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STRICT AND NON-STRICT INEQUALITIES FOR IMPLICIT FIRST ORDER CAUSAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, some fundamental strict and non-strict differential inequalities for the implicit perturbations of nonlinear first order ordinary causal differential equations have been established.

1. INTRODUCTION

Differential inequalities are crucial in the qualitative study of nonlinear differential equations and an extensive literature of differential inequalities along with some nice applications may be found in the research monographs of Lakshmikantham and Leela [6]. Similarly, the differential inequalities and comparison theorems involving the causal operators have been established in McNabb and Weir [8]. Thus, the differential inequalities for nonlinear initial and boundary value problems of ordinary differential equations of different orders have already discussed in the literature. Very recently the differential inequalities for implicit perturbations of first order initial value problems of ordinary differential equations have been studied in Dhage [3, 4], however, to the best of our knowledge, the differential inequalities for implicit perturbations of second type of causal differential equations have not been so far studied in the literature. In the present note, we establish strict and non-strict differential inequalities for nonlinear initial value problems of nonlinear implicit first order ordinary causal differential equations.

Given a bounded interval $J = [t_0, t_0 + a)$ in $\mathbb{R}$ for some $t_0, a \in \mathbb{R}$ with $a > 0$, let $C(J, \mathbb{R})$ be the class of continuous real-valued functions defined on J. An operator $Q : C(J, \mathbb{R}) = E \to E$ is said to be causal or nonanticipative if for any $x, y \in E$ with $x(s) = y(s)$, $t_0 \le s \le t$, we have that $(Qx)(s) = (Qy)(s)$ for $t_0 \le s \le t < t_0 + a$. Note that the sum and product of two causal operators is again a causal operator. Again, if $\{Q_n\}$ is a sequence of causal operators in E such that

$$\lim_{n\to\infty} (Q_n x)(t) = (Qx)(t)$$

for $(t, x) \in J \times E$, then Q is again a causal operator on E into itself.

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Given a causal operator $Q : E \to E$, consider a initial value problems for first order hybrid causal differential equation (in short HCDE) given by

$$\left.\begin{aligned} \frac{d}{dt}[f(t,x(t))] &= (Qx)(t),\ t \in J \\ x(t_0) &= x_0 \in \mathbb{R} \end{aligned}\right\} \tag{1.1}$$

where, $f \in C(J \times \mathbb{R}, \mathbb{R})$.

By a *solution* of the ICDE (1.1) we mean a function $x \in C(J, \mathbb{R})$ that satisfies the equations in (1.1).

The ICDE (1.1) can be discussed for qualitative and quantitative properties via different approaches, but to the knowledge of author, there is no such result available in this direction. It is clear that the ICDE (1.1) is a implicit perturbation of second type of the well-known initial value problems of nonlinear first order ordinary differential equations (DE),

$$\left.\begin{aligned} x'(t) &= (Qx)(t),\ t \in J \\ x(t_0) &= x_0 \in \mathbb{R}. \end{aligned}\right\} \tag{1.2}$$

The details of causal differential equations appears in a recent monograph of Lakshmikantham *et al.* [7] and the details of different types of perturbations of the differential equations (1.1) appears in Dhage [2]. In this note we prove strict and non-strict differential inequalities related to the ICDE (1.1).

2. Strict and non-strict differential inequalities

We need the following definition in the subsequent development of the paper.

Definition 2.1. *A function $f : \mathbb{R} \to \mathbb{R}$ is called nondecreasing if for all $x, y \in \mathbb{R}$, $x \leq y$ implies that $f(x) \leq f(y)$. Again, f is called increasing if $x < y$ implies $f(x) < f(y)$ for all $x, y \in \mathbb{R}$. Similarly, the nonincreasing and decreasing functions on $\mathbb{R}$ into itself are defined.*

We also consider the following hypothesis in what follows.

