

# Solvability of multi-point boundary value problem of nonlinear impulsive fractional differential equation at resonance

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**Abstract.** In this paper, we investigate the existence of solutions for multi-point boundary value problems of impulsive fractional differential equations at resonance by using the coincidence degree theory due to Mawhin.

**Keywords and Phrases.** resonance; impulsive; fractional derivative; fractional integral

**MSC (2010):** 34B10; 34B37; 34K37

## 1. Introduction

Differential equation with fractional order have recently proved valuable tools in the modeling of many phenomena in various fields of science and engineering [1-5]. Recently, many researchers paid attention to existence result of solution of the boundary value problems for fractional differential equations at nonresonance, see for examples [6-15]. But, there are few papers which consider the boundary value problem at resonance for nonlinear ordinary differential equations of fractional order. In [16], N. Kosmatov studied the boundary value problems of fractional differential equations at resonance with  $\dim \text{Ker} L = 1$ . More recently, Jiang [17] investigated the existence of solutions for the fractional differential equation at resonance with  $\dim \text{Ker} L = 2$  :

$$D_{0+}^{\alpha} u(t) = f(t, u(t), D^{\alpha-1} u(t)), \quad \text{a.e. } t \in [0, 1],$$
$$u(0) = 0, \quad D_{0+}^{\alpha-1} u(0) = \sum_{j=1}^m a_j D_{0+}^{\alpha-1} u(\xi_j), \quad D_{0+}^{\alpha-2} u(1) = \sum_{j=1}^n b_j D_{0+}^{\alpha-2} u(\eta_j),$$

where  $2 < \alpha < 3$ ,  $0 < \xi_1 < \xi_2 < \dots < \xi_m < 1$ ,  $\sum_{i=1}^m a_i = 1$ ,  $\sum_{j=1}^n b_j = 1$ ,  $\sum_{j=1}^n b_j \eta_j = 1$ ,  $f : [0, 1] \times R \times R \rightarrow R$  satisfies Caratheodory condition.

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To the best of the author knowledge, the solvability of resonance boundary value problems for impulsive fractional differential equations has not been well studied till now. We will fill this gap in the literature. Motivated by the excellent results of [17], [18], [19] and [20], in this paper, we investigate the existence of solutions for boundary value problems of nonlinear impulsive fractional differential equation at resonance

$$D_{0+}^{\alpha} u(t) = f(t, u(t), D_{0+}^{\alpha-1} u(t)), \quad t \in (0, 1), \quad t \neq t_i, \quad i = 1, \dots, k, \quad (1.1)$$

$$\lim_{t \rightarrow 0+} t^{2-\alpha} u(t) = \sum_{j=1}^n a_j u(\xi_j), \quad u(1) = \sum_{j=1}^n b_j u(\eta_j), \quad (1.2)$$

$$\Delta u(t_i) = I_i(u(t_i), D_{0+}^{\alpha-1} u(t_i)), \quad \Delta D_{0+}^{\alpha-1} u(t_i) = J_i(u(t_i), D_{0+}^{\alpha-1} u(t_i)), \quad (1.3)$$

where  $D_{0+}^{\alpha}$  is the standard Riemann-Liouville fractional derivative,  $1 < \alpha < 2$ ,  $f : [0, 1] \times R^2 \rightarrow R$ , and  $I_i, J_i : R \times R \rightarrow R$  are continuous,  $k$  is a fixed positive integer,  $t_i$  ( $i = 1, 2, \dots, k$ ) are fixed points with  $0 < t_1 < t_2 < \dots < t_k < 1$ ,  $\Delta u(t_i) = u(t_i + 0) - u(t_i - 0)$ ,  $\Delta D_{0+}^{\alpha} u(t_i) = D_{0+}^{\alpha} u(t_i + 0) - D_{0+}^{\alpha} u(t_i - 0)$ ,  $i = 1, \dots, k$ ,  $\xi_j, \eta_j \in (0, 1)$  ( $j = 1, \dots, n$ ) be given  $0 < \xi_1 < \dots < \xi_n < 1$ ,  $0 < \eta_1 < \dots < \eta_n < 1$ , and  $\xi_j, \eta_j \neq t_i$  ( $1 \leq j \leq n, 1 \leq i \leq k$ ),  $\sum_{j=1}^n a_j \xi_j^{\alpha-2} = \sum_{j=1}^n b_j \eta_j^{\alpha-2} = 1$ ,  $\sum_{j=1}^n a_j \xi_j^{\alpha-1} = 0$ , and  $\sum_{j=1}^n b_j \eta_j^{\alpha-1} = 1$ .

The BVP (1.1)-(1.3) happens to be at resonance in the sense that its associated linear homogeneous nonimpulse boundary value problem

$$D_{0+}^{\alpha} u(t) = 0, \quad 0 < t < 1, \quad (1.4)$$

$$\lim_{t \rightarrow 0+} t^{2-\alpha} u(t) = \sum_{j=1}^n a_j u(\xi_j), \quad u(1) = \sum_{j=1}^n b_j u(\eta_j), \quad (1.5)$$

has  $u(t) = h_1 t^{\alpha-1} + h_2 t^{\alpha-2}$ ,  $h_1, h_2 \in R$  as a nontrivial solution.

By the way, the theory of impulsive differential equation may be seen in [21] and [22].

The rest of this paper is organized as follows. In Section 2, we give some notations and lemmas. In Section 3, we establish an existence theorem for boundary value problem (1.1)-(1.3) at resonance case.

## 2. Preliminaries

For the convenience of the reader, we first briefly recall some fundamental tools of fractional calculus and the coincidence degree theory.

The Riemann-Liouville fractional integral of order  $\alpha > 0$  of a function  $u : (0, \infty) \rightarrow R$  is given by

$$I_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds$$

provided the right side is pointwise defined on  $(0, \infty)$ . The Riemann-Liouville fractional derivative of order  $\alpha > 0$  of a function  $u : (0, \infty) \rightarrow R$  is given by

$$D_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dt} \right)^n \int_0^t \frac{u(s)}{(t-s)^{\alpha-n+1}} ds,$$

where  $n = [\alpha] + 1$ , provided the right side is pointwise defined on  $(0, \infty)$ .

We make use of two relationships between  $D_{0+}^{\alpha} u$  and  $I_{0+}^{\alpha} u$  that are stated in the following lemma (see [3, 9]).

**Lemma 2.1.** *Assume that  $u \in C(0, 1) \cap L^1(0, 1)$ . Then*

$$(1) \quad I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \dots + c_n t^{\alpha-n},$$

for some  $c_i \in R$ ,  $i = 1, 2, \dots, n$ , where  $\alpha > 0$  and  $n = [\alpha] + 1$ .

$$(2) \quad D_{0+}^{\beta} I_{0+}^{\alpha} u(t) = I_{0+}^{\alpha-\beta} u(t), \quad \alpha \geq \beta \geq 0.$$

$$(3) \quad D_{0+}^{\alpha} t^{\alpha-i} = 0, \quad i = 1, 2, \dots, [\alpha] + 1.$$

Consider an operator equation

$$Lx = Nx, \tag{2.1}$$

where  $L : \text{dom}L \cap X \rightarrow Z$  is a linear operator,  $N : X \rightarrow Z$  is a non-linear operator,  $X$  and  $Z$  are Banach spaces. If  $\dim \text{Ker}L = \dim(Z/\text{Im}L) < +\infty$  and  $\text{Im}L$  is closed in  $Z$ , then  $L$  will be called a Fredholm mapping of index 0, and at the same time there exist continuous projectors  $P : X \rightarrow X$  and  $Q : Z \rightarrow Z$  such that  $\text{Im}P = \text{Ker}Q$ . It follows that  $L|_{\text{Dom}L \cap \text{Ker}P} : \text{dom}L \cap \text{Ker}P \rightarrow \text{Im}L$  is invertible. We denote the inverse of this map by  $K_P$ . Let  $\Omega$  be an open bounded subset of  $X$ . The map  $N$  will be called  $L$ -compact on  $\overline{\Omega}$  if  $QN(\overline{\Omega})$  is bounded and  $K_P(I - Q) : \overline{\Omega} \rightarrow X$  is compact. Since  $\text{Im}Q$  is compact. Since  $\text{Im}Q$  is isomorphic to  $\text{Ker}L$  there exists an isomorphism  $: \text{Im}Q \rightarrow \text{Ker}L$ .

