\begin{cases} D^\alpha w(t) + Mw(t) \geq 0, \quad t \in (0, T], \\ t^{1-\alpha} w(t)|_{t=0} \geq 0, \end{cases} \tag{2.12}$$

where $M \in \mathbb{R}$ is a constant. Then $w(t) \geq 0, \quad t \in (0, T]$.

Proof By Lemma 2.3, we know that $E_{\alpha,\ \alpha}(-Mt^\alpha) > 0, \ t \in (0, T]$. Hence $e_\alpha(-M, t) > 0, \ t \in (0, T]$. Let $t^{1-\alpha}w(t)|_{t=0} = w_0, \ \ D^\alpha w(t) + Mw(t) = \sigma(t), \ \ t \in (0, T]$. Then $w_0 \geq 0, \ \ \sigma(t) \geq 0, \ \ t \in (0, T]$. By the formula (1.11) of Lemma 1.1, we obtain that $w(t) \geq 0, \ \ t \in (0, T]$. □

Remark 2.1 In this result, we delete the condition $M > -\dfrac{\Gamma(1+\alpha)}{T^\alpha}$ of the Lemma 2.1 of paper [4], so this result is an essential improvement of the paper [4].

Lemma 2.5. (a comparison result) If $w \in C_{1-\alpha}([0, T])$ and satisfies the relations

$$\begin{cases} D^\alpha w(t) + Mw(t) \geq 0, \quad t \in (0, T], \\ t^{1-\alpha} w(t)|_{t=0} = t^{1-\alpha} w(t)|_{t=T}, \end{cases} \tag{2.13}$$

where $M > 0$ is a constant. Then $w(t) \geq 0, \quad t \in (0, T]$.

Proof Let $t^{1-\alpha}w(t)|_{t=0} = w_0, \ \ D^\alpha w(t) + Mw(t) = \sigma(t), \ \ t \in (0, T]$. Then $\sigma(t) \geq 0, \ \ t \in (0, T]$. By the proof of Lemma 1.2, we have

$$w(x) = \Gamma(\alpha) w_0 e_\alpha(-M, x) + [e_\alpha(-M, t) * \sigma(t)](x), \tag{2.14}$$

where

$$w_0 = \frac{T^{1-\alpha}}{[1 - \Gamma(\alpha)E_{\alpha,\alpha}(-MT^\alpha)]} \int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(-M(T-s)^\alpha)\sigma(s)\ \mathrm{d}s.$$

By Lemma 2.3, we know that $0 < E_{\alpha,\ \alpha}(-MT^\alpha) < \dfrac{1}{\Gamma(\alpha)}$ and $E_{\alpha,\ \alpha}(-Mt^\alpha) > 0, \ t \in (0, T]$. Hence $e_\alpha(-M, t) > 0, \ t \in (0, T]$ and $w_0 \geq 0$. The (2.13) and (2.14) imply that $w(t) \geq$

$0, \quad t \in (0,T]$. $\square$

Lemma 2.6. (a comparison result) If $w \in C^{\alpha}_{1-\alpha}([0,T])$ and satisfies the relations

$$\begin{cases} (\mathscr{D}^{2\alpha}_{0+}w)(x) + N\mathscr{D}^{\alpha}_{0+}w(x) + Mw(x) = \sigma(x) \geq 0, \quad x \in (0,T], \\ x^{1-\alpha}w(x)|_{x=0} = w_0 = x^{1-\alpha}w(x)|_{x=T}, \\ x^{1-\alpha}(\mathscr{D}^{\alpha}_{0+}w)(x)|_{x=0} = w_1 = x^{1-\alpha}(\mathscr{D}^{\alpha}_{0+}w)(x)|_{x=T}, \end{cases} \tag{2.15}$$

where $N > 0$, $M \in \mathbb{R}$, $N^2 > 4M$ are constants such that

$$\lambda_2 = \frac{-N - \sqrt{N^2 - 4M}}{2} < \lambda_1 = \frac{-N + \sqrt{N^2 - 4M}}{2} < 0. \tag{2.16}$$

Then $w(t) \geq 0, \quad t \in (0,T]$.