- (A_0) The function $x \mapsto f(t,x)$ is increasing in $\mathbb{R}$ for all $t \in J$.
- (B_0) The causal operator Q is semi-nondecreasing, that is,

  $$x(t_1) = y(t_1),\ \ x(t) < y(t),\ t_0 \leq t < t_1$$

  implies

  $$(Qx)(t_1) = (Qy)(t_1),\ \ (Qx)(t) \leq (Qy)(t)$$

  for $t_0 \leq t < t_1 < t_0 + a$.

There do exist functions f satisfying the hypothesis (A_0). In fact the function $f(t,x) = t + x$, $x \in \mathbb{R}$ satisfies the hypothesis (A_0). Similarly, the hypothesis (B_0) has been widely used in the analysis of functional causal equations (see Corduneanu [1] and the references therein).

Our first differential inequality related to the ICDE (2.1) is

Theorem 2.1. *Assume that the hypotheses (A_0)-(B_0) hold. Suppose that there exist functions $y, z \in C(J, \mathbb{R})$ such that*

$$\frac{d}{dt}[f(t, y(t))] < (Qy)(t), \tag{2.1}$$

and

$$\frac{d}{dt}[f(t, z(t))] \geq (Qz)(t), \tag{2.2}$$

for all $t \in J$. Then,

$$y(t_0) < z(t_0) \tag{2.3}$$

implies

$$y(t) < z(t) \tag{2.4}$$

for all $t \in J$.

Proof. Suppose that the inequality (2.4) is false. Then the set Z^* defined by

$$Z^* = \{t \in J \mid y(t) \geq z(t),\ t \in J\} \tag{2.5}$$

is non-empty. Denote $t_1 = \inf Z^*$. Without loss of generality, we may assume that $y(t_1) = z(t_1)$ and $y(t) < z(t)$ for all $t < t_1$. Since Q is semi-nondecreasing, one has

$$(Qx)(t_1) = (Qy)(t_1),\ \ (Qx)(t) \leq (Qy)(t)$$

for $t_0 \leq t < t_1 < t_0 + a$.

Define the function Y and Z on J by

$$Y(t) = f(t, y(t)) \quad \text{and} \quad Z(t) = f(t, z(t))$$

for all $t \in J$. Then, in view of (A_0), we have

$$Y(t_1) = Z(t_1) \tag{2.6}$$

and by virtue of hypothesis (A_0), we get

$$Y(t) < Z(t) \tag{2.7}$$

for all $t < t_1$.

From (2.7) it follows that

$$\frac{Y(t_1 + h) - Y(t_1)}{h} > \frac{Z(t_1 + h) - Z(t_1)}{h}$$

for small $h < 0$. The above inequality implies that

$$Y'(t_1) \geq Z'(t_1)$$

or

$$(Qy)(t_1) > (Qz)(t_1).$$

This is a contradiction and hence the the set Z^* is empty. As a result, the inequality (2.4) holds for all $t \in J$. This completes the proof. □

Similarly, we can prove

Theorem 2.2. *Assume that the hypotheses (A_0)-(B_0) hold. Suppose that there exist functions $y, z \in C(J, \mathbb{R})$ such that*

$$\frac{d}{dt}[f(t, y(t))] \leq (Qy)(t), \tag{2.8}$$

and

$$\frac{d}{dt}[f(t, z(t))] > (Qz)(t), \tag{2.9}$$

for all $t \in J$. Then, the inequality (2.3) implies the inequality (2.4) on J.

Theorem 2.3. *Assume that the hypotheses (A_0)-(B_0) hold. Let the function $u \in C(J, \mathbb{R})$ satisfies* (2.1) *with y replaced by u and let the function $v \in C(J, \mathbb{R})$ satisfies* (2.8) *with z replaced by v on J. If w is any solution of the ICDE (1.1) existing on J with*

$$u(t_0) < w(t_0) < v(t_0), \tag{2.10}$$

then

$$u(t) < w(t) < v(t) \tag{2.11}$$

for all $t \in J$.