**Theorem 2.2** ([23]). *Suppose that  $L$  is a Fredholm operator of index 0 and  $N$  is  $L$ -compact on  $\overline{\Omega}$ , where  $\Omega$  is an open bounded subset of  $X$ . If the following conditions are satisfied:*

$$(i) \quad Lx \neq \lambda Nx \text{ for every } (x, \lambda) \in [(\text{dom}L \setminus \text{Ker}L) \cap \partial\Omega] \times (0, 1);$$

(ii)  $Nx \notin \text{Im}L$  for every  $x \in \text{Ker}L \cap \partial\Omega$ ;

(iii)  $\deg(QN|_{\text{Ker}L}, \Omega \cap \text{Ker}L, 0) \neq 0$ , where  $Q : Y \rightarrow Y$  is a projection such that  $\text{Im}L = \text{Ker}Q$ .  
Then the equation  $Lx = Nx$  has at least one solution in  $\text{dom}L \cap \overline{\Omega}$ .

In the following, in order to obtain the existence theorem of (1.1)-(1.3), we use the classical Banach space

$$PC[0,1] = \{x : x|_{(t_i, t_{i+1}]} \in C(t_i, t_{i+1}], \text{ there exist} \\ x(t_i^-) \text{ and } x(t_i^+) \text{ with } x(t_i^-) = x(t_i), i = 1, 2, \dots, k\}$$

with norm

$$\|x\|_{PC} = \sup\{|x(t)| : t \in [0, 1]\}.$$

Let  $u_\alpha(t) = t^{2-\alpha}u(t)$ . Take

$$X = \{u|u_\alpha, D_{0+}^{\alpha-1}u \in PC[0,1]\}, \quad Y = PC[0,1] \times R^{2k}.$$

It is easy to check that  $X$  is a Banach space with norm  $\|u\| = \max\{\|u_\alpha\|_{PC}, \|D_{0+}^{\alpha-1}u\|_{PC}\}$ ,  $Y$  is a Banach space with norm

$$\|y\|_Y = \max\{\|z\|_{PC}, |c|\}, \quad \forall y = (z, c) \in Y.$$

Define operator  $L = D_{0+}^\alpha$  with

$$\text{dom}L = \{u \in X \mid \lim_{t \rightarrow 0^+} t^{2-\alpha}u(t) = \sum_{j=1}^n a_j u(\xi_j), \quad u(1) = \sum_{j=1}^n b_j u(\eta_j)\}.$$

Let

$$L : \text{dom}L \rightarrow Y, \quad u \rightarrow (D_{0+}^\alpha u, \Delta u(t_1), \dots, \Delta u(t_k), \Delta D_{0+}^{\alpha-1}u(t_1), \dots, \Delta D_{0+}^{\alpha-1}u(t_k)),$$

$$N : X \rightarrow Y, \quad u \rightarrow (f(t, u, D_{0+}^{\alpha-1}u), I_1(u(t_1), D_{0+}^{\alpha-1}u(t_1)), \dots, I_k(u(t_k), D_{0+}^{\alpha-1}u(t_k)),$$

$$J_1(u(t_1), D_{0+}^{\alpha-1}u(t_1)), \dots, J_k(u(t_k), D_{0+}^{\alpha-1}u(t_k)).$$

Then problem (1.1)-(1.3) can be written as

$$Lu = Nu, \quad u \in \text{dom}L.$$

In this paper, we will always suppose the following conditions hold.

(H<sub>1</sub>)  $0 < \xi_1 < \xi_2 < \dots < \xi_n < 1$ ,  $0 < \eta_1 < \eta_2 < \dots < \eta_m < 1$ ,  $a_j, b_j$  ( $1 \leq j \leq n$ ) are

non-negative real constant numbers, and

$$\sum_{j=1}^n a_j \xi_j^{\alpha-2} = \sum_{j=1}^n b_j \eta_j^{\alpha-2} = 1, \quad \sum_{j=1}^n a_j \xi_j^{\alpha-1} = 0, \quad \sum_{j=1}^n b_j \eta_j^{\alpha-1} = 1. \quad (2.2)$$

$$(H_2) \quad \sigma = \begin{vmatrix} \sigma_1 & \sigma_2 \\ \sigma_3 & \sigma_4 \end{vmatrix} \neq 0, \text{ where}$$

$$\sigma_1 = \frac{1}{\alpha(\alpha+1)} \left( 1 - \sum_{j=1}^n b_j \eta_j^{\alpha+1} \right), \quad \sigma_2 = \frac{1}{\alpha(\alpha+1)} \sum_{j=1}^n a_j \xi_j^{\alpha+1},$$

$$\sigma_3 = \frac{1}{\alpha} \sum_{j=1}^n \left( 1 - \sum_{j=1}^n b_j \eta_j^\alpha \right), \quad \sigma_4 = \frac{1}{\alpha} \sum_{j=1}^n a_j \xi_j^\alpha.$$

**Remark 2.1.** If  $(H_1)$  holds, then the BVP (1.4), (1.5) has a nontrivial solution  $u(t) = h_1 t^{\alpha-1} + h_2 t^{\alpha-2}$ , where  $h_1, h_2 \in R$ .

**Lemma 2.3.** If  $(H_1)$  and  $(H_2)$  hold, then mapping  $L : \text{dom}L \subset X \rightarrow Y$  is a Fredholm mapping of index zero. Moreover,

$$\text{Ker}L = \{h_1 t^{\alpha-1} + h_2 t^{\alpha-2} : h_1, h_2 \in R\}, \quad (2.3)$$

and

$$\begin{aligned} \text{Im}L &= \{(z, c_1, \dots, c_k, d_1, \dots, d_k) : D_{0+}^\alpha u(t) = z(t), \Delta u(t_i) = c_i, \\ &\quad \Delta D_{0+}^{\alpha-1}(t_i) = d_i, i = 1, 2, \dots, k, \text{ for some } u(t) \in \text{dom}L\} \\ &= \left\{ (z, c_1, \dots, c_k, d_1, \dots, d_k) : \sum_{j=1}^n a_j \int_0^{\xi_j} (\xi_j - s)^{\alpha-1} z(s) ds + \sum_{j=1}^n a_j \xi_j^{\alpha-1} \sum_{t_i < \xi_j} d_i \right. \\ &\quad \left. + \Gamma(\alpha) \sum_{j=1}^n a_j \xi_j^{\alpha-2} \sum_{t_i < \xi_j} c_i t_i^{2-\alpha} - \sum_{j=1}^n a_j \xi_j^{\alpha-2} \sum_{t_i < \xi_j} d_i t_i = 0 \text{ and} \right. \\ &\quad \left. \int_0^1 (1-s)^{\alpha-1} z(s) ds - \sum_{j=1}^n b_j \eta_j^{2-\alpha} \int_0^{\eta_j} (\eta_j - s)^{\alpha-1} z(s) ds \right. \\ &\quad \left. + \frac{1}{\Gamma(\alpha)} \sum_{j=1}^n b_j \eta_j \sum_{\eta_j < t_i < 1} d_i + \sum_{j=1}^n b_j \sum_{\eta_j < t_i < 1} c_i t_i^{2-\alpha} - \frac{1}{\Gamma(\alpha)} \sum_{j=1}^n b_j \sum_{\eta_j < t_i < 1} d_i t_i = 0 \right\}. \end{aligned} \quad (2.4)$$