Proof By means of Lemma 2.3, we know that $e_\alpha(\lambda_1, x) > 0, \quad e_\alpha(\lambda_2, x) > 0, \ x \in (0,T]$. Therefore, $[e_\alpha(\lambda_2,t) * e_\alpha(\lambda_1,t)](x) \geq 0, \quad [e_\alpha(\lambda_2,t) * e_\alpha(\lambda_1,t) * \sigma(t)](x) \geq 0, \ x \in (0,T]$. Since $\lambda_2 < \lambda_1 < 0$, by the proof of Lemmas 1.3, 2.3 and 2.5, we obtain that

$$\bar{y}_0 = w_1 - \lambda_2 w_0 = \frac{T^{1-\alpha}(e_\alpha(\lambda_1,t) * \sigma(t))(T)}{1 - \Gamma(\alpha)E_{\alpha,\alpha}(\lambda_1 T^\alpha)} \geq 0, \tag{2.17}$$

$$u_0 = \frac{T^{1-\alpha}(e_\alpha(\lambda_2,t) * y(t))(T)}{1 - \Gamma(\alpha)E_{\alpha,\alpha}(\lambda_2 T^\alpha)} \geq 0, \tag{2.18}$$

where $y(t)$ is given by (1.20). Hence, from (1.18), $w(t) \geq 0, \quad t \in (0,T]$. The proof of Lemma 2.6 is completed. $\square$

3. Main results

On the basis of Lemmas 1.2-1.4 and 2.3-2.6, using the method of upper and lower solutions and Schauder fixed point theorem, we shall show the existence theorem of solutions for PBVP (1.4). For convenience, we list the following conditions:

(H1): there exist constants $N > 0$, $M \in \mathbb{R}$, $N^2 > 4M$ such that

$$f(t,\ q,\ \mathscr{D}^{\alpha}_{0+}q) - f(t,\ p,\ \mathscr{D}^{\alpha}_{0+}p) \geq -N(\mathscr{D}^{\alpha}_{0+}q - \mathscr{D}^{\alpha}_{0+}p) - M(q-p), \tag{3.1}$$

$p,\ q \in C^{\alpha}_{1-\alpha}([0,T])$ are lower and upper solutions of problem (1.4);

(H2): there exist constants $N > 0$, $M \in \mathbb{R}$, $N^2 > 4M$ such that (H1) holds, and for $x \in (0,T]$, $p(x) \leq y_2 \leq y_1 \leq q(x)$, $D_1(x) \leq z_i \leq D_2(x)$, $i = 1,\ 2$ such that

$$f(x,\ y_1,\ z_1) - f(t,\ y_2,\ z_2) \geq -N(z_1 - z_2) - M(y_1 - y_2), \tag{3.2}$$

where

$$D_1(x) = (\mathscr{D}^{\alpha}_{0+}p)(x) + \lambda_2(q(x) - p(x)), \quad D_2(x) = (\mathscr{D}^{\alpha}_{0+}q)(x) - \lambda_2(q(x) - p(x)), \ x \in (0,T], \tag{3.3}$$

$$\lambda_2 = \frac{-N - \sqrt{N^2 - 4M}}{2} < \lambda_1 = \frac{-N + \sqrt{N^2 - 4M}}{2} < 0. \tag{3.4}$$

In view of (3.2), the function

$$f(t,u,v)+Mu+Nv$$

is monotone nondecreasing in $u,\ v$ for $u,\ v\in C_{1-\alpha}([0,T])$.

Lemma 3.1. Let (H1) be satisfied. Then

$$\mathscr{D}^{\alpha}_{0+}(q-p)(x)-\lambda_2(q-p)(x)\geq 0,\quad x\in(0,T]. \tag{3.5}$$

Hence,

$$\mathscr{D}^{\alpha}_{0+}(q)(x)-\lambda_2(q-p)(x)\geq \mathscr{D}^{\alpha}_{0+}(p)(x)\geq \mathscr{D}^{\alpha}_{0+}(p)(x)+\lambda_2(q-p)(x),\quad x\in(0,T],$$

where $\lambda_2<0$ is given by (3.4).