Corollary 2.1. *Assume that the hypothesis (A_0) holds. Let the causal operators $Q_1, Q_2 : C(J, \mathbb{R}) \to C(J, \mathbb{R})$ satisfy (B_0) and*

$$(Q_1 x)(t) < (Q_2 x)(t) \tag{2.12}$$

for all $t \in J$ and $x \in C(J, \mathbb{R})$. If u_1 and u_2 are any two solutions respectively of the ICDEs

$$\frac{d}{dt}[f(t, u(t))] = (Q_1 u)(t), \tag{2.13}$$

and

$$\frac{d}{dt}[f(t, u(t))] = (Q_2 u)(t), \tag{2.14}$$

for all $t \in J$ satisfying

$$u_1(t_0) < u_2(t_0). \tag{2.15}$$

Then,

$$u_1(t) < u_2(t) \tag{2.16}$$

for all $t \in J$.

Our next result is about the non-strict differential inequality related to the ICDE (1.1). Here, we use the one-sided Lipschitz type condition on the functions involved in the ICDE (1.1).

Theorem 2.4. *Assume that the hypotheses* (A_0)-(B_0) *hold and there exists a real number* $L > 0$ *such that*

$$(Qy)(t) - (Qz)(t) \le L \sup_{t_0 \le s \le t} \big[f(s, y(s)) - f(s, z(s))\big] \tag{2.17}$$

whenever $y(s) \ge z(s)$, $t_0 \le s \le t$. *Suppose that there exist functions* $y, z \in C(J, \mathbb{R})$ *such that*

$$\left.\begin{aligned} \frac{d}{dt}[f(t, y(t))] \le (Qy)(t),\ t \in J,\\ y(t_0) \le x_0, \end{aligned}\right\} \tag{2.18}$$

and

$$\left.\begin{aligned} \frac{d}{dt}[f(t, z(t))] \ge (Qz)(t),\ t \in J,\\ y(t_0) \ge x_0. \end{aligned}\right\} \tag{2.19}$$

Then,

$$y(t) \le z(t) \tag{2.20}$$

for all $t \in J$.

Proof. Let $\epsilon > 0$ and let a real number $L > 0$ be given. Set

$$f(t, z_\epsilon) = f(t, z) + \epsilon e^{2Lt} \tag{2.21}$$

so that

$$f(t, z_\epsilon) > f(t, z) \Longrightarrow z_\epsilon > z.$$

Let $Z_\epsilon = f(t, z_\epsilon)$ so that $Z = f(t, z)$ for $t \in J$. Then, by inequality (2.19),

$$Z'_\epsilon = Z' + 2L\epsilon e^{2Lt} \ge (Qz)(t) + 2L\epsilon e^{2Lt}. \tag{2.22}$$

Since

$$(Qz_\epsilon)(t) - (Qz)(t) \le L\,(f(t, z_\epsilon) - f(t, z))$$

for all $t \in J$, one has

$$Z'_\epsilon(t) \ge (Qz_\epsilon)(t) - L\epsilon e^{2Lt} + 2L\epsilon e^{2Lt} > (Qz_\epsilon)(t),$$

or

$$\frac{d}{dt}\,[f(t, z_\epsilon(t))] > (Qz_\epsilon)(t) \tag{2.23}$$

for all $t \in J$. Also, we have

$$z_\epsilon(t_0) > z(t_0) > y(t_0).$$

Hence, by an application of Theorem 2.1 yields that

$$y(t) < z_\epsilon(t) \tag{2.24}$$

for all $t \in J$. Taking the limit as $\epsilon \to 0$, we get $y(t) \le z(t)$ for all $t \in J$. □

Remark 2.1. The conclusion of Theorems 2.1, 2.2 and 2.4 also remains true if we replace the derivatives in the inequalities (2.1)- (2.2) and (2.18)- (2.19) by Dini-derivative D_{-} of the function $f(t, x(t))$ on the bounded interval J.

Open Problems. Finally, the ICDE (1.1) is open for the study of other aspects of the solutions such as existence, uniqueness and stability theory etc. We claim that the differential inequalities proved in this paper will be useful in settling down some of these problems under some suitable conditions. Some of the results in theses directions will be reported elsewhere.

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