**Proof.** It is easy to see that (2.3) holds. Next, we will show that (2.4) holds. If  $(z, c_1, \dots, c_k, d_1, \dots, d_k)$

$\in \text{Im}L$ , then there exists  $u \in \text{dom}L$  such that

$$\begin{cases} D_{0+}^{\alpha} u(t) = z(t), \\ \Delta u(t_i) = c_i, \quad i = 1, 2, \dots, k, \\ \Delta D_{0+}^{\alpha-1}(t_i) = d_i, \quad i = 1, 2, \dots, k \end{cases} \quad (2.5)$$

has solution  $u(t)$  satisfying

$$\lim_{t \rightarrow 0^+} t^{2-\alpha} u(t) = \sum_{j=1}^n a_j u(\xi_j), \quad (2.6)$$

and

$$u(1) = \sum_{j=1}^n b_j u(\eta_j). \quad (2.7)$$

From (2.5) and Lemma 2.1, we obtain

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z(s) ds + \left( h_1 + \frac{1}{\Gamma(\alpha)} \sum_{t_i < t} d_i \right) t^{\alpha-1} \\ &\quad + \left( h_2 + \sum_{t_i < t} c_i t_i^{2-\alpha} - \frac{1}{\Gamma(\alpha)} \sum_{t_i < t} d_i t_i \right) t^{\alpha-2}, \end{aligned} \quad (2.8)$$

where  $h_1, h_2$  are two arbitrary constants. Substitute the boundary condition (2.6) into (2.8), one has

$$\begin{aligned} &\sum_{j=1}^n a_j \int_0^{\xi_j} (\xi_j - s)^{\alpha-1} z(s) ds + \sum_{j=1}^n a_j \xi_j^{\alpha-1} \sum_{t_i < \xi_j} d_i \\ &\quad + \Gamma(\alpha) \sum_{j=1}^n a_j \xi_j^{\alpha-2} \sum_{t_i < \xi_j} c_i t_i^{2-\alpha} - \sum_{j=1}^n a_j \xi_j^{\alpha-2} \sum_{t_i < \xi_j} d_i t_i = 0. \end{aligned} \quad (2.9)$$

Moreover, substitute condition (2.7) into (2.8), we obtain

$$\begin{aligned} &\int_0^1 (1-s)^{\alpha-1} z(s) ds - \sum_{j=1}^n b_j \int_0^{\eta_j} (\eta_j - s)^{\alpha-1} z(s) ds + \sum_{j=1}^n b_j \eta_j^{\alpha-1} \sum_{\eta_j < t_i < 1} d_i \\ &\quad + \Gamma(\alpha) \sum_{j=1}^n b_j \eta_j^{\alpha-2} \sum_{\eta_j < t_i < 1} c_i t_i^{2-\alpha} - \sum_{j=1}^n b_j \eta_j^{\alpha-2} \sum_{\eta_j < t_i < 1} d_i t_i = 0. \end{aligned} \quad (2.10)$$

Conversely, if (2.9) and (2.10) hold, setting

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z(s) ds + \frac{1}{\Gamma(\alpha)} \sum_{t_i < t} d_i t^{\alpha-1} + \left( \sum_{t_i < t} c_i t_i^{2-\alpha} - \frac{1}{\Gamma(\alpha)} \sum_{t_i < t} d_i t_i \right) t^{\alpha-2},$$

then it is easy to check that  $u(t)$  is a solution of (2.5) and satisfies (2.6), (2.7). Hence, (2.4) holds.

For convenience, let  $Z = (z, c_1, \dots, c_k, d_1, \dots, d_k)$ . Define operators  $T_1, T_2 : Y \rightarrow Y$  as follows :

$$T_1 Z = \left( \sum_{j=1}^n a_j \int_0^{\xi_j} (\xi_j - s)^{\alpha-1} z(s) ds + \sum_{j=1}^n a_j \xi_j^{\alpha-1} \sum_{t_i < \xi_j} d_i \right. \\ \left. + \Gamma(\alpha) \sum_{j=1}^n a_j \xi_j^{\alpha-2} \sum_{t_i < \xi_j} c_i t_i^{2-\alpha} - \sum_{j=1}^n a_j \xi_j^{\alpha-2} \sum_{t_i < \xi_j} d_i t_i, 0, \dots, 0 \right), \quad (2.11)$$

$$T_2 Z = \left( \int_0^1 (1-s)^{\alpha-1} z(s) ds - \sum_{j=1}^n b_j \int_0^{\eta_j} (\eta_j - s)^{\alpha-1} z(s) ds + \sum_{j=1}^n b_j \eta_j^{\alpha-1} \sum_{\eta_j < t_i < 1} d_i \right. \\ \left. + \Gamma(\alpha) \sum_{j=1}^n b_j \eta_j^{\alpha-2} \sum_{\eta_j < t_i < 1} c_i t_i^{2-\alpha} - \sum_{j=1}^n b_j \eta_j^{\alpha-2} \sum_{\eta_j < t_i < 1} d_i t_i, 0, \dots, 0 \right). \quad (2.12)$$

From (2.4), we have

$$\text{Im}L = \{Z \in Y | T_1 Z = T_2 Z = 0\}. \quad (2.13)$$

Define operator  $Q : Y \rightarrow Y$  as follows :

$$QZ = Q_1 Z + Q_2 Z \cdot t,$$

where

$$Q_1 Z = \frac{1}{\sigma} (\sigma_1 T_1 Z - \sigma_2 T_2 Z) := (\bar{z}_1, 0, \dots, 0),$$

$$Q_2 Z = -\frac{1}{\sigma} (\sigma_3 T_1 Z - \sigma_4 T_2 Z) := (z_1^*, 0, \dots, 0),$$

and  $\sigma, \sigma_i$  ( $i = 1, \dots, 4$ ) are as in  $(H_2)$ . Then

$$T_1(Q_1 Z) = \left( \frac{1}{\alpha} \sum_{j=1}^n a_j \xi_j^\alpha \bar{z}_1, 0, \dots, 0 \right) = \frac{1}{\alpha} \sum_{j=1}^n a_j \xi_j^\alpha (\bar{z}_1, 0, \dots, 0) \\ = \frac{1}{\alpha} \sum_{j=1}^n a_j \xi_j^\alpha \cdot Q_1 Z = \sigma_4 \cdot Q_1 Z,$$