Proof Let $z(x)=\mathscr{D}^{\alpha}_{0+}(q-p)(x)-\lambda_2(q-p)(x),\quad x\in(0,T]$. Then

$$\begin{cases}\mathscr{D}^{\alpha}_{0+}z(x)-\lambda_1 z(x)=\mathscr{D}^{2\alpha}_{0+}(q-p)(x)-(\lambda_1+\lambda_2)\mathscr{D}^{\alpha}_{0+}(q-p)(x)+\lambda_1\lambda_2(q-p)(x)\\ \geq f(x,\ q,\ \mathscr{D}^{\alpha}_{0+}q)-f(x,\ p,\ \mathscr{D}^{\alpha}_{0+}p)+N\mathscr{D}^{\alpha}_{0+}(q-p)(x)+M(q-p)(x)\geq 0,\quad x\in(0,T],\\ x^{1-\alpha}z(x)|_{x=0}=q_1-p_1-\lambda_2(q_0-p_0)\geq 0.\end{cases}$$

By Lemma 2.4, we have that $z(x)\geq 0,\quad x\in(0,T]$. This complete the proof of Lemma 3.1. □

Lemma 3.2. Let (H1) be satisfied. Then

$$\Omega=\{\ \eta\in[p,q]:\ D_1(x)\leq(\mathscr{D}^{\alpha}_{0+}\eta)(x)\leq D_2(x),\quad x\in(0,T]\ \}. \tag{3.6}$$

is a convex closed set, where $D_1(x),\quad D_2(x)$ are given by (3.3).

Theorem 3.1. Assume that $p,\ q\in C^{\alpha}_{1-\alpha}([0,T])$ are lower and upper solutions of problem (1.4), such that (1.8) holds, and $f\in C([0,T]\times\mathbb{R}\times\mathbb{R})$ satisfies (H1) and (H2). Then there exists one solution u of PBVP (1.4) such that

$$p(x)\leq u(x)\leq q(x),\qquad D_1(x)\leq(\mathscr{D}^{\alpha}_{0+}u)(x)\leq D_2(x),\quad x\in(0,T],$$

where $D_1(x),\quad D_2(x)$ are given by (3.3).

Proof of Theorem 3.1. Let

$$\sigma(\eta)(x)=f(x,\eta(x),\mathscr{D}^{\alpha}_{0+}\eta(x))+N\mathscr{D}^{\alpha}_{0+}\eta(x)+M\eta(x),\quad x\in(0,T],\quad\forall\ \eta\in\Omega. \tag{3.7}$$

For any $\eta\in\Omega$, consider the linear PBVP

$$\begin{cases}(\mathscr{D}^{2\alpha}_{0+}u)(x)+N\mathscr{D}^{\alpha}_{0+}u(x)+Mu(x)=\sigma(\eta)(x),\quad x\in(0,T],\\ x^{1-\alpha}u(x)|_{x=0}=x^{1-\alpha}u(x)|_{x=T},\\ x^{1-\alpha}(\mathscr{D}^{\alpha}_{0+}u)(x)|_{x=0}=x^{1-\alpha}(\mathscr{D}^{\alpha}_{0+}u)(x)|_{x=T}.\end{cases} \tag{3.8}$$

By Lemma 1.3, (3.8) has exactly one solution $u\in C^{\alpha}_{1-\alpha}([0,T])$ given by

$$\begin{aligned}u(x)=(A\eta)(x)\quad&=\Gamma(\alpha)u_0(\eta)e_\alpha(\lambda_2,x)+\Gamma(\alpha)\bar{y}_0(\eta)[e_\alpha(\lambda_2,t)*e_\alpha(\lambda_1,t)](x)\\&+[e_\alpha(\lambda_2,t)*e_\alpha(\lambda_1,t)*\sigma(\eta)(t)](x),\end{aligned} \tag{3.9}$$

and

$$
\begin{aligned}
(\mathscr{D}_{0+}^{\alpha}A\eta)(x) = \ & \Gamma(\alpha)\left(u_0(\eta)\lambda_2 e_\alpha(\lambda_2,x)+\bar{y}_0(\eta)\frac{\Big[\lambda_1 e_\alpha(\lambda_1,t)-\lambda_2 e_\alpha(\lambda_2,t)\Big](x)}{\lambda_1-\lambda_2}\right) \\
&+\frac{1}{\lambda_1-\lambda_2}\Big[\lambda_1 e_\alpha(\lambda_1,t)*\sigma(\eta)(t)-\lambda_2 e_\alpha(\lambda_2,t)*\sigma(\eta)(t)\Big](x),
\end{aligned}
\tag{3.10}
$$

where

$$
\bar{y}_0(\eta)=(u_1-\lambda_2 u_0)(\eta)=\frac{(e_\alpha(\lambda_1,t)*\sigma(\eta)(t))(T)}{T^{\alpha-1}-\Gamma(\alpha)e_\alpha(\lambda_1,T)},\tag{3.11}
$$

$$
u_0(\eta)=\frac{(e_\alpha(\lambda_2,t)*y(\eta)(t))(T)}{T^{\alpha-1}-\Gamma(\alpha)e_\alpha(\lambda_2,T)}.\tag{3.12}
$$

$$
y(\eta)(x)=\Gamma(\alpha)\bar{y}_0 e_\alpha(\lambda_1,x)+[e_\alpha(\lambda_1,t)*\sigma(\eta)(t)](x),\quad x\in(0,T].\tag{3.13}
$$

And then A is an operator from Ω into $C_{1-\alpha}^{\alpha}([0,T])$ and η is a solution of PBVP(1.4) if and only if $\eta=A\eta$.

Let $w(x)=(Ap-p)(x),\ \ x\in(0,T]$. Then by (1.6), $w(x)$ satisfies the relations

$$
\begin{cases}
(\mathscr{D}_{0+}^{2\alpha}w)(x)+N\mathscr{D}_{0+}^{\alpha}w(x)+Mw(x)=(\mathscr{D}_{0+}^{2\alpha}Ap)(x)+N\mathscr{D}_{0+}^{\alpha}Ap(x)+MAp(x)\\
-[(\mathscr{D}_{0+}^{2\alpha}p)(x)+N\mathscr{D}_{0+}^{\alpha}p(x)+Mp(x)]\\
=f(t,p,\mathscr{D}_{0+}^{\alpha}p\ )(x)-(\mathscr{D}_{0+}^{2\alpha}p)(x)\geq 0,\quad x\in(0,T],\\
x^{1-\alpha}w(x)|_{x=0}=x^{1-\alpha}w(x)|_{x=T},\quad x^{1-\alpha}(\mathscr{D}_{0+}^{\alpha}w)(x)|_{x=0}=x^{1-\alpha}(\mathscr{D}_{0+}^{\alpha}w)(x)|_{x=T}.
\end{cases}
$$

By means of Lemma 2.6, we obtain that $w(x)\geq 0,\ \ x\in(0,T]$. Hence, $p(x)\leq(Ap)(x),\ \ x\in(0,T]$. Similarly, by (1.7) we can easily obtain that $(Aq)(x)\leq q(x),\ \ x\in(0,T]$.

By (3.2), we have

$$
\begin{cases}
f(x,\ p,\ \mathscr{D}_{0+}^{\alpha}p)+N(\mathscr{D}_{0+}^{\alpha}p)(x)+Mp(x)\leq f(x,\ \eta,\ \mathscr{D}_{0+}^{\alpha}\eta)+N(\mathscr{D}_{0+}^{\alpha}\eta)(x)+M\eta(x)\\
\leq f(x,\ q,\ \mathscr{D}_{0+}^{\alpha}q)+N(\mathscr{D}_{0+}^{\alpha}q)(x)+Mq(x),\quad x\in(0,T],\quad \forall\eta\in\Omega.
\end{cases}
$$

Hence, by means of Lemma 2.6, (1.6), (1.7), (3.2), (3.9) and (3.10), we can obtain

$$
p\leq Ap\leq A\eta\leq Aq\leq q,\qquad \forall\eta\in\Omega,\tag{3.14}
$$

and

$$
\begin{cases}
\text{if}\quad p\leq\eta_1\leq\eta_2\leq q,\quad \eta_i\in\Omega,\quad i=1,\ 2,\quad \text{then}\\
\sigma(\eta_1)\leq\sigma(\eta_2),\quad u_0(\eta_1)\leq u_0(\eta_2),\quad \bar{y}_0(\eta_1)\leq\bar{y}_0(\eta_2),\\
\text{and}\quad A\eta_1\leq A\eta_2.
\end{cases}
\tag{3.15}
$$

By the proof of Lemma 3.1, we know that

$$
z_1(x)=\mathscr{D}_{0+}^{\alpha}(A\eta-p)(x)-\lambda_2(A\eta-p)(x)\geq 0,\quad x\in(0,T],\quad \forall\eta\in\Omega.
$$