$$\begin{aligned}
T_2(Q_1Z) &= \left( \frac{1}{\alpha} \left( 1 - \sum_{j=1}^n b_j \eta_j^\alpha \right) \bar{z}_1, 0, \dots, 0 \right) = \frac{1}{\alpha} \left( 1 - \sum_{j=1}^n b_j \eta_j^\alpha \right) (\bar{z}_1, 0, \dots, 0) \\
&= \frac{1}{\alpha} \left( 1 - \sum_{j=1}^n b_j \eta_j^\alpha \right) Q_1Z = \sigma_3 \cdot Q_1Z, \\
T_1(Q_2Z \cdot t) &= \left( \frac{1}{\alpha(\alpha+1)} \sum_{j=1}^n a_j \xi_j^{\alpha+1} z_1^*, 0, \dots, 0 \right) = \frac{1}{\alpha(\alpha+1)} \sum_{j=1}^n a_j \xi_j^{\alpha+1} (z_1^*, 0, \dots, 0) \\
&= \frac{1}{\alpha(\alpha+1)} \sum_{j=1}^n a_j \xi_j^{\alpha+1} \cdot Q_2Z = \sigma_2 \cdot Q_2Z, \\
T_2(Q_2Z \cdot t) &= \left( \frac{1}{\alpha(\alpha+1)} \left( 1 - \sum_{j=1}^n b_j \eta_j^{\alpha+1} \right) z_1^*, 0, \dots, 0 \right) \\
&= \frac{1}{\alpha(\alpha+1)} \left( 1 - \sum_{j=1}^n b_j \eta_j^{\alpha+1} \right) (z_1^*, 0, \dots, 0) \\
&= \frac{1}{\alpha(\alpha+1)} \left( 1 - \sum_{j=1}^n b_j \eta_j^{\alpha+1} \right) Q_2Z = \sigma_1 \cdot Q_2Z.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
Q_1^2Z &= \frac{1}{\sigma} (\sigma_1 T_1(Q_1Z) - \sigma_2 T_2(Q_1Z)) = \frac{1}{\sigma} (\sigma_1 \sigma_4 - \sigma_2 \sigma_3) Q_1Z = Q_1Z, \\
Q_2(Q_1Z) &= -\frac{1}{\sigma} (\sigma_3 T_1(Q_1Z) - \sigma_4 T_2(Q_1Z)) = -\frac{1}{\sigma} (\sigma_3 \sigma_4 - \sigma_4 \sigma_3) Q_1Z = 0, \\
Q_1(Q_2Z \cdot t) &= \frac{1}{\sigma} (\sigma_1 T_1(Q_2Z \cdot t) - \sigma_2 T_2(Q_2Z \cdot t)) = \frac{1}{\sigma} (\sigma_1 \sigma_2 - \sigma_2 \sigma_1) Q_2Z = 0, \\
Q_2(Q_2Z \cdot t) &= -\frac{1}{\sigma} (\sigma_3 T_1(Q_2Z \cdot t) - \sigma_4 T_2(Q_2Z \cdot t)) = -\frac{1}{\sigma} (\sigma_3 \sigma_2 - \sigma_4 \sigma_1) Q_2Z = Q_2Z.
\end{aligned}$$

Hence,

$$Q^2Z = Q_1(Q_1Z + Q_2Z \cdot t) + Q_2(Q_1Z + Q_2Z \cdot t)t = Q_1Z + Q_2Z \cdot t = QZ,$$

which implies the operator  $Q$  is a projector.

Now, we show that  $\text{Ker}Q = \text{Im}L$ . Obviously,  $\text{Ker}Q \subset \text{Im}L$ . On the other hand, if  $Z \in \text{Im}L$ , from  $QZ = 0$ , we have



$$\begin{cases} \sigma_1 T_1 Z - \sigma_2 T_2 Z = 0, \\ \sigma_3 T_1 Z - \sigma_4 T_2 Z = 0. \end{cases}$$

Since  $\sigma = \begin{vmatrix} \sigma_1 & \sigma_2 \\ \sigma_3 & \sigma_4 \end{vmatrix} \neq 0$ , we get  $T_1 Z = T_2 Z = 0$ , which yields  $Z \in \text{Ker}Q$ . Hence,  $\text{Ker}Q = \text{Im}L$ .

For  $Z \in Y$ , set  $Z = (Z - QZ) + QZ$ . Then,  $Z - QZ \in \text{Ker}Q = \text{Im}L$ ,  $QZ \in \text{Im}Q$ , we have  $Y = \text{Im}L + \text{Im}Q$ . Moreover, it follows from  $\text{Ker}Q = \text{Im}L$  and  $Q^2 Z = QZ$  that  $\text{Im}Q \cap \text{Im}L = \{0\}$ . So,  $Y = \text{Im}L \oplus \text{Im}Q$ . Since  $\dim \text{Ker}L = \dim \text{Im}Q = \text{codim} \text{Im}L = 2$ ,  $L$  is a Fredholm map of index zero.

Define  $P : X \rightarrow X$  by

$$Pu(t) = \lim_{t \rightarrow 0^+} t^{2-\alpha} u(t) \cdot t^{\alpha-2} + \frac{1}{\Gamma(\alpha)} D_{0^+}^{\alpha-1} u(0) \cdot t^{\alpha-1}.$$

Moreover, we define operator  $K_P : \text{Im}L \rightarrow X$  as follows :

$$\begin{aligned} & K_P(z, c_1, \dots, c_k, d_1, \dots, d_k) \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z(s) ds + \frac{1}{\Gamma(\alpha)} \sum_{t_i < t} d_i \cdot t^{\alpha-1} + \left( \sum_{t_i < t} c_i t_i^{2-\alpha} - \frac{1}{\Gamma(\alpha)} \sum_{t_i < t} d_i t_i \right) t^{\alpha-2}. \end{aligned} \tag{2.14}$$

**Lemma 2.4.**  $P : X \rightarrow X$  is a linear continuous projector operator and  $K_P$  is the inverse of  $L|_{\text{dom}L \cap \text{Ker}P}$ .

**Proof.** Obviously,  $\text{Im}P = \text{Ker}L$  and

$$\begin{aligned} (P^2 u)(t) &= P(Pu(t)) = \lim_{t \rightarrow 0^+} t^{2-\alpha} Pu(t) \cdot t^{\alpha-2} + \frac{1}{\Gamma(\alpha)} D_{0^+}^{\alpha-1} Pu(0) \cdot t^{\alpha-1} \\ &= \lim_{t \rightarrow 0^+} t^{2-\alpha} u(t) \cdot t^{\alpha-2} + \frac{1}{\Gamma(\alpha)} D_{0^+}^{\alpha-1} u(0) \cdot t^{\alpha-1} = (Pu)(t), \end{aligned}$$

since

$$\begin{aligned} D_{0^+}^{\alpha-1} Pu(t) &= \frac{1}{\Gamma(2-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{1-\alpha} \left( \lim_{t \rightarrow 0^+} t^{2-\alpha} u(t) \cdot \tau^{\alpha-2} + \frac{1}{\Gamma(\alpha)} D_{0^+}^{\alpha-1} u(0) \cdot \tau^{\alpha-1} \right) d\tau \\ &= \frac{1}{\Gamma(2-\alpha)} \frac{d}{dt} \left[ \lim_{t \rightarrow 0^+} t^{2-\alpha} u(t) \cdot \Gamma(2-\alpha) \Gamma(\alpha-1) + \Gamma(2-\alpha) D_{0^+}^{\alpha-1} u(0) \cdot t \right] \\ &= D_{0^+}^{\alpha-1} u(0). \end{aligned}$$

Hence,  $P : X \rightarrow X$  is a continuous linear projector. It follows from  $u = (u - Pu) + Pu$  that  $X = \text{Ker}P + \text{Ker}L$ . Moreover, we can easily obtain that  $\text{Ker}L \cap \text{Ker}P = \{0\}$ . Thus, we have

$$X = \text{Ker}L \oplus \text{Ker}P.$$

By some calculation, it is easy to check that  $K_P(\text{Im}L) \subset \text{Ker}P \cap \text{dom}L$ . In the following, we will prove that  $K_P$  is the inverse of  $L|_{\text{dom}L \cap \text{Ker}P}$ .