Hence,

$$
\begin{aligned}
\mathscr{D}_{0+}^{\alpha}(A\eta)(x)\ &\geq\mathscr{D}_{0+}^{\alpha}(p)(x)+\lambda_2(A\eta-p)(x)\\
&\geq\mathscr{D}_{0+}^{\alpha}(p)(x)+\lambda_2(q-p)(x)=D_1(x),\ x\in(0,T],\quad \forall\eta\in\Omega.
\end{aligned}
$$

Similarly, we can obtain that $\mathscr{D}_{0+}^{\alpha}(A\eta)(x) \le D_2(x),\ x \in (0,T],\ \forall \eta \in \Omega$. Therefore, $A(\Omega) \subset \Omega$.

In the following, we shall show that $A(\Omega)$ is a relatively compact set in $C_{1-\alpha}^{\alpha}[0,T]$. For any $\eta \in [p,q]$, by (1.6), (1.7) and (3.2), we have

$$\begin{cases} (\mathscr{D}_{0+}^{2\alpha}p)(x) + N(\mathscr{D}_{0+}^{\alpha}p)(x) + Mp(x) \le f(x,\ p,\ \mathscr{D}_{0+}^{\alpha}p) + N(\mathscr{D}_{0+}^{\alpha}p)(x) + Mp(x) \\ \le f(x,\ \eta,\ \mathscr{D}_{0+}^{\alpha}\eta) + N(\mathscr{D}_{0+}^{\alpha}\eta)(x) + M\eta(x) \le f(x,\ q,\ \mathscr{D}_{0+}^{\alpha}q) + N(\mathscr{D}_{0+}^{\alpha}q)(x) + Mq(x) \\ \le (\mathscr{D}_{0+}^{2\alpha}q)(x) + N(\mathscr{D}_{0+}^{\alpha}q)(x) + Mq(x), \quad x \in (0,T]. \end{cases}$$

Since $\Omega \subset C_{1-\alpha}^{\alpha}[0,T]$ are bounded sets, therefore, $\{\sigma(\eta)(t) = f(x,\ \eta,\ \mathscr{D}_{0+}^{\alpha}\eta) + N(\mathscr{D}_{0+}^{\alpha}\eta)(x) + M\eta(x) \mid \eta \in \Omega\}$ is a bounded set also. Hence, there exists a constant $L > 0$ such that

$$\begin{cases} \|\sigma(\eta)\| = \max\limits_{0\le t\le T} |t^{1-\alpha}\sigma(\eta)(t)| \le L, \quad \forall\ \eta \in \Omega, \\ \Longleftrightarrow |\sigma(\eta)(t)| \le Lt^{\alpha-1}, \quad \forall\ \ t \in (0,T],\ \forall\ \eta \in \Omega, \end{cases} \tag{3.16}$$

$$\begin{cases} |u_0(\eta)| \le \dfrac{\Big(|\bar{y}_0| + \dfrac{LT^{\alpha}\Gamma(\alpha)}{\Gamma(2\alpha)}\Big)T^{\alpha}\Gamma(\alpha)}{\Gamma(2\alpha)\,[1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda_2T^{\alpha})]}, \quad \forall\ \eta \in \Omega, \\ |\bar{y}_0(\eta)| \le \dfrac{LT^{\alpha}\Gamma(\alpha)}{\Gamma(2\alpha)\,[1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda_1T^{\alpha})]}, \quad \forall\ \eta \in \Omega. \end{cases} \tag{3.17}$$

On the other hand, from (1.27), $\{(A\eta)(t) \mid \forall\ \eta \in \Omega\}$ satisfies (3.9) and (3.10). Let

$$G(\lambda_i, t) = t^{1-\alpha}[e_{\alpha}(\lambda_i, t) * \sigma(\eta)(t)], \quad t \in [0,T], \quad i = 1,2. \tag{3.18}$$

(Without loss of generality, we assume $0 \le t_1 < t_2 \le T$.) Since $\lambda_2 < \lambda_1 < 0$, we have

$$|G(\lambda_i,t_1) - G(\lambda_i,t_2)| \le \frac{L\Gamma(\alpha)}{|\lambda_1|}\big|E_{\alpha,\alpha}(\lambda_i t_1^{\alpha}) - E_{\alpha,\alpha}(\lambda_i t_2^{\alpha})\big| + \frac{2L\Gamma(\alpha)}{\Gamma(2\alpha)}(t_2-t_1)^{\alpha}, \quad i = 1,2. \tag{3.19}$$