If  $Z \in \text{Im}L$ , then  $LK_PZ = Z$ . On the other hand, for  $u \in \text{dom}L \cap \text{Ker}P$ , we have by (2.14) that

$$\begin{aligned} (K_PL)u(t) &= K_P(D_{0+}^\alpha u(t), c_1, \dots, c_k, d_1, \dots, d_k) \\ &= u(t) + \left( h_1 + \frac{1}{\Gamma(\alpha)} \sum_{t_i < t} d_i \right) t^{\alpha-1} + \left( h_2 + \sum_{t_i < t} c_i t_i^{2-\alpha} - \frac{1}{\Gamma(\alpha)} \sum_{t_i < t} d_i t_i \right) t^{\alpha-2}, \end{aligned} \quad (2.15)$$

where  $c_i = \Delta u(t_i)$ ,  $d_i = \Delta D_{0+}^{\alpha-1} u(t_i)$ ,  $i = 1, 2, \dots, k$ , and  $h_1, h_2$  are two arbitrary constants. Noting that  $D_{0+}^{\alpha-1} t^{\alpha-2} = 0$  and  $D_{0+}^{\alpha-1} t^{\alpha-1} = \Gamma(\alpha)$ , we get by (2.15) that

$$D_{0+}^{\alpha-1} K_PLu(t) = D_{0+}^{\alpha-1} u(t) + \Gamma(\alpha) h_1 + \sum_{t_i < t} d_i. \quad (2.16)$$

From  $u \in \text{Ker}P$  and  $K_PLu \in \text{Ker}P$ , we obtain

$$\begin{aligned} \lim_{t \rightarrow 0^+} t^{2-\alpha} u(t) &= D_{0+}^{\alpha-1} u(0) = 0, \\ \lim_{t \rightarrow 0^+} t^{2-\alpha} K_PLu(t) &= \lim_{t \rightarrow 0^+} t^{2-\alpha} u(t) + h_2 + \sum_{t_i < t} c_i t_i^{2-\alpha} - \frac{1}{\Gamma(\alpha)} \sum_{t_i < t} d_i t_i = 0, \\ D_{0+}^{\alpha-1} K_PLu(0) &= D_{0+}^{\alpha-1} u(0) + \Gamma(\alpha) h_1 + \sum_{t_i < t} d_i = 0, \quad (\text{by (2.16)}) \end{aligned}$$

which imply that

$$h_1 + \frac{1}{\Gamma(\alpha)} \sum_{t_i < t} d_i = 0, \quad h_2 + \sum_{t_i < t} c_i t_i^{2-\alpha} - \frac{1}{\Gamma(\alpha)} \sum_{t_i < t} d_i t_i = 0.$$

So,  $K_PLu = u$ . Thus  $K_P = (L|_{\text{dom}L \cap \text{Ker}P})^{-1}$ .

**Lemma 2.5.** Assume that  $\Omega \subset X$  is an open bounded subset with  $\text{dom}L \cap \overline{\Omega} \neq \emptyset$ , then  $N$  is  $L$ -compact on  $\overline{\Omega}$ .

**Proof.** From Lemma 2.4, we know that  $K_P$  is the inverse of  $L|_{\text{dom}L \cap \text{Ker}P}$ . By (2.11) and (2.12), we have

$$T_1Nu = (T_1Nu^{(1)}, 0, \dots, 0), \quad T_2Nu = (T_2Nu^{(1)}, 0, \dots, 0),$$

where

$$\begin{aligned} T_1Nu^{(1)} &= \sum_{j=1}^n a_j \int_0^{\xi_j} (\xi_j - s)^{\alpha-1} f(s, u(s), D_{0+}^{\alpha-1}u(s)) ds + \sum_{j=1}^n a_j \xi_j^{\alpha-1} \sum_{t_i < \xi_j} J_i \\ &\quad + \Gamma(\alpha) \sum_{j=1}^n a_j \xi_j^{\alpha-2} \sum_{t_i < \xi_j} I_i t_i^{2-\alpha} - \sum_{j=1}^n a_j \xi_j^{\alpha-2} \sum_{t_i < \xi_j} J_i t_i, \end{aligned} \quad (2.17)$$

$$\begin{aligned} T_2Nu^{(1)} &= \int_0^1 (1-s)^{\alpha-1} f(s, u(s), D_{0+}^{\alpha-1}u(s)) ds \\ &\quad - \sum_{j=1}^n b_j \int_0^{\eta_j} (\eta_j - s)^{\alpha-1} f(s, u(s), D_{0+}^{\alpha-1}u(s)) ds + \sum_{j=1}^n b_j \eta_j^{\alpha-1} \sum_{\eta_j < t_i < 1} J_i \\ &\quad + \Gamma(\alpha) \sum_{j=1}^n b_j \eta_j^{\alpha-2} \sum_{\eta_j < t_i < 1} I_i t_i^{2-\alpha} - \sum_{j=1}^n b_j \eta_j^{\alpha-2} \sum_{\eta_j < t_i < 1} J_i t_i. \end{aligned} \quad (2.18)$$

Here,  $I_i = I_i(u(t_i), D_{0+}^{\alpha-1}u(t_i))$ ,  $J_i = J_i(u(t_i), D_{0+}^{\alpha-1}u(t_i))$ ,  $i = 1, \dots, k$ . Thus, we have

$$QNu = (u^*, 0, \dots, 0), \quad (2.19)$$

where

$$\begin{aligned} u^* &= \frac{\sigma_1 - \sigma_3 t}{\sigma} \left( \sum_{j=1}^m a_j \int_0^{\xi_j} (\xi_j - s)^{\alpha-1} f(s, u(s), D_{0+}^{\alpha-1}u(s)) ds + \sum_{j=1}^m a_j \xi_j^{\alpha-1} \sum_{t_i < \xi_j} J_i \right. \\ &\quad \left. + \Gamma(\alpha) \sum_{j=1}^m a_j \xi_j^{\alpha-2} \sum_{t_i < \xi_j} I_i t_i^{2-\alpha} - \sum_{j=1}^m a_j \xi_j^{\alpha-2} \sum_{t_i < \xi_j} J_i t_i \right) \\ &\quad + \frac{\sigma_4 t - \sigma_2}{\sigma} \left( \int_0^1 (1-s)^{\alpha-1} f(s, u(s), D_{0+}^{\alpha-1}u(s)) ds \right. \\ &\quad - \sum_{j=1}^n b_j \int_0^{\eta_j} (\eta_j - s)^{\alpha-1} f(s, u(s), D_{0+}^{\alpha-1}u(s)) ds + \sum_{j=1}^n b_j \eta_j^{\alpha-1} \sum_{\eta_j < t_i < 1} J_i \\ &\quad \left. + \Gamma(\alpha) \sum_{j=1}^n b_j \eta_j^{\alpha-2} \sum_{\eta_j < t_i < 1} I_i t_i^{2-\alpha} - \sum_{j=1}^n b_j \eta_j^{\alpha-2} \sum_{\eta_j < t_i < 1} J_i t_i \right), \end{aligned}$$

and

$$K_P(I - Q)Nu$$

$$\begin{aligned}
&= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s), D_{0+}^{\alpha-1} u(s)) ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \sum_{t_i < t} J_i \cdot t^{\alpha-1} + \left( \sum_{t_i < t} I_i t_i^{2-\alpha} - \frac{1}{\Gamma(\alpha)} \sum_{t_i < t} J_i t_i \right) t^{\alpha-2} \\
&\quad + \frac{t^\alpha}{\Gamma(\alpha+1)\sigma} \left( \sigma_1 - \frac{\sigma_3 t}{\alpha+1} \right) \left( \sum_{j=1}^m a_j \int_0^{\xi_j} (\xi_j - s)^{\alpha-1} f(s, u(s), D_{0+}^{\alpha-1} u(s)) ds \right. \\
&\quad \quad \left. + \sum_{j=1}^m a_j \xi_j^{\alpha-1} \sum_{t_i < \xi_j} J_i + \Gamma(\alpha) \sum_{j=1}^m a_j \xi_j^{\alpha-2} \sum_{t_i < \xi_j} I_i t_i^{2-\alpha} - \sum_{j=1}^m a_j \xi_j^{\alpha-2} \sum_{t_i < \xi_j} J_i t_i \right) \\
&\quad + \frac{t^\alpha}{\Gamma(\alpha+1)\sigma} \left( \frac{\sigma_4 t}{\alpha+1} - \sigma_2 \right) \left( \int_0^1 (1-s)^{\alpha-1} f(s, u(s), D_{0+}^{\alpha-1} u(s)) ds \right. \\
&\quad \quad - \sum_{j=1}^n b_j \int_0^{\eta_j} (\eta_j - s)^{\alpha-1} f(s, u(s), D_{0+}^{\alpha-1} u(s)) ds + \sum_{j=1}^n b_j \eta_j^{\alpha-1} \sum_{\eta_j < t_i < 1} J_i \\
&\quad \quad \left. + \Gamma(\alpha) \sum_{j=1}^n b_j \eta_j^{\alpha-2} \sum_{\eta_j < t_i < 1} I_i t_i^{2-\alpha} - \sum_{j=1}^n b_j \eta_j^{\alpha-2} \sum_{\eta_j < t_i < 1} J_i t_i \right).
\end{aligned}$$

By using the Ascoli-Arzelà theorem, we can prove that  $QN(\overline{\Omega})$  is bounded and  $K_P(I - Q)N : \overline{\Omega} \rightarrow X$  is compact. Hence,  $N$  is  $L$ -compact on  $\overline{\Omega}$ .

### 3. Main result

Denote by  $L^1[0, 1]$  the space of all Lebesgue integrable functions on  $[0, 1]$ . It is well known that  $L^1[0, 1]$  is a Banach space with norm  $\|u\|_1 = \int_0^1 |u(t)| dt$ .

To obtain our main result, we need the following conditions.

(H<sub>3</sub>) There exist positive numbers  $p_{i1}, p_{i2}, q_{i1}, q_{i2}$  ( $i = 1, \dots, k$ ) such that

$$|I_i(x, y)| \leq p_{i1}|x| + p_{i2}|y|,$$

$$|J_i(x, y)| \leq q_{i1}|x| + q_{i2}|y|.$$

(H<sub>4</sub>) There exist functions  $\phi, \beta, \gamma \in C[0, 1]$  such that

$$|f(t, x, y)| \leq |\phi(t)| + t^{2-\alpha} |\beta(t)| |x| + |\gamma(t)| |y|, \quad \forall (t, x, y) \in [0, 1] \times \mathbb{R}^2.$$

(H<sub>5</sub>) For  $u \in \text{dom}L$ , there exist two constants  $a^* \in (0, 1]$  and  $M^* > 0$  such that if  $|D_{0+}^{\alpha-1} u(t)| > M^*$  for all  $t \in [0, a^*]$ , then either

$$D_{0+}^{\alpha-1}u(t) \cdot T_1Nu^{(1)} > 0 \quad \text{or} \quad D_{0+}^{\alpha-1}u(t) \cdot T_1Nu^{(1)} < 0,$$

where  $T_1Nu^{(1)}$  is as in (2.17).

(H<sub>6</sub>) For  $u \in \text{dom}L$ , there exist two constants  $a_* \in (0, 1)$  and  $M_* > 0$  such that if  $|u(t)| > M_*$  for all  $t \in [a_*, 1]$ , then either

$$u(t) \cdot T_2Nu^{(1)} > 0 \quad \text{or} \quad u(t) \cdot T_2Nu^{(1)} < 0,$$

where  $T_2Nu^{(1)}$  is as in (2.18).

**Remark 3.1.** If (H<sub>5</sub>) holds, then  $T_1Nu(t) \neq (0, 0, \dots, 0)$ ,  $\forall t \in [0, a^*]$ . And if (H<sub>6</sub>) holds, then  $T_2Nu(t) \neq (0, 0, \dots, 0)$ ,  $\forall t \in [a_*, 1]$ .

**Theorem 3.1.** Let  $f : [0, 1] \times R^2 \rightarrow R$  and  $I_i, J_i : R \rightarrow R$  ( $i = 1, \dots, k$ ) be continuous. Assume (H<sub>1</sub>) – (H<sub>4</sub>) hold. In addition, suppose that either the first part of (H<sub>5</sub>) and (H<sub>6</sub>) hold or the second part of (H<sub>5</sub>) and (H<sub>6</sub>) hold. Then the boundary value problem (1.1)-(1.3) has at least one solution in  $X$  provided that

$$B\bar{A} < (1 - A)(1 - \bar{B}), \tag{3.1}$$

where

$$A = \frac{4}{\Gamma(\alpha)} \|\beta\|_1 + \frac{2}{\Gamma(\alpha)} \sum_{i=1}^k q_{i1}(t_i^{\alpha-1} + 2t_i^{\alpha-2}) + 2 \sum_{i=1}^k p_{i1} < 1, \tag{3.2}$$

$$B = \frac{4}{\Gamma(\alpha)} \|\gamma\|_1 + \frac{2}{\Gamma(\alpha)} \sum_{i=1}^k q_{i2}(2 + t_i) + 2 \sum_{i=1}^k p_{i2}, \tag{3.3}$$

$$\bar{A} = 2\|\beta\|_1 + 2 \sum_{i=1}^k q_{i1}t_i^{\alpha-2}, \quad \bar{B} = 2\|\gamma\|_1 + 2 \sum_{i=1}^k q_{i2} < 1. \tag{3.4}$$

**Proof.** Set

$$\Omega_1 = \{u \in \text{dom}L \setminus \text{Ker}L : Lu = \lambda Nu, \text{ for some } \lambda \in (0, 1)\}.$$