From $E_{\alpha,\alpha}(t) \in C[0,T]$, $\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon)$, when $|t_1 - t_2| < \delta$ (without loss of generality, we assume $0 \le t_1 < t_2 \le T$), we have

$$\Big|E_{\alpha,\alpha}(\lambda_1 t_1^{\alpha}) - E_{\alpha,\alpha}(\lambda_1 t_2^{\alpha})\Big| < \frac{\varepsilon}{8L_1}, \tag{3.20}$$

$$\Big|E_{\alpha,\alpha}(\lambda_2 t_1^{\alpha}) - E_{\alpha,\alpha}(\lambda_2 t_2^{\alpha})\Big| < \frac{\varepsilon}{8L_2}, \tag{3.21}$$

$$(t_2 - t_1)^{\alpha} < \frac{\varepsilon}{8L_3}, \tag{3.22}$$

where

$$L_1 = \max\left\{ \frac{|\Gamma(\alpha)\bar{y}_0(\eta)\lambda_1|}{|\lambda_1 - \lambda_2|},\ \frac{L\Gamma(\alpha)}{|\lambda_1 - \lambda_2|} \right\},$$

$$L_2 = \max\left\{ |\Gamma(\alpha)u_0(\eta)\lambda_2|,\ \ \frac{|\Gamma(\alpha)\bar{y}_0(\eta)\lambda_1|}{|\lambda_1 - \lambda_2|},\ \frac{L\Gamma(\alpha)}{|\lambda_1 - \lambda_2|} \right\},$$

$$L_3 = \frac{2L\Gamma(\alpha)}{\Gamma(2\alpha)|\lambda_1 - \lambda_2|}\Big(|\lambda_2| + |\lambda_1|\Big).$$

From (3.10), (3.16)–(3.22) and by a direct computation, we obtain that

$$\begin{aligned}&\left|t_1^{1-\alpha}\mathscr{D}_{0+}^{\alpha}(A\eta)(t_1)-t_2^{1-\alpha}\mathscr{D}_{0+}^{\alpha}(A\eta)(t_2)\right|\le|\Gamma(\alpha)u_0(\eta)\lambda_2|\Big|E_{\alpha,\alpha}(\lambda_2t_1^{\alpha})-E_{\alpha,\alpha}(\lambda_2t_2^{\alpha})\Big|\\&+\frac{|\Gamma(\alpha)\bar{y}_0(\eta)|}{|\lambda_1-\lambda_2|}\Big[|\lambda_1|\big|E_{\alpha,\alpha}(\lambda_1t_1^{\alpha})-E_{\alpha,\alpha}(\lambda_1t_2^{\alpha})\big|+|\lambda_2|\big|E_{\alpha,\alpha}(\lambda_2t_1^{\alpha})-E_{\alpha,\alpha}(\lambda_2t_2^{\alpha})\big|\Big]\\&+\frac{L\Gamma(\alpha)}{|\lambda_1-\lambda_2|}\left[\big|E_{\alpha,\alpha}(\lambda_1t_1^{\alpha})-E_{\alpha,\alpha}(\lambda_1t_2^{\alpha})\big|+\big|E_{\alpha,\alpha}(\lambda_2t_1^{\alpha})-E_{\alpha,\alpha}(\lambda_2t_2^{\alpha})\big|\right]\\&+\frac{2L\Gamma(\alpha)}{\Gamma(2\alpha)|\lambda_1-\lambda_2|}\Big(|\lambda_2|+|\lambda_1|\Big)(t_2-t_1)^{\alpha}<\varepsilon.\end{aligned}$$

This means $A(\Omega)$ is equi-continuity in $C_{1-\alpha}^{\alpha}[0,T]$, by means of the Arzela-Ascoli theorem, we have that $A(\Omega)$ is a relatively compact set of $C_{1-\alpha}^{\alpha}[0,T]$.