For  $u \in \Omega_1$ , we have  $u \notin \text{Ker}L$  and  $Nu \in \text{Im}L$ . By (2.13), we get that  $T_1Nu = T_2Nu = 0$ . Thus, from (H<sub>5</sub>), (H<sub>6</sub>) and Remark 3.1, we obtain that there exist constants  $t_* \in [a_*, 1]$  and  $t^* \in [0, a^*]$  such that

$$|u(t_*)| \leq M_*, \quad |D_{0+}^{\alpha-1}u(t^*)| \leq M^*. \tag{3.5}$$

It follows from  $Lu = \lambda Nu$  that

$$D_{0+}^{\alpha} u(t) = \lambda f(t, u(t), D_{0+}^{\alpha-1} u(t)), \quad t \neq t_i, \quad (3.6)$$

$$\Delta u(t_i) = \lambda I_i(u(t_i), D_{0+}^{\alpha-1} u(t_i)), \quad i = 1, \dots, k, \quad (3.7)$$

$$\Delta D_{0+}^{\alpha-1} u(t_i) = \lambda J_i(u(t_i), D_{0+}^{\alpha-1} u(t_i)), \quad i = 1, \dots, k. \quad (3.8)$$

From (3.6)-(3.8) and noticing that  $u \in \text{dom}L$ , we have by (2.5) and (2.8) that

$$\begin{aligned} u(t) &= \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s), D_{0+}^{\alpha-1} u(s)) ds + \left( h_1 + \frac{\lambda}{\Gamma(\alpha)} \sum_{t_i < t} J_i \right) t^{\alpha-1} \\ &\quad + \left( h_2 + \lambda \sum_{t_i < t} I_i t_i^{2-\alpha} - \frac{\lambda}{\Gamma(\alpha)} \sum_{t_i < t} J_i t_i \right) t^{\alpha-2}. \end{aligned} \quad (3.9)$$

From (3.9) and Lemma 2.1, we get

$$D_{0+}^{\alpha-1} u(t) = \lambda \int_0^t f(s, u(s), D_{0+}^{\alpha-1} u(s)) ds + h_1 \Gamma(\alpha) + \lambda \sum_{t_i < t} J_i. \quad (3.10)$$

By (3.5), (3.9) and (3.10), we have

$$\begin{aligned} |h_1| &= \frac{1}{\Gamma(\alpha)} \left| D_{0+}^{\alpha-1} u(t^*) - \lambda \int_0^{t^*} f(s, u(s), D_{0+}^{\alpha-1} u(s)) ds - \lambda \sum_{t_i < t^*} J_i \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \left( M^* + \int_0^1 |f(s, u(s), D_{0+}^{\alpha-1} u(s))| ds + \sum_{i=1}^k |J_i| \right), \end{aligned} \quad (3.11)$$

and

$$\begin{aligned} |h_2| &= \left| t_*^{2-\alpha} u(t_*) - \frac{\lambda}{\Gamma(\alpha)} t_*^{2-\alpha} \int_0^{t_*} (t_*-s)^{\alpha-1} f(s, u(s), D_{0+}^{\alpha-1} u(s)) ds \right. \\ &\quad \left. - \left( h_1 + \frac{\lambda}{\Gamma(\alpha)} \sum_{t_i < t_*} J_i \right) t_* - \lambda \sum_{t_i < t_*} I_i t_i^{2-\alpha} + \frac{\lambda}{\Gamma(\alpha)} \sum_{t_i < t_*} J_i t_i \right| \\ &\leq M_* + \frac{1}{\Gamma(\alpha)} \int_0^1 |f(s, u(s), D_{0+}^{\alpha-1} u(s))| ds + |h_1| + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^k |J_i| (1+t_i) + \sum_{i=1}^k |I_i| t_i^{2-\alpha}. \end{aligned} \quad (3.12)$$

Substitute (3.11) and (3.12) into (3.9), we have by (H<sub>3</sub>) and (H<sub>4</sub>) that

$$|t^{2-\alpha} u(t)| \leq \frac{1}{\Gamma(\alpha)} t^{2-\alpha} \int_0^t (t-s)^{\alpha-1} |f(s, u(s), D_{0+}^{\alpha-1} u(s))| ds + \left( |h_1| + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^k |J_i| \right) t$$

$$\begin{aligned}
& + |h_2| + \sum_{i=1}^k |I_i| t_i^{2-\alpha} + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^k |J_i| t_i \\
\leq & \frac{1}{\Gamma(\alpha)} \int_0^1 |f(s, u(s), D_{0+}^{\alpha-1} u(s))| ds + |h_1| + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^k |J_i| (1 + t_i) + |h_2| + \sum_{i=1}^k |I_i| t_i^{2-\alpha} \\
\leq & \frac{4}{\Gamma(\alpha)} \int_0^1 |f(s, u(s), D_{0+}^{\alpha-1} u(s))| ds + \frac{2}{\Gamma(\alpha)} M^* + M_* \\
& + \frac{2}{\Gamma(\alpha)} \sum_{i=1}^k |J_i| (2 + t_i) + 2 \sum_{i=1}^k |I_i| t_i^{2-\alpha} \\
\leq & \frac{4}{\Gamma(\alpha)} \int_0^1 [|\phi(s)| + s^{2-\alpha} |\beta(s)| |u(s)| + |\gamma(s)| |D_{0+}^{\alpha-1} u(s)|] ds + \frac{2}{\Gamma(\alpha)} M^* + M_* \\
& + \frac{2}{\Gamma(\alpha)} \sum_{i=1}^k (q_{i1} |u(t_i)| + q_{i2} |D_{0+}^{\alpha-1} u(t_i)|) (2 + t_i) \\
& + 2 \sum_{i=1}^k (p_{i1} |u(t_i)| + p_{i2} |D_{0+}^{\alpha-1} u(t_i)|) t_i^{2-\alpha} \\
\leq & \frac{4}{\Gamma(\alpha)} \left[ \|\phi\|_1 + \int_0^1 |\beta(s)| ds \cdot \|u_\alpha\|_{PC} + \int_0^1 |\gamma(s)| ds \cdot \|D_{0+}^{\alpha-1} u\|_{PC} \right] + \frac{2}{\Gamma(\alpha)} M^* + M_* \\
& + \frac{2}{\Gamma(\alpha)} \sum_{i=1}^k q_{i1} (t_i^{\alpha-1} + 2t_i^{\alpha-2}) \|u_\alpha\|_{PC} + \frac{2}{\Gamma(\alpha)} \sum_{i=1}^k q_{i2} (2 + t_i) \|D_{0+}^{\alpha-1} u\|_{PC} \\
& + 2 \sum_{i=1}^k p_{i1} \|u_\alpha\|_{PC} + 2 \sum_{i=1}^k p_{i2} \|D_{0+}^{\alpha-1} u\|_{PC} \\
= & A \|u_\alpha\|_{PC} + B \|D_{0+}^{\alpha-1} u\|_{PC} + C, \tag{3.13}
\end{aligned}$$

where  $A, B$  are as in (3.2), (3.3), respectively, and  $C = \frac{4}{\Gamma(\alpha)} \|\phi\|_1 + \frac{2}{\Gamma(\alpha)} M^* + M_*$ .

Moreover, we have

$$\begin{aligned}
|D_{0+}^{\alpha-1} u(t)| & = \left| \lambda \int_0^t f(s, u(s), D_{0+}^{\alpha-1} u(s)) ds + \Gamma(\alpha) h_1 + \lambda \sum_{t_i < t} J_i \right| \\
& \leq 2 \int_0^1 |f(s, u(s), D_{0+}^{\alpha-1} u(s))| ds + M^* + 2 \sum_{i=1}^k |J_i| \\
& \leq \bar{A} \|u_\alpha\|_{PC} + \bar{B} \|D_{0+}^{\alpha-1} u\|_{PC} + \bar{C}, \tag{3.14}
\end{aligned}$$

where  $\bar{A}$  and  $\bar{B}$  are as in (3.4), and  $\bar{C} = M^* + 2\|\phi\|_1$ .