By the assumption of function f, the function σ is continuous. Hence $A:\Omega\longrightarrow\Omega$ is continuous and completely continuous. By means of Schauder fixed point theorem, A has a fixed point $\rho\in\Omega$, that is, ρ satisfies the integral equation

$$\begin{aligned}\rho(x)=(A\rho)(x)=\ &\Gamma(\alpha)u_0(\rho)e_{\alpha}(\lambda_2,x)+\Gamma(\alpha)(u_1-\lambda_2u_0)(\rho)[e_{\alpha}(\lambda_2,t)*e_{\alpha}(\lambda_1,t)](x)\\&+[e_{\alpha}(\lambda_2,t)*e_{\alpha}(\lambda_1,t)*\sigma(\rho)(t)](x),\quad x\in(0,T].\end{aligned}\tag{3.23}$$

That is, $\rho(x)$ is an integral representation of the solution to problem (3.8), that is, $\rho(t)$ is an integral representation of the solution to problem (1.4). By assumptions of functions f and Lemma 1.3, ρ is a classical solution of periodic boundary value problem (1.4). Thus, we complete this proof. □

Example Consider the following PBVP

$$\begin{cases}(\mathscr{D}_{0+}^{2\alpha}u)(x)=9x^{2(\alpha-1)}(1-x/T)^{1/4}-\dfrac{1}{8}u^2+u^{\delta_1}(\mathscr{D}_{0+}^{\alpha}u)^{\delta_2},\quad x\in(0,T],\\ x^{1-\alpha}u(x)|_{x=0}=x^{1-\alpha}u(x)|_{x=T},\quad x^{1-\alpha}(\mathscr{D}_{0+}^{\alpha}u)(x)|_{x=0}=x^{1-\alpha}(\mathscr{D}_{0+}^{\alpha}u)(x)|_{x=T},\end{cases}\tag{3.24}$$

where $0<\alpha<1,\ 0<\delta_1,\ 0<\delta_2$. Then PBVP (3.24) has a solution u such that $0\le u(x)\le q(x),\ \ x\in(0,T]$, where

$$p(x)=0,\quad q(x)=9x^{\alpha-1},\qquad x\in(0,T],$$

p is a lower solution and q an upper solution. The proof is omitted.

Acknowledgments The authors would like to thank the referee for some valuable suggestions and comments.

References

[1] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam, 2006.

[2] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives. Theory and Applications, Gordon and Breach, Yverdon, 1993.

[3] K.S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley and Sons, New York, 1993

[4] Shuqin Zhang, Monotone iterative method for initial value problem involving Riemann–Liouville fractional derivatives, Nonlinear Analysis 71 (2009) 2087–2093.

[5] D. Băleanu, O. G. Mustafa, On the global existence of solutions to a class of fractional differential equations, Computers and Mathematics with Applications 59 (2010) 1835–1841.

[6] D. Băleanu, O. G. Mustafa, On the asymptotic integration of a class of sublinear fractional differential equations, Journal of Mathematical Physics 50 (12)(2009) Article Number: 123520.

[7] Zhongli Wei, Qingdong Li, Junling Che, Initial value problems for fractional differential equations involving Riemann–Liouville sequential fractional derivative, J. Math. Anal. Appl. 367(1) (2010) 260-272.

[8] R.P. Agarwal, V. Lakshmikantham, J. J. Nieto, On the concept of solution for fractional differential equations with uncertainty. Nonlinear Analysis 72 (6) (2010) 2859-2862.

[9] D. Băleanu, S. I. Muslih, E. M. Rabei, On fractional Euler–Lagrange and Hamilton equations and the fractional generalization of total time derivative, Nonlinear Dyn 53 (2008) 67–74.

[10] Varsha Daftardar-Gejji, Sachin Bhalekar, Solving fractional boundary value problems with Dirichlet boundary conditions using a new iterative method, Computers and Mathematics with Applications 59 (2010) 1801–1809.

[11] Kehua Shi, Yongjin Wang, On a stochastic fractional partial differential equation driven by a Lévy space-time white noise. Journal of Mathematical Analysis and Applications 364 (1) (2010) 119-129.

[12] Junyi Liu, Mingyu Xu, Some exact solutions to Stefan problems with fractional differential equations. Journal of Mathematical Analysis and Applications 351 (2) (2009) 536-542.

[13] B. Bonilla, M. Rivero, L. Rodríguez-Germá, J.J. Trujillo, Fractional differential equations as alternative models to nonlinear differential equations, Applied Mathematics and Computation 187 (2007) 79–88.

[14] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.

(Received May 15, 2011)