Hence,

$$\|D_{0+}^{\alpha-1}u\|_{PC} \leq \frac{1}{1-\bar{B}}(\bar{C} + \bar{A}\|u_\alpha\|_{PC}). \quad (3.15)$$

Substitute (3.15) into (3.13), we obtain

$$\|u_\alpha\|_{PC} \leq \frac{B\bar{C} + C(1-\bar{B})}{(1-A)(1-\bar{B}) - B\bar{A}}. \quad (3.16)$$

Therefore, by (3.1) and (3.16), we know that  $\Omega_1$  is bounded.

Let

$$\Omega_2 = \{u \in \text{Ker}L : Nu \in \text{Im}L\}.$$

For  $u \in \Omega_2$ , we have  $u(t) = h_1t^{\alpha-1} + h_2t^{\alpha-2}$ ,  $h_1, h_2 \in R$  and  $T_1Nu^{(1)} = T_2Nu^{(1)} = 0$ . From (H<sub>6</sub>), we get  $|D_{0+}^{\alpha-1}u(t)| = |h_1|\Gamma(\alpha) \leq M^*$ , that is  $|h_1| \leq \frac{M^*}{\Gamma(\alpha)}$ . Moreover, by (H<sub>5</sub>), there exists  $t_* \in [a_*, 1]$  such that  $|u(t_*)| \leq M_*$ . Thus, we obtain  $|h_2| \leq M_*t_*^{2-\alpha} + |h_1|t_* \leq M_* + \frac{M^*}{\Gamma(\alpha)}$ . So,

$$\|u_\alpha\|_{PC} = \|h_1t + h_2\|_{PC} \leq M_* + \frac{2M^*}{\Gamma(\alpha)}, \quad \|D_{0+}^{\alpha-1}u\|_{PC} = |\Gamma(\alpha)h_1| \leq M^*, \quad (3.17)$$

which implies that  $\Omega_2$  is bounded in  $X$ .

If the first part of (H<sub>5</sub>) and (H<sub>6</sub>) hold, set

$$\Omega_3 = \{u \in \text{Ker}L : \lambda\Lambda u + (1-\lambda)QNu = 0, \lambda \in [0, 1]\},$$

here  $\Lambda : \text{Ker}L \rightarrow \text{Im}Q$  is the linear isomorphism given by

$$\Lambda(h_1t^{\alpha-1} + h_2t^{\alpha-2}) = \left( \frac{1}{\sigma}(\sigma_1h_1 - \sigma_2h_2) + \frac{1}{\sigma}(-\sigma_3h_1 + \sigma_4h_2)t, 0, \dots, 0 \right),$$

where  $h_1, h_2 \in R$ . For  $u_* = h_1t^{\alpha-1} + h_2t^{\alpha-2} \in \Omega_3$ , we have

$$\begin{aligned} \lambda\Lambda u_* + (1-\lambda)QNu_* &= (\lambda\Lambda u_*^{(1)} + (1-\lambda)(Q_1Nu_* + Q_2Nu_* \cdot t)^{(1)}, 0, \dots, 0) \\ &= (0, 0, \dots, 0) \end{aligned}$$

which implies that

$$\begin{aligned} &\lambda \left( \frac{1}{\sigma}(\sigma_1h_1 - \sigma_2h_2) + \frac{1}{\sigma}(-\sigma_3h_1 + \sigma_4h_2)t \right) \\ &+ (1-\lambda) \left[ \frac{1}{\sigma}(\sigma_1T_1Nu_*^{(1)} - \sigma_2T_2Nu_*^{(1)}) - \frac{1}{\sigma}(\sigma_3T_1Nu_*^{(1)} - \sigma_4T_2Nu_*^{(1)}) \cdot t \right] = 0. \end{aligned}$$



Thus, we obtain

$$\lambda h_1 + (1 - \lambda)T_1Nu_*^{(1)} = 0, \quad (3.18)$$

$$\lambda h_2 + (1 - \lambda)T_2Nu_*^{(1)} = 0. \quad (3.19)$$

In (3.18), if  $\lambda = 1$ , then  $|h_1| = 0 \leq \frac{M^*}{\Gamma(\alpha)}$ . Otherwise, if  $|h_1| > \frac{M^*}{\Gamma(\alpha)}$ , that is  $|D_{0+}^{\alpha-1}u_*| > M^*$ , then we have by the first part of (H<sub>5</sub>) that

$$D_{0+}^{\alpha-1}u_* \cdot T_1Nu_*^{(1)} > 0. \quad (3.20)$$

Multiplying (3.18) by  $h_1$ , we have

$$\lambda h_1^2 + (1 - \lambda)h_1 \cdot T_1Nu_*^{(1)} = \lambda h_1^2 + \frac{1 - \lambda}{\Gamma(\alpha)}D_{0+}^{\alpha-1}u_* \cdot T_1Nu_*^{(1)} = 0$$

which contradicts to (3.20). Thus, we obtain that  $|h_1| \leq \frac{M^*}{\Gamma(\alpha)}$ . Similarly, by the first part of (H<sub>6</sub>) and (3.19), we can show that  $|h_2| \leq M_* + \frac{M^*}{\Gamma(\alpha)}$ . Thus,  $\Omega_3$  is bounded.

If the second part of (H<sub>5</sub>) and (H<sub>6</sub>) hold, then define the set

$$\Omega_3 = \{u \in \text{Ker}L : -\lambda\Lambda u + (1 - \lambda)QNu = 0, \lambda \in [0, 1]\},$$

where  $\Lambda$  as in above. Similar to above argument, we can show that  $\Omega_3$  is bounded too.

Finally, set  $\Omega$  be a bounded open set of  $X$  such that  $\cup_{i=1}^3 \overline{\Omega}_i \subset \Omega$ . By Lemma 2.4,  $N$  is  $L$ -compact on  $\overline{\Omega}$ . Then by the above arguments, we have

- (i)  $Lx \neq \lambda Nx$  for every  $(x, \lambda) \in [(\text{dom}L \setminus \text{Ker}L \cap \partial\Omega) \times (0, 1)]$ ;
- (ii)  $Nx \notin \text{Im}L$  for every  $x \in \text{Ker}L \cap \partial\Omega$ .

In the following, we need only to prove that (iii) of Theorem 2.2 is satisfied. Let  $H(u, \lambda) = \pm\lambda\Lambda u + (1 - \lambda)QNu$ . According to the above argument, we know

$$H(u, \lambda) \neq 0, \quad \text{for all } u \in \text{Ker}L \cap \partial\Omega,$$

thus, by the homotopy property of degree

$$\begin{aligned} \deg(QN|_{\text{Ker}L}, \Omega \cap \text{Ker}L, 0) &= \deg(H(\cdot, 0), \Omega \cap \text{Ker}L, 0) \\ &= \deg(H(\cdot, 1), \Omega \cap \text{Ker}L, 0) = \deg(\pm\Lambda, \Omega \cap \text{Ker}L, 0) \neq 0. \end{aligned}$$

Then by Theorem 2.2,  $Lu = Nu$  has at least one solution in  $\text{dom}L \cap \overline{\Omega}$ , so that the problem (1.1)-(1.3) has one solution in  $X$ . The proof is complete.

## Acknowledgements

The author thanks the referees for their suggestions and comments which improved the presentation of this paper.

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(Received August 7, 2